Representation of Bipartite Graphs by OBDDs

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Abstract: We show upper and lower bounds for the worst-case OBDD size of certain bipartite graphs such as bipartite permutation graphs, biconvex graphs, convex graphs, 2-directional orthogonal ray graphs, and orthogonal ray graphs.

Keywords: Counting, Graph representation, OBDD, (Two-directional) orthogonal ray graphs, Permutation graphs

1. Introduction

In some applications such as nano-circuit design, we have to handle such huge graphs that the usual explicit representation by adjacency list or adjacency matrices is infeasible. To deal with such huge graphs, some implicit representations of graphs have been proposed. The *Ordered Binary Decision Diagram* (OBDD) [5], [18] has been considered as a promising implicit representation of graphs. Nunkesser and Woelfel [11] considered the space complexity of the OBDD representation of certain graphs as follows:

- The worst-case OBDD size of graphs is $O(N^2/\log N)$ and $O(M \log N)$;
- The worst-case OBDD size of cographs and related graphs is $\Theta(N \log N)$;
- The worst-case OBDD size of unit interval graphs is $O(N/\sqrt{\log N})$ and $\Omega(N/\log N)$;
- The worst-case OBDD size of interval graphs is $O(N^{3/2}/\log^{3/4} N)$ and $\Omega(N)$;
- The worst-case OBDD size of bipartite graphs is $\Omega(N^2/\log N)$,

where N and M are the number of vertices and edges of a graph, respectively.

This paper considers the OBDD size of some classes of bipartite graphs. We show in Section 4.2 and 4.3 that the worst-case OBDD size of bipartite permutation graphs and biconvex graphs is $O(N/\sqrt{\log N})$ and $\Omega(N/\log N)$. We also show in Section 4.4 through 4.6 that the worst-case OBDD size of convex graphs, 2-directional orthogonal ray graphs, and orthogonal ray graphs is $O(N^{3/2}/\log^{3/4} N)$ and $\Omega(N)$. We further show in Section 5 that the worst-case OBDD size of (not necessarily bipartite) permutation graphs is $O(N^{3/2}/\log^{3/4} N)$ and $\Omega(N)$.

2. Graph Representation by OBDDs

Let $X_n = \{x_1, \ldots, x_n\}$ be a set of Boolean variables, and B_n be the set of Boolean functions on X_n. A *variable ordering* π on X_n is a bijection $\pi : \{1, \ldots, n\} \to X_n$, leading to the ordered list $\pi(1), \ldots, \pi(n)$ of the variables. A π-OBDD on X_n for a variable ordering π is a single-root directed acyclic graph with two sinks labeled by 0 and 1, respectively. Each inner node, i.e., non-sink node, is labeled by a variable from X_n and has two outgoing edges, one of them is labeled by 0, and the other by 1. If an edge leads from an x_i-node to an x_j-node then $\pi^{-1}(x_i) < \pi^{-1}(x_j)$. For an input $\boldsymbol{a} = (a_{n-1}, \ldots, a_0) \in \{0,1\}^n$, the *computation path* of $\boldsymbol{a}$ is the unique root-to-sink path such that if it reaches an $\pi(i)$-node then it follows the edge with label a_{n-i}, for any i. A π-OBDD is said to *represent* $f \in B_n$ if $f(\boldsymbol{a})$ is the label of the sink reached by the computation path of $\boldsymbol{a}$ for any $\boldsymbol{a} \in \{0,1\}^n$. The *size* of a π-OBDD is the number of its nodes. The π-*OBDD size* of $f \in B_n$ is the minimal size of a π-OBDD representing f. The *OBDD size* of $f \in B_n$ is the minimal π-OBDD size of f over all variable orderings. Notice that the minimal π-OBDD representing $f \in B_n$ can be found in almost linear time [18], while it is NP-hard to compute the OBDD size of f [3].

Let G be an N-vertex graph with the vertex set $V(G)$ and edge set $E(G)$, and $n = \lceil \log N \rceil$. We assign a label $\boldsymbol{v} \in \{0,1\}^n$ to each vertex $v \in V(G)$ such that $\boldsymbol{u} \neq \boldsymbol{v}$ if $u \neq v$. Let $\chi_G : \{0,1\}^{2n} \to \{0,1\}$ be a Boolean function such that $\chi_G(\boldsymbol{u}, \boldsymbol{v}) = 1$ if and only if $(u, v) \in E(G)$. χ_G is called a characteristic function of G. A π-OBDD representing χ_G is said to represent G. The (π-)OBDD size of a graph G is the minimal of the (π-)OBDD size of a characteristic function of G. The *worst-case OBDD size* of a graph class $\mathcal{G}_N$ of N-vertex graphs is the maximal OBDD size of a graph in $\mathcal{G}_N$.

3. Classes of Bipartite Graphs

A bipartite graph (bigraph) G with a bipartition (U, V) is a *grid intersection graph* if there exist a set of horizontal line segments L_u, $u \in U$, on the xy-plain and a set of vertical line segments L_v, $v \in V$, such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if L_u and L_v intersect. A grid intersection graph G is a *unit grid intersection graph* if every L_w, $w \in U \cup V$, has the same length. The grid intersection graph was introduced in [9].

A bigraph G is a *chordal bipartite graph* (chordal bigraph) if it

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contains no cycle of length at least 6 as an induced subgraph. The chordal bigraph was introduced in [8].

A bigraph G with a bipartition (U, V) is an *orthogonal ray graph* if there exist a set of horizontal (leftward and rightward) rays (half-lines) R_u, $u \in U$, on the xy-plain and a set of vertical (upward and downward) rays R_v, $v \in V$, such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. The set $\mathcal{R}(G) = \{R_u, R_v \mid u \in U, v \in V\}$ is called an *orthogonal ray representation* of G. An orthogonal ray graph G is a *2-directional orthogonal ray graph* if G has an orthogonal ray representation consisting of only rightward rays and downward rays. The (2-directional) orthogonal ray graph was introduced in [13], [14].

Let G be a bigraph with a bipartition (U, V). A *convex ordering* of U is a total ordering such that for every $v \in V$, the vertices in $\Gamma_G(v)$ occur consecutively in the ordering, where $\Gamma_G(v)$ is the set of vertices adjacent to v in G. If no confusion arises, we will omit the index. A bigraph G is a *convex graph* if it has a convex ordering. A *biconvex ordering* of G is a pair of convex orderings of U and V. A bigraph G is a *biconvex graph* if it has a biconvex ordering. The convex graph was introduced in [7].

A graph G with vertex set $V(G) = \{v_1, \ldots, v_N\}$ is a *permutation graph* if there exists a permutation σ on $\{1, \ldots, N\}$ such that for every $i, j \in \{1, \ldots, N\}$, $(v_i, v_j) \in E(G)$ if and only if $(i - j)(\sigma(i) - \sigma(j)) < 0$. σ is called a *realizer* of G. A permutation graph G is a *bipartite permutation graph* (permutation bigraph) if it is bipartite. A *strong ordering* of a bigraph G with a bipartition (U, V) is a pair of total orderings $(u_0, \ldots, u_{p-1})$ of U and $(v_0, \ldots, v_{q-1})$ of V such that for any $i, j, k, l (0 \leq i < j \leq p-1, 0 \leq k < l \leq q-1)$, $(u_i, v_l) \in E(G)$ and $(u_j, v_k) \in E(G)$ imply $(u_i, v_k) \in E(G)$ and $(u_j, v_l) \in E(G)$. For any $u_i, u_j \in U$, we denote $u_i \leq_s u_j$ if $i \leq j$. For any $v_i, v_j \in V$, we denote $v_i \leq_s v_j$ if $i \leq j$. It is shown in [17] that a bigraph G is a permutation bigraph if and only if G has a strong ordering. It is easy to see that a strong ordering is a biconvex ordering.

The following relationships between bigraph classes have been known [14] :

{Bipartite Permutation Graphs}
$\subset$ {Biconvex Graphs}
$\subset$ {Convex Graphs}
$\subset$ {2-directional Orthogonal Ray Graphs}
$\subset$ {Chordal Bipartite Graphs},

and

{2-directional Orthogonal Ray Graphs}
$\subset$ {Orthogonal Ray Graphs}
$\subset$ {Unit Grid Intersection Graphs}
$\subset$ {Grid Intersection Graphs}.

Comprehensive surveys with many other results can be found in [4], [16].

4. OBDD Size of Bipartite Graphs

We use the following easy observation to prove upper bounds for the worst-case OBDD size.

Lemma 4.1. *The number of nodes labeled by $\pi(i)$ in the minimal π-OBDD representing $f \in B_n$ is bounded by the number of non-constant subfunctions obtained from f by replacing variable $\pi(j)$ by a constant for any $j < i$.* □

4.1 Lower Bounds with Counting Arguments

We follow the arguments used in [11]. It is shown in [18] that OBDDs on X_n of size s can represent at most

$$sn^s(s+1)^{2s}/s! = 2^{s \log s + s \log n + \Theta(s)}$$

different functions. Since $n = \lceil \log N \rceil$, the number of characteristic functions needed to represent graphs in $\mathcal{G}_N$ is at least $|\mathcal{G}_N|$. The following is implicit in [11].

Theorem 4.1. *The worst-case OBDD size of $\mathcal{G}_N$ is*

$$\begin{cases} \Omega(N/\log N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N)}, \\ \Omega(N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N \log N)}, \\ \Omega(N \log N) & \text{if } |\mathcal{G}_N| = 2^{\Omega(N \log^2 N)}. \end{cases}$$

□

4.2 Bipartite Permutation Graphs

4.2.1 Upper Bound

For a binary string $\boldsymbol{a} = (a_{n-1}, \ldots, a_0) \in \{0, 1\}^n$, let

$$|\boldsymbol{a}| = \sum_{i=0}^{n-1} a_i 2^i.$$

Let G be an N-vertex permutation bigraph with a bipartition (U, V) and a strong ordering $(u_0, \ldots, u_{p-1})$ and $(v_0, \ldots, v_{q-1})$. For each vertex $u_i \in U$, we assign a label $\boldsymbol{u}_i \in \{0, 1\}^n$ such that $|\boldsymbol{u}_i| = i$, and for each vertex $v_i \in V$, we assign a label $\boldsymbol{v}_i \in \{0, 1\}^n$ such that $|\boldsymbol{v}_i| = p + i$. We consider a π-OBDD representing a characteristic function $\chi_G(\boldsymbol{u}, \boldsymbol{v})$ with a variable ordering π such that

$$(\pi(1), \ldots, \pi(2n)) = (a_{n-1}, b_{n-1}, \ldots, a_0, b_0),$$

where $\boldsymbol{u} = (a_{n-1}, \ldots, a_0)$ and $\boldsymbol{v} = (b_{n-1}, \ldots, b_0)$.

Let s_k, $0 \leq k < n$, be the number of nodes labeled by u_{n-k-1}, and t_k, $0 \leq k < n$, be the number of nodes labeled by v_{n-k-1}. Notice that $t_k \leq 2s_k$. If k is large, we have the following upper bound by Lemma 4.1:

$$s_k \leq 2^{2^{2n-2k}}, \quad (1)$$

since there are 2^{2^m} Boolean functions in m variables. If k is small, we need a better upper bound. Let

$$V(\boldsymbol{\gamma}) = \{w \in V(G) \mid \boldsymbol{\gamma} \in \{0,1\}^k \text{ is a prefix of } \boldsymbol{w}\}, \text{ for } k \leq n, \text{ and}$$

$$\mathcal{S} = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha}, \boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}} \text{ is a non-constant subfunction} \end{array} \right\},$$

where $\chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a subfunction of χ_G such that

$$\begin{aligned} &\chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}}(a_{n-k-1}, b_{n-k-1}, \ldots, a_0, b_0) \\ &= \chi_G(\alpha_{k-1}, \beta_{k-1}, \ldots, \alpha_0, \beta_0, a_{n-k-1}, b_{n-k-1}, \ldots, a_0, b_0). \end{aligned}$$

If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{S}$, for any $w \in V(\boldsymbol{\alpha})$ and $z \in V(\boldsymbol{\beta})$, we have $|\boldsymbol{w}| < |\boldsymbol{z}|$, since $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$. Since $|\boldsymbol{u}| < |\boldsymbol{v}|$ for any $u \in U$ and $v \in V$, we have $V(\boldsymbol{\alpha}) \cap U \neq \emptyset$ and $V(\boldsymbol{\beta}) \cap V \neq \emptyset$, for otherwise $\chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a 0-function (constant function with value 0), a contradiction.

Let $l(\boldsymbol{\alpha})$ and $r(\boldsymbol{\alpha})$ be the vertices in $V(\boldsymbol{\alpha}) \cap U$ with the smallest and largest label, respectively, and let $l(\boldsymbol{\beta})$ and $r(\boldsymbol{\beta})$ be the vertices in $V(\boldsymbol{\beta}) \cap V$ with the smallest and largest label, respectively. Since $\chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is not a 1-function (constant function with value 1), we conclude that $(r(\boldsymbol{\alpha}), l(\boldsymbol{\beta})) \notin E(G)$ or $(l(\boldsymbol{\alpha}), r(\boldsymbol{\beta})) \notin E(G)$, for otherwise $(w, z) \in E(G)$ for any $w \in V(\boldsymbol{\alpha})$ and $z \in V(\boldsymbol{\beta})$ by the definition of strong ordering, and so $\chi_G|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a 1-function, a contradiction. Let

$$\begin{aligned}\mathcal{S}_1 &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid (r(\boldsymbol{\alpha}), l(\boldsymbol{\beta})) \notin E(G)\},\\ \mathcal{S}_2 &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid (l(\boldsymbol{\alpha}), r(\boldsymbol{\beta})) \notin E(G)\},\\ c_1 &= |\mathcal{S}_1|, \text{ and } c_2 = |\mathcal{S}_2|.\end{aligned}$$

Let $((\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1), \ldots, (\boldsymbol{\alpha}_{c_1},\boldsymbol{\beta}_{c_1}))$ be a lexicographic ordering of $\mathcal{S}_1$.

Lemma 4.2. $|\boldsymbol{\beta}_i| \le |\boldsymbol{\beta}_{i+1}|$ *for any* $i(1 \le i \le c_1)$.

Proof. It is trivial if $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i+1}$. If $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_{i+1}$ then we have $|\boldsymbol{\alpha}_i| < |\boldsymbol{\alpha}_{i+1}|$. Suppose contrary $|\boldsymbol{\beta}_i| > |\boldsymbol{\beta}_{i+1}|$. Since $\chi_G|\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i$ is not a 0-function, there exist $c(\boldsymbol{\alpha}_i) \in V(\boldsymbol{\alpha}_i)$ and $c(\boldsymbol{\beta}_i) \in V(\boldsymbol{\beta}_i)$ such that $(c(\boldsymbol{\alpha}_i), c(\boldsymbol{\beta}_i)) \in E(G)$. Since $\chi_G|\boldsymbol{\alpha}_{i+1},\boldsymbol{\beta}_{i+1}$ is not a 0-function, there exist $c(\boldsymbol{\alpha}_{i+1}) \in V(\boldsymbol{\alpha}_{i+1})$ and $c(\boldsymbol{\beta}_{i+1}) \in V(\boldsymbol{\beta}_{i+1})$ such that $(c(\boldsymbol{\alpha}_{i+1}), c(\boldsymbol{\beta}_{i+1})) \in E(G)$. Since $c(\boldsymbol{\alpha}_i) \le_s r(\boldsymbol{\alpha}_i) <_s c(\boldsymbol{\alpha}_{i+1})$ and $c(\boldsymbol{\beta}_{i+1}) <_s l(\boldsymbol{\beta}_i) \le_s c(\boldsymbol{\beta}_i)$, we conclude that $(r(\boldsymbol{\alpha}_i), l(\boldsymbol{\beta}_i)) \in E(G)$ by the definition of strong ordering, contradicting to the definition of $\mathcal{S}_1$. □

From Lemma 4.2 above, we conclude that

$$c_1 \le |\boldsymbol{\alpha}_{c_1}| + |\boldsymbol{\beta}_{c_1}| + 1.$$

Similarly, we have

$$c_2 \le |\boldsymbol{\alpha}_{c_2}| + |\boldsymbol{\beta}_{c_2}| + 1.$$

Thus we have

$$s_k \le 2(c_1 + c_2) = O(2^k). \qquad (2)$$

By Equations (1) and (2), the π-OBDD size of χ_G is:

$$\begin{aligned}&\sum_{k=0}^{n-1}(s_k + t_k)\\ \le\ & 3\sum_{k=0}^{n-\lfloor\frac{\log n}{2}\rfloor} O(2^k) + 3\sum_{n-\lfloor\frac{\log n}{2}\rfloor+1}^{n-1} 2^{2^{2(n-k)}}\\ =\ & O(N/\sqrt{\log N}).\end{aligned}$$

Therefore, we have a following.

Theorem 4.2. *The worst-case OBDD size of N-vertex permutation bigraphs is* $O(N/\sqrt{\log N})$. □

4.2.2 Lower Bound

The following is shown in [12].

Theorem I. *For* $N \ge 2$, *the number of unlabeled connected N-vertex permutation bigraphs is given by*

$$\begin{cases}\frac{1}{4}\left(C(N-1) + C(N/2-1) + \binom{N}{N/2}\right) & \text{if } N \text{ is even}\\ \frac{1}{4}\left(C(N-1) + \binom{N-1}{(N-1)/2}\right) & \text{if } N \text{ is odd}\end{cases}$$

where $C(N) = \frac{1}{N+1}\binom{2N}{N}$ *is called the Nth Catalan number.* □

The following is immediate from Theorem I.

Lemma 4.3. *The number of unlabeled connected N-vertex permutation bigraphs is* $2^{\Theta(N)}$. □

From Theorem 4.1 and Lemma 4.3, we have the following.

Theorem 4.3. *The worst-case OBDD size of N-vertex permutation bigraphs is* $\Omega(N/\log N)$. □

4.3 Biconvex Graphs

4.3.1 Upper Bound

The following is shown in [1].

Theorem II. *A connected biconvex graph G with a bipartition* (U, V) *has a biconvex ordering* $(u_0, \ldots, u_{p-1})$ *and* $(v_0, \ldots, v_{q-1})$ *such that for some vertices* $v_l, v_r \in V$ $(0 \le l \le r \le q-1)$*, the following conditions are satisfied:*

- $G[U \cup V_C]$ *is a connected permutation bigraph with a strong ordering* $(u_0, \ldots, u_{p-1})$ *and* $(v_l, \ldots, v_r)$*, where*

$$V_C = \{v_i \in V \mid l \le i \le r\},$$

and $G[X]$ *is a subgraph of G induced by* $X \subseteq V(G)$.

- $\Gamma(v_i) \subseteq \Gamma(v_j)$ *for any* i, j *such that* $0 \le i < j \le l$ *or* $r \le j < i \le q-1$.

□

Let $U_{ij} = \{u_k \in U \mid i \le k \le j\}$ and $V_{ij} = \{v_k \in V \mid i \le k \le j\}$. The following is implicit in [1], [19].

Theorem III. *For any* i, j, k, l $(0 \le i < j \le p-1, 0 \le k < l \le q-1)$, $G[U_{ij} \cup V_{kl}]$ *is a permutation bigraph with a strong ordering* $(u_i, \ldots, u_j)$ *and* $(v_k, \ldots, v_l)$ *if and only if* $(u_i, v_k) \in E(G)$ *and* $(u_j, v_l) \in E(G)$. □

Let u_x be a vertex in $\Gamma(v_0)$ and u_y be a vertex in $\Gamma(v_{q-1})$. Let

$$\begin{aligned}U_L &= \{u_i \in U \mid 0 \le i \le y\},\\ U_R &= \{u_i \in U \mid x \le i \le q-1\},\\ V_L &= \{v_i \in V \mid 0 \le i \le l\}, \text{ and}\\ V_R &= \{v_i \in V \mid r \le i \le q-1\}.\end{aligned}$$

By Theorem III,

- $G[U_L \cup V_R]$ is a permutation bigraph with a strong ordering $(u_0, \ldots, u_y)$ and $(v_r, \ldots, v_{q-1})$;
- $G[U_R \cup V_L]$ is a permutation bigraph with a strong ordering $(u_x, \ldots, u_{p-1})$ and $(v_0, \ldots, v_l)$;
- $G[U_L \cup V_L]$ is a permutation bigraph with a strong ordering $(u_0, \ldots, u_y)$ and $(v_l, \ldots, v_0)$; and
- $G[U_R \cup V_R]$ is a permutation bigraph with a strong ordering $(u_x, \ldots, u_{p-1})$ and $(v_{q-1}, \ldots, v_r)$.

For each vertex $u_i \in U$, we assign a label $\boldsymbol{u}_i \in \{0,1\}^n$ such that $|\boldsymbol{u}_i| = i$, and for each vertex $v_i \in V$, we assign a label $\boldsymbol{v}_i \in \{0,1\}^n$ such that $|\boldsymbol{v}_i| = p + i$. We consider a π-OBDD representing a characteristic function $\chi_G(\boldsymbol{u}, \boldsymbol{v})$ with a variable ordering π such that

$$(\pi(1), \ldots, \pi(2n)) = (a_{n-1}, b_{n-1}, \ldots, a_0, b_0),$$

where $\boldsymbol{u} = (a_{n-1}, \ldots, a_0)$ and $\boldsymbol{v} = (b_{n-1}, \ldots, b_0)$. Let

$$\begin{aligned}
V(\boldsymbol{\gamma}) &= \{w \in V(G) \mid \boldsymbol{\gamma} \in \{0,1\}^k \text{ is a prefix of } \boldsymbol{w}\}, \text{ for } k \le n,\\
\mathcal{S} &= \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha},\boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|,\\ \chi_G|_{\boldsymbol{\alpha}\boldsymbol{\beta}} \text{ is a non-constant subfunction} \end{array} \right\},\\
\mathcal{S}_{LL} &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid u_i \in V(\boldsymbol{\alpha}) \cap V_L, v_i \in V(\boldsymbol{\beta}) \cap V_L\},\\
\mathcal{S}_{LR} &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid u_i \in V(\boldsymbol{\alpha}) \cap V_L, v_i \in V(\boldsymbol{\beta}) \cap V_R\},\\
\mathcal{S}_{C} &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid v_i \in V(\boldsymbol{\beta}) \cap V_C\},\\
\mathcal{S}_{RL} &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid u_i \in V(\boldsymbol{\alpha}) \cap V_R, v_i \in V(\boldsymbol{\beta}) \cap V_L\}, \text{ and}\\
\mathcal{S}_{RR} &= \{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{S} \mid u_i \in V(\boldsymbol{\alpha}) \cap V_R, v_i \in V(\boldsymbol{\beta}) \cap V_R\}.
\end{aligned}$$

Since $G[U_L \cup V_L]$, $G[U_L \cup V_R]$, $G[U \cup V_C]$, $G[U_R \cup V_L]$, and $G[U_R \cup V_R]$ are permutation bigraphs, we have $|\mathcal{S}_{LL}| = O(2^k)$, $|\mathcal{S}_{LR}| = O(2^k)$, $|\mathcal{S}_C| = O(2^k)$, $|\mathcal{S}_{RL}| = O(2^k)$, and $|\mathcal{S}_{RR}| = O(2^k)$. Thus we conclude that

$$s_k \le 2(|\mathcal{S}_{LL}| + |\mathcal{S}_{LR}| + |\mathcal{S}_C| + |\mathcal{S}_{RL}| + |\mathcal{S}_{RR}|) = O(2^k). \qquad (3)$$

By Equations (1) and (3), the π-OBDD size of χ_G is:

$$\begin{aligned}
&\sum_{k=0}^{n-1}(s_k + t_k)\\
\le\ & 3\sum_{k=0}^{n-\lfloor\frac{\log n}{2}\rfloor} O(2^k) + 3\sum_{n-\lfloor\frac{\log n}{2}\rfloor+1}^{n-1} 2^{2^{2(n-k)}}\\
=\ & O(N/\sqrt{\log N}).
\end{aligned}$$

Therefore, we have a following.

Theorem 4.4. *The worst-case OBDD size of N-vertex biconvex graphs is* $O(N/\sqrt{\log N})$. □

4.3.2 Lower Bound

Lemma 4.4. *, The number of unlabeled connected N-vertex biconvex graphs is* $2^{\Theta(N)}$.

Proof. Let $\mathcal{BCG}_N$ and $\mathcal{PB}_N$ be the classes of unlabeled N-vertex biconvex graphs and permutation bigraphs, respectively. Lemma 4.3 implies $|\mathcal{BCG}_N| = 2^{\Omega(N)}$. From Theorem II and Lemma 4.3, we have

$$\begin{aligned}
|\mathcal{BCG}_N| &\le \sum_{i=0}^{N}\sum_{j=0}^{N-i} |\mathcal{PB}_{N-i-j}| \binom{N+2i-1}{2i}\binom{N+2j-1}{2j}\\
&\le N^2 2^{O(N)} \binom{3N}{3N/2}^2\\
&= 2^{O(N)}.
\end{aligned}$$

□

From Theorem 4.1 and Lemma 4.4, we have the following.

Theorem 4.5. *The worst-case OBDD size of N-vertex biconvex graphs is* $\Omega(N/\log N)$. □

4.4 Convex Graphs

We have the following as a corollary of Theorem 4.9, which will be shown in the next section, since the class of convex graphs is a proper subset of the class of 2-directional orthogonal ray graphs.

Theorem 4.6. *The worst-case OBDD size of N-vertex convex graphs is* $O(N^{3/2}/\log^{3/4} N)$. □

Now, we show a lower bound. A graph G is an *interval graph* if there exists a set of intervals I_v, $v \in V(G)$, on the real line such that for any $u, v \in V(G)$, $(u,v) \in E(G)$ if and only if I_u and I_v intersect. The set $\mathcal{I}(G) = \{I_v \mid v \in V(G)\}$ is called an *interval representation* of G. The following is shown in [6].

Theorem IV. *The number of unlabeled connected N-vertex interval graphs is* $2^{N\log N - O(N)}$. □

Lemma 4.5. *The number of unlabeled connected N-vertex convex graphs is* $2^{\Omega(N\log N)}$.

Proof. Let $\mathcal{CG}_{N_U,N_V}$ be a class of N-vertex connected convex graphs with a bipartition (U,V) and a convex ordering of U, such that $|U| = N_U$ and $|V| = N_V$. Let $\mathcal{IG}_N$ be a class of N-vertex connected interval graphs. Assume w.l.o.g. that N can be divide by 3. It suffices to show that there exists a surjection $\phi : \mathcal{CG}_{2N/3,N/3} \to \mathcal{IG}_{N/3}$.

For any $G \in \mathcal{CG}_{2N/3,N/3}$ with a bipartition (U,V), we define that $\phi(G)$ is a graph such that

$$\begin{aligned}
V(\phi(G)) &= V,\\
E(\phi(G)) &= \{(v,v') \mid \Gamma_G(v) \cap \Gamma_G(v') \ne \emptyset\}.
\end{aligned}$$

It is easy to see that $\phi(G)$ is in $\mathcal{IG}_{N/3}$.

Let $H \in \mathcal{IG}_{N/3}$ with an interval representation $\mathcal{I}(H)$. For each $I_v \in \mathcal{I}(H)$, there exist the left and right boundaries. Assume w.l.o.g. that every boundary is not overlapped. For each boundary b, define vertex u_b. Let G_H be a bigraph with a bipartition (U_H, V_H) such that

$$\begin{aligned}
U_H &= \{u_b \mid b \text{ is a boundary of some } I_v \in \mathcal{I}(H)\},\\
V_H &= V(H),\\
E(G_H) &= \{(u_b, v) \mid b \in I_v\}.
\end{aligned}$$

It is easy to see that G_H is in $\mathcal{CG}_{2N/3,N/3}$ and $\phi(G_H) = H$ for any $H \in \mathcal{IG}_{N/3}$. □

From Theorem 4.1 and Lemma 4.5, we have the following.

Theorem 4.7. *The worst-case OBDD size of N-vertex convex graphs is* $\Omega(N)$. □

4.5 Two-Directional Orthogonal Ray Graphs

We have the following as a corollary of Theorem 4.7.

Theorem 4.8. *The worst-case OBDD size of N-vertex 2-directional orthogonal ray graphs is* $\Omega(N)$. □

Now, we show an upper bound. Let G be an N-vertex 2-directional orthogonal ray graph with a bipartition (U,V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_u, R_v \mid u \in U, v \in V\}$. Let (x_w, y_w) be the endpoint of $R_w \in \mathcal{R}(G)$, and assume w.l.o.g. that every x_w and y_w, $w \in U \cup V$ is distinct. Notice that for any $u \in U$ and $v \in V$, $(u,v) \in E(G)$ if and only if $x_u < x_v$ and $y_u < y_v$. For each vertex $w \in U \cup V$, we assign a label $\boldsymbol{w} \in \{0,1\}^n$ such that for any vertices $w_i, w_j \in U$ $[V]$, $\boldsymbol{w}_i^e < \boldsymbol{w}_j^e$ imply $x_{w_i} < x_{w_j}$ and $\boldsymbol{w}_i^o < \boldsymbol{w}_j^o$ imply $y_{w_i} < y_{w_j}$, and for any $u \in U$ and $v \in V$, $\boldsymbol{u} < \boldsymbol{v}$. Here $\boldsymbol{w}^e$ and $\boldsymbol{w}^o$ are the substring of $\boldsymbol{w}$ that consists of the bits with even and odd index, respectively. We consider a π-OBDD representing a characteristic function $\chi_G(\boldsymbol{u},\boldsymbol{v})$ with a variable ordering π such that

$$(\pi(1), \ldots, \pi(2n)) = (a_{n-1}, b_{n-1}, \ldots, a_0, b_0),$$

where $\boldsymbol{u} = (a_{n-1}, \ldots, a_0)$ and $\boldsymbol{v} = (b_{n-1}, \ldots, b_0)$. Let

$$V(\boldsymbol{\gamma}) = \{w \in V(G) \mid \boldsymbol{\gamma} \in \{0,1\}^k \text{ is a prefix of } \boldsymbol{w}\}, \text{ for } k \le n, \text{ and}$$
$$\mathcal{S} = \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha},\boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_G|_{\boldsymbol{\alpha}\boldsymbol{\beta}} \text{ is a non-constant subfunction} \end{array} \right\}.$$

Lemma 4.6. *For every i, j ($1 \le i < j \le c$),*

$$(\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j) \wedge (\boldsymbol{\alpha}^o_i < \boldsymbol{\alpha}^o_j) \Rightarrow (\boldsymbol{\beta}^e_i \le \boldsymbol{\beta}^e_j) \vee (\boldsymbol{\beta}^o_i \le \boldsymbol{\beta}^o_j). \quad (4)$$

Proof. Since $\chi_G|_{\boldsymbol{\alpha}_j\boldsymbol{\beta}_j}$ is not a 0-function, there exist $u \in \mathcal{V}(\boldsymbol{\alpha}_j) \cap U$ and $v \in \mathcal{V}(\boldsymbol{\beta}_j) \cap V$ such that $(u,v) \in E(G)$, i.e., $x_u < x_v$ and $y_u < y_v$. Since $\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j$ and $\boldsymbol{\alpha}^o_i < \boldsymbol{\alpha}^o_j$, for every vertex $w \in V(\boldsymbol{\alpha}_i) \cap U$, $x_w < x_u$ and $y_w < y_u$. Suppose contrary $\boldsymbol{\beta}^e_i > \boldsymbol{\beta}^e_j$ and $\boldsymbol{\beta}^o_i > \boldsymbol{\beta}^o_j$. Since for every vertex $z \in V(\boldsymbol{\beta}_i) \cap V$, $x_v < x_z$ and $y_v < y_z$, we conclude that $\chi_G|_{\boldsymbol{\alpha}_i\boldsymbol{\beta}_i}$ is a 1-function, a contradiction. □

The number of tuples $(\boldsymbol{\alpha}^e_i, \boldsymbol{\alpha}^o_i, \boldsymbol{\beta}^e_i, \boldsymbol{\beta}^o_i)$ satisfying Equation (4) is bounded by $2 \cdot 2^{\lceil \frac{k}{2} \rceil} c'$, where c' is the number of tuples $(\boldsymbol{\alpha}^e_i, \boldsymbol{\alpha}^o_i, \boldsymbol{\beta}^e_i)$ satisfying

$$(\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j) \wedge (\boldsymbol{\alpha}^o_i < \boldsymbol{\alpha}^o_j) \Rightarrow \boldsymbol{\beta}^e_i \le \boldsymbol{\beta}^e_j.$$

Furthermore, c' is bounded by $2 \cdot 2^{\lceil \frac{k}{2} \rceil} c''$, where c'' is the number of tuples $(\boldsymbol{\alpha}^e_i, \boldsymbol{\beta}^e_i)$ satisfying

$$\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j \Rightarrow \boldsymbol{\beta}^e_i \le \boldsymbol{\beta}^e_j.$$

Since $c'' \le |\boldsymbol{\alpha}^e| + |\boldsymbol{\beta}^e| + 1$, we conclude that

$$s_k \le 2c \le 2(2 \cdot 2^{\lceil \frac{k}{2} \rceil}(2 \cdot 2^{\lceil \frac{k}{2} \rceil}(2 \cdot 2^{\lceil \frac{k}{2} \rceil} + 1))) = O(2^{\frac{3}{2}k}). \quad (5)$$

By Equation (1) and (5), The π-OBDD size of χ_G is:

$$\sum_{k=0}^{n-1}(s_k + t_k)$$
$$\le 3 \sum_{k=0}^{n-\lfloor \frac{2\log n - 1}{4} \rfloor} O(2^{\frac{3}{2}k}) + 3 \sum_{n-\lfloor \frac{2\log n-1}{4} \rfloor + 1}^{n-1} 2^{2^{2(n-k)}}$$
$$= O(N^{3/2} / \log^{3/4} N).$$

Therefore, we have a following.

Theorem 4.9. *The worst-case OBDD size of N-vertex 2-directional orthogonal ray graphs is $O(N^{3/2} / \log^{3/4} N)$.* □

4.6 Orthogonal Ray Graphs

We have the following as a corollary of Theorem 4.7.

Theorem 4.10. *The worst-case OBDD size of N-vertex orthogonal ray graphs is $\Omega(N)$.* □

Now, we show an upper bound. Let G be an N-vertex orthogonal ray graph with a bipartition (U,V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_u, R_v \mid u \in U, v \in V\}$. Let (x_w, y_w) be the endpoint of $R_w \in \mathcal{R}(G)$, and assume w.l.o.g. that every x_w and y_w, $w \in U \cup V$ is distinct. Let

$$U_l = \{u \in U \mid R_u \text{ is a leftward ray }\},$$
$$U_r = \{u \in U \mid R_u \text{ is a rightward ray }\},$$
$$V_u = \{v \in V \mid R_v \text{ is a upward ray }\},$$
$$V_d = \{v \in V \mid R_v \text{ is a downward ray }\}.$$

For each vertex $w \in U \cup V$, we assign a label $\boldsymbol{u} \in \{0,1\}^n$ such that for any vertices $w_i, w_j \in U_l$ [U_r, V_u, or V_d], $\boldsymbol{w}^e_i < \boldsymbol{w}^e_j$ imply $x_{w_i} < x_{w_j}$ and $\boldsymbol{w}^o_i < \boldsymbol{w}^o_j$ imply $y_{w_i} < y_{w_j}$, and for any $u_l \in U_l$, $u_r \in U_r$, $v_u \in V_u$, and $v_d \in V_d$, $\boldsymbol{u}_l < \boldsymbol{u}_r < \boldsymbol{v}_u < \boldsymbol{v}_d$. Here $\boldsymbol{w}^e$ and $\boldsymbol{w}^o$ are the substring of $\boldsymbol{w}$ that consists of the bits with even and odd index, respectively.

Since subgraphs of G induced by $U_l \cup V_u$, $U_l \cup V_d$, $U_r \cup V_u$, and $U_r \cup V_d$ are 2-directional orthogonal ray graph, similar arguments as in Section 4.5 show the following.

Theorem 4.11. *The worst-case OBDD size of N-vertex orthogonal ray graphs is $O(N^{3/2} / \log^{3/4} N)$.* □

5. OBDD Size of Permutation Graphs

The following is shown in [2].

Theorem V. *The number of unlabeled connected N-vertex interval graphs is $2^{\Omega(N \log N)}$.* □

From Theorem 4.1 and V, we have the following.

Theorem 5.1. *The worst-case OBDD size of N-vertex permutation graphs is $\Omega(N)$.* □

Now, we show an upper bound. Let G be an N-vertex permutation graph with a realizer σ. For each vertex $v \in V(G)$, we assign a label $\boldsymbol{v}_i \in \{0,1\}^n$ such that $\boldsymbol{v}^e_i < \boldsymbol{v}^e_j$ imply $i < j$ and $\boldsymbol{v}^o_i < \boldsymbol{v}^o_j$ imply $\sigma(i) < \sigma(j)$. Here $\boldsymbol{v}^e$ and $\boldsymbol{v}^o$ are the substring of $\boldsymbol{v}$ that consists of the bits with even and odd index, respectively. We consider a π-OBDD representing a characteristic function $\chi_G(\boldsymbol{u},\boldsymbol{v})$ with a variable ordering π such that

$$(\pi(1), \ldots, \pi(2n)) = (a_{n-1}, b_{n-1}, \ldots, a_0, b_0),$$

where $\boldsymbol{u} = (a_{n-1}, \ldots, a_0)$ and $\boldsymbol{v} = (b_{n-1}, \ldots, b_0)$. Let

$$V(\boldsymbol{\gamma}) = \{w \in V(G) \mid \boldsymbol{\gamma} \in \{0,1\}^k \text{ is a prefix of } \boldsymbol{w}\}, \text{ for } k \le n, \text{ and}$$
$$\mathcal{S} = \left\{ (\boldsymbol{\alpha},\boldsymbol{\beta}) \;\middle|\; \begin{array}{l} \boldsymbol{\alpha},\boldsymbol{\beta} \in \{0,1\}^k, |\boldsymbol{\alpha}| < |\boldsymbol{\beta}|, \\ \chi_G|_{\boldsymbol{\alpha}\boldsymbol{\beta}} \text{ is a non-constant subfunction} \end{array} \right\}.$$

We have the following, which is the same as claim 4.6.

Lemma 5.1. *For every i, j ($1 \le i < j \le c$),*

$$(\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j) \wedge (\boldsymbol{\alpha}^o_i < \boldsymbol{\alpha}^o_j) \Rightarrow (\boldsymbol{\beta}^e_i \le \boldsymbol{\beta}^e_j) \vee (\boldsymbol{\beta}^o_i \le \boldsymbol{\beta}^o_j).$$

Proof. Since $\chi_G|_{\boldsymbol{\alpha}_j\boldsymbol{\beta}_j}$ is not a 1-function, there exist $v_a \in \mathcal{V}(\boldsymbol{\alpha}_j)$ and $v_b \in \mathcal{V}(\boldsymbol{\beta}_j)$ such that $a < b$ and $\sigma(a) < \sigma(b)$. Since $\boldsymbol{\alpha}^e_i < \boldsymbol{\alpha}^e_j$ and $\boldsymbol{\alpha}^o_i < \boldsymbol{\alpha}^o_j$, for every vertex $v_k \in \mathcal{V}(\boldsymbol{\alpha}_i)$, $k < a$ and $\sigma(k) < \sigma(a)$. Suppose contrary $\boldsymbol{\beta}^e_i > \boldsymbol{\beta}^e_j$ and $\boldsymbol{\beta}^o_i > \boldsymbol{\beta}^o_j$. Since for every vertex $v_l \in \mathcal{V}(\boldsymbol{\beta}_i)$, $b < l$ and $\sigma(b) < \sigma(l)$, we conclude that $\chi_G|_{\boldsymbol{\alpha}_i\boldsymbol{\beta}_i}$ is a 0-function, a contradiction. □

Therefore, we have a following by similar arguments as in Section 4.5.

Theorem 5.2. *The worst-case OBDD size of N-vertex permutation graphs is $O(N^{3/2} / \log^{3/4} N)$.* □

6. Concluding Remarks

- It is shown in [15] that the number of N-vertex chordal bigraphs is $2^{\Theta(N \log^2 N)}$. Thus we conclude that the worst-case OBDD size of N-vertex chordal bigraphs is $\Omega(N \log N)$.
- It is shown in [6] that the number of labeled N-vertex interval graphs is $2^{O(N \log N)}$. We can show by similar arguments that the number of labeled N-vertex grid intersection graphs and

permutation graphs is $2^{O(N \log N)}$. Thus we conclude that the number of unlabeled and labeled N-vertex convex graphs, 2-directional orthogonal ray graphs, orthogonal ray graphs, unit grid intersection graphs, grid intersection graphs, and permutation graphs is $2^{\Theta(N \log N)}$. Therefore, we can draw a line between biconvex and convex graphs as to whether the number of N-vertex unlabeled graphs in the class is $2^{\Theta(N)}$ or $2^{\Theta(N \log N)}$, and we can also draw a line between 2-directional orthogonal ray graphs and chordal bigraphs as to whether it is $2^{\Theta(N \log N)}$ or $2^{\Theta(N \log^2 N)}$.

- Upper bounds for the worst-case OBDD size of chordal bigraphs, unit grid intersection graphs, and grid intersection graphs are still open. Also, the bounds we presented are not tight, and closing the gaps between upper and lower bounds for the worst-case OBDD size of graphs are another open problems.
- It is shown in [10] that increasing the length of vertex labels can reduce the worst-case OBDD size as follows:
  - The worst-case OBDD size of graphs of bounded tree-width is $O(\log N)$ if we use $O(\log N)$-bit vertex label;
  - The worst-case OBDD size of graphs of bounded clique-width is $O(N)$ if we use $O(N)$-bit vertex label;
  - The worst-case OBDD size of graphs of bounded clique-width such that there is a clique-width expression whose associated binary tree is of depth $O(\log N)$ is $O(N)$ if we use $O(\log N)$-bit vertex label;
  - The worst-case OBDD size of cographs is $O(N)$ if we use $O(\log N)$-bit vertex label,

  where N is the number of vertices in a graph.

  We have no lower bounds, however, if we use more than $\lceil \log N \rceil$ bits for vertex labels. Moreover, as mentioned in [11], it is easy to see that the worst-case OBDD size of general graphs is $O(N)$ if we use $2N$-bit vertex label.

  Many researchers assume $\lceil \log N \rceil$-bit vertex labels, but it is worth considering whether increasing the length of vertex labels is a good strategy for implicit representation of graphs, and if it is, relationships between length of vertex labels and OBDD size, especially when we use $O(\log N)$ or $\log N + O(1)$ bits for vertex labels, may become interesting questions.

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