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On Two-Directional Orthogonal Ray Graphs

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An orthogonal ray graph is an intersection graph of horizontal and vertical rays(half-lines) in the xy-plane. An orthogonal ray graph is a 2-directional orthogonal ray graph if all the horizontal rays extend in the positive x-direction and all the vertical rays extend in the positive y-direction. We show several characterizations of 2-directional orthogonal ray graphs. We first show a forbidden submatrix characterization of 2-directional orthogonal ray graphs. A characterization in terms of a vertex ordering follows immediately. Next, we show that 2-directional orthogonal ray graphs are exactly those bipartite graphs whose complements are circular arc graphs. This characterization provides polynomial-time recognition and isomorphism algorithms for 2-directional orthogonal ray graphs. It also leads to a characterization of 2-directional orthogonal ray graphs by a list of forbidden induced subgraphs. Our results settle an open question on the recognition of certain forbidden submatrices.

1. Introduction

A bipartite graph G with a bipartition (U, V) is called an *orthogonal ray graph* if there exist a family of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting rays $R_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. An orthogonal ray graph G is called a *2-directional orthogonal ray graph* if $R_u = \{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v = \{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where a_w and b_w are real numbers for any $w \in U \cup V$. We introduced orthogonal ray graphs[14] in connection with defect tolerance schemes for nano-programmable logic arrays[13),17]. In this paper, we provide several characterizations of 2-directional orthogonal ray graphs and their consequences on the recognition and isomorphism problems of such graphs and an open question posed by Klinz, Rudolf, and Woeginger[6].

Let G be a bipartite graph with a bipartition (U, V). A $(0, 1)$-matrix $M = [m_{ij}]$ is called a *bipartite adjacency matrix* of G if the rows of M correspond to the vertices of U, the columns of M correspond to the vertices of V, and $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$, where $u_i \in U$ is a vertex corresponding to row i and $v_j \in V$ is a vertex corresponding to column j. Let A and B be matrices. A is said to be *B-free* if A does not contain B as a submatrix. For a set S of matrices, A is said to be *S-free* if A is M-free for every $M \in S$. A is said to be *S-freeable* if there exist a permutation of rows of A and a permutation of columns of A such that the permuted matrix is S-free. Let

$$\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

We show in Section 3.1 that a bipartite graph G is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of G is γ-freeable.

A bipartite graph G with bipartition (U, V) is said to be *weakly orderable* if there exist an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V and an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U such that for every i, i', j, j' $(1 \leq i < i' \leq |U|, 1 \leq j < j' \leq |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$. We show in Section 3.2 that a graph G is a 2-directional orthogonal ray graph if and only if G is weakly-orderable.

A graph G is a *circular arc graph* if there exists a collection of circular arcs $A_u, u \in V(G)$ on a fixed circle, such that two arcs A_v and A_w intersect if and only if $(v, w) \in E(G)$. We show in Section 3.3 that a bipartite graph G is a 2-directional orthogonal ray graph if and only if the complement of G is a circular arc graph. This characterization implies polynomial-time recognition and isomorphism algorithms for 2-directional orthogonal ray graphs, thereby settling an open question of deciding whether a matrix is γ-freeable raised by Klinz, Rudolf, and Woeginger[6].

An *edge-asteroid* is a set of edges $e_0, e_1, \ldots, e_{2k}$ such that for each $i = 0, 1, \ldots, 2k$, there is a path joining e_i and e_{i+1}, and containing both e_i and e_{i+1}, that avoids the neighbors of $e_{i+k+1 (\bmod 2k+1)}$. We obtain from a result by Feder, Hell, and Huang[2] that a graph G is a 2-directional orthogonal ray graph if and only if it contains no induced cycles of length at least 6 and no edge-asteroids.

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Let $(X, \leq)$ be a partially ordered set (poset). For $x, y \in X$, we shall use the notation $x < y$ to mean $x \leq y$ and $x \neq y$. The *interval dimension* of a poset $(X, \leq)$ is the least positive integer k for which there exists a function F which assigns to each $x \in X$, a sequence $\{F(x)(i) : 1 \leq i \leq k\}$ of k closed intervals on the real line so that $x < y$ if and only if $F(x)(i)$ lies completely to the left of $F(y)(i)$ for all $1 \leq i \leq k$. A *bipartite poset* is a triple $(X, Y, \leq)$ where X and Y are disjoint sets and $\leq$ is a partial order on $X \cup Y$ with $\{(x, y) | x < y\} \subseteq X \times Y$. With a bipartite graph G with bipartition (X, Y), we associate a bipartite poset $P_G = (X, Y, \leq)$, where $x < y$ if and only if $(x, y) \in E(G)$, for every $x \in X$ and $y \in Y$. We obtain from a result by Trotter and Moore[18], that G is a 2-directional orthogonal ray graph if and only if P_G is a bipartite poset of interval dimension at most 2. This connection allows us to characterize two-directional orthogonal ray graphs by a list of forbidden induced subgraphs.

The 3-*claw* is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. In our earlier work[14], we showed that a tree T is a 2-directional orthogonal ray graph if and only if T does not contain 3-claw as a subtree. It follows that we can decide in linear time whether a given tree is a 2-directional orthogonal ray graph.

2. Related Graph Classes

A bipartite graph G with a bipartition (U, V) is called a *grid intersection graph* if there exist a family of non-intersecting line segments $L_u, u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting line segments $L_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V, (u, v) \in E(G)$ if and only if L_u and L_v intersect. Let

$$X = \left\{ \begin{bmatrix} w & 1 & x \\ 1 & 0 & 1 \\ y & 1 & z \end{bmatrix} \middle| \, w, x, y, z \in \{0, 1\} \right\}.$$

Hartman, Newman, and Ziv[3] showed that a bipartite graph G is a grid intersection graph if and only if a bipartite adjacency matrix of G is X-freeable. Kratochvil[8] showed that the recognition problem for grid intersection graphs is NP-complete.

A grid intersection graph is said to be *unit* if all the line segments corresponding to the vertices have the same length. Otachi, Okamoto, and Yamazaki[12] showed that a bipartite graph G is a unit grid intersection graph if a bipartite adjacency matrix of G is γ-freeable.

A bipartite graph is *chordal bipartite* if it contains no cycle of length at least 6 as an induced subgraph. A graph G is chordal bipartite if and only if a bipartite adjacency matrix of G is Γ-freeable (see for example[6]), where

$$\Gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Lubiw[9] showed a polynomial-time recognition algorithm for chordal bipartite graphs based on Γ-free matrices.

A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is called a *permutation graph* if there exists a pair of permutations π_1 and π_2 on $N = \{1, 2, \ldots, n\}$ such that for all $i, j \in N$, $(v_i, v_j) \in E(G)$ if and only if

$$(\pi_1^{-1}(i) - \pi_1^{-1}(j))(\pi_2^{-1}(i) - \pi_2^{-1}(j)) < 0.$$

A bipartite graph G with bipartition (U, V) is said to be *strongly orderable* if there exist an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U and an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that for any integers i, i', j, j' $(1 \leq i < i' \leq |U|,\ 1 \leq j < j' \leq |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$ and $(u_{i'}, v_{j'}) \in E(G)$. Spinrad, Brandstadt, and Stewart[15] showed that a bipartite graph G is a permutation graph if and only if G is strongly orderable, and gave a linear-time recognition algorithm for bipartite permutation graphs based on this characterization. Let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

It also follows from the characterization that a bipartite graph G is a permutation graph if and only if a bipartite adjacency matrix of G is β-freeable as shown by Chen and Yesha[1].

A bipartite graph G with a bipartition (U, V) is called an *interval bigraph* if every vertex $w \in U \cup V$ can be assigned an interval I_w on the real line so that for all $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if I_u and I_v intersect. The class of interval bigraphs, which properly contains the class of bipartite permutation graphs, have been extensively studied. Muller[11] noted that the class of interval bigraphs is a proper subset of the class of chordal bipartite graphs and provided a polynomial-time recognition algorithm for interval bigraphs. Hell and Huang[4] showed that G is an interval bigraph if and only if the complement of G is a circular arc graph which has a circular arc representation in which no two arcs together cover the whole circle. They showed that an interval bigraph contains no induced cycles of length at least 6 and no edge-asteroids. They also provided a characterization of interval bigraphs in terms of a vertex ordering.

3. Characterizations of Two-Directional Orthogonal Ray Graphs

In this section, we give several characterizations of 2-directional orthogonal ray graphs.

3.1 Bipartite Adjacency Matrix Characterization

The following is obvious from the definition of γ.

Lemma 1 An $m \times n$ matrix $M = [m_{ij}]$ is γ-free if and only if for any integers i, i', j, j' ($1 \leq i < i' \leq m$, $1 \leq j' < j \leq n$), $m_{ij'} = 1$ and $m_{i'j} = 1$ imply $m_{ij} = 1$. □

We can characterize the 2-directional orthogonal ray graphs as follows.

Theorem 1 A bipartite graph G is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of G is γ-freeable.

Proof: Let G be a bipartite graph with a bipartition (U, V). Suppose that a bipartite adjacency matrix of G is γ-freeable, and let $M = [m_{ij}]$ be a bipartite adjacency matrix of G which is γ-free. We denote by $u_i \in U$ the vertex corresponding to row i, and by $v_j \in V$ the vertex corresponding to column j. For each row i of M, define $l(i)$ to be the column which contains the leftmost 1 in that row. Then define ray $R_{u_i} = \{(x, |U| - i + 1) \mid x \geq l(i)\}$. Similarly for each column j, define $b(j)$ to be the row which contains the bottommost 1 in that column. Define ray $R_{v_j} = \{(j, y) \mid y \geq |U| - b(j) + 1\}$. Note that from this definition, two rays R_{u_i} and R_{v_j} intersect if and only if $l(i) \leq j$ and $b(j) \geq i$. (See Figure 1.) We are now ready to show that R_{u_i} and R_{v_j} intersect if and only if $(u_i, v_j) \in E(G)$. Suppose first that $(u_i, v_j) \in E(G)$. Then $m_{ij} = 1$, which means that $l(i) \leq j$ and $b(j) \geq i$. Therefore, rays R_{u_i} and R_{v_j} intersect. Suppose next that $(u_i, v_j) \notin E(G)$. Then $m_{ij} = 0$. Since M is γ-free, we have $m_{i'j} = 0$ for

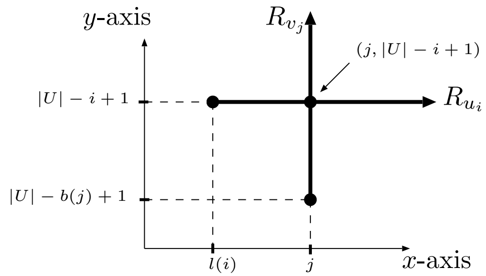

Fig. 1 Rays R_{u_i} and R_{v_j} intersect if and only if $l(i) \leq j$ and $b(j) \geq i$.

every $i' > i$ or $m_{ij'} = 0$ for every $j' < j$, by Lemma 1. This means that $l(i) > j$ or $b(j) < i$, which implies that R_{u_i} and R_{v_j} do not intersect. Thus we conclude that G is a 2-directional orthogonal ray graph for rays $\{R_{u_i} | u_i \in U\} \cup \{R_{v_j} | v_j \in V\}$.

Conversely, suppose that G is a 2-directional orthogonal ray graph, and $\{R_u \mid u \in U\} \cup \{R_v \mid v \in V\}$ is the set of rays corresponding to the vertices. Let $(u_1, u_2, \ldots, u_{|U|})$ be the ordering of U such that for any integers i, i' ($1 \leq i < i' \leq |U|$), R_{u_i} is above $R_{u_{i'}}$ in the xy-plane. Similarly, let $(v_1, v_2, \ldots, v_{|V|})$ be the ordering of V such that for any integers j, j' ($1 \leq j < j' \leq |V|$), R_{v_j} is to the left of $R_{v_{j'}}$. Construct a bipartite adjancency matrix $M = [m_{ij}]$ of G such that $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$. We shall show that M is γ-free. For some integers i, i', j, j',($1 \leq i < i' \leq |U|, 1 \leq j' < j \leq |V|$), suppose $m_{i'j} = 1$ and $m_{ij'} = 1$. Since ray R_{u_i} is above ray $R_{u_{i'}}$ and $R_{v'_j}$ is to the left of R_{v_j}, R_{u_i} must intersect with R_{v_j} implying that $m_{ij} = 1$. Thus from Lemma 1, M is γ-free. □

3.2 Vertex Order Characterization

The following corollary is immediate from Theorem 1.

Corollary 1 A bipartite graph G is a 2-directional orthogonal ray graph if and only if G is weakly orderable. □

3.3 Characterization in Terms of Circular Arc Graphs

An arc A on a circle O can be denoted by a pair of its endpoints $(s(A), t(A))$, where A is obtained by traversing O clockwise from its counterclockwise endpoint $s(A)$ to its clockwise endpoint $t(A)$.

Lemma 2 The complement of a 2-directional orthogonal ray graph is a circular arc graph.

Proof: Suppose a bipartite graph G with bipartition (U, V) is a 2-directional orthogonal ray graph. G has a γ-free bipartite adjacency matrix $M = [m_{ij}]$, by Theorem 1. For each row $i (1 \le i \le |U|)$ of M, define $l(i)$ to be the column which contains the leftmost 1 in that row, and for each column $j (1 \le j \le |V|)$, define $b(j)$ to be the row which contains the bottommost 1 in that column. Let O be a circle and let

$$p, r'_1, c_1, r'_2, c_2, \ldots, r'_{|U|}, c_{|U|}, q, c'_{|V|}, r_{|V|}, c'_{|V-1|}, r_{|V-1|}, \ldots, c'_1, r_1 \qquad (1)$$

be $2|U| + 2|V| + 2$ distinct points on O in the order of their occurrence in a clockwise traversal of O starting from p. Corresponding to each row i, define arc R_i to be $(r_{l(i)}, r'_i)$ and corresponding to each column j, define arc C_j to be $(c_{b(j)}, c'_j)$. (An example is shown in Figure 2.) We shall now show that two arcs R_i and C_j intersect if and only if $m_{ij} = 0$. Suppose first that $m_{ij} = 1$, which implies $i \le b(j)$ and $l(i) \le j$. Since $i \le b(j)$, we can see that r'_i precedes $c_{b(j)}$ in Sequence (1). Since we have defined the clockwise endpoint of R_i to be r'_i and the counterclockwise endpoint of C_j to be $c_{b(j)}$, we can deduce that they do not intersect on arc (p, q). Similarly, we can show that $l(i) \le j$ implies R_i and C_j do not intersect on arc (q, p) either. Next suppose $m_{ij} = 0$. Since M is γ-free, we have $m_{i'j} = 0$ for every $i' > i$ or $m_{ij'} = 0$ for every $j' < j$, by Lemma 1. This means that $l(i) > j$ or $b(j) < i$. Then from Sequence (1), we can see that both R_i and C_j contain the arc $(r_{l(i)}, c'(j))$ or the arc $(c_{b(j)}, r'(i))$. Finally, all R_i intersect at p, and all C_j intersect at q, and therefore we can conclude that the complement of G is a circular arc graph for the family of arcs $\{R_i | 1 \le i \le |U|\} \cup \{C_j | 1 \le j \le |V|\}$. $\square$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

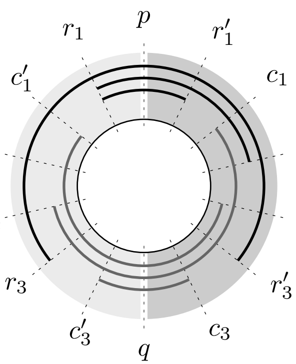

Fig. 2 An example of a family of circular arcs corresponding to a γ-free bipartite adjacency matrix.

Spinrad[16] showed the following.

Lemma 3 For a circular arc graph G that can be partitioned into cliques U and V, there exist two points p, q on a circle and a representation by arcs A_w, $w \in V(G)$ on the same circle such that for every $u \in U$, A_u contains p but not q and A_v contains q but not p. $\square$

Lemma 4 A bipartite graph is a 2-directional orthogonal ray graph if its complement is a circular arc graph.

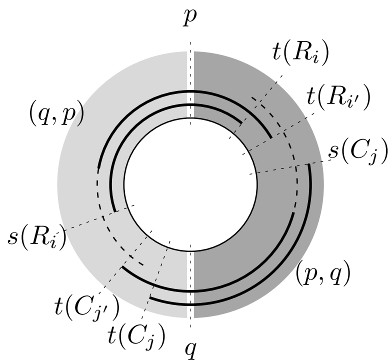

Fig. 3 Arcs $R_i, R_{i'}, C_j$, and $C_{j'}$.

Proof: Let G be a bipartite graph with bipartition (U, V). Suppose $\overline{G}$, the complement of G, is a circular arc graph. Let p and q be two points on a circle O, and let $\mathcal{R}_U$ and $\mathcal{C}_V$ be the set of arcs on O corresponding to the vertices in U and V, respectively, such that all arcs in $\mathcal{R}_U$ contain p but not q, and all arcs in $\mathcal{C}_V$ contain q but not p, by Lemma 3. Let $R_1, R_2, \ldots, R_{|U|}$ be the arcs in $\mathcal{R}_U$ in the order of the occurrence of their clockwise endpoints when moving around O in the clockwise direction starting from p, and let $C_1, C_2, \ldots, C_{|V|}$ be the arcs in $\mathcal{C}_V$ in the order of the occurrence of their clockwise endpoints when moving around C in the counterclockwise direction starting from p. Let $M = [m_{ij}]$ be a

$|U| \times |V|$ $(0,1)$-matrix defined as $m_{ij} = 1$ if and only if R_i and C_j do not intersect. Obviously, M is a bipartite adjacency matrix of G. We shall show that M is γ-free. For some integers i, i', j, j', $(1 \leq i < i' \leq |U|, 1 \leq j' < j \leq |V|)$, suppose $m_{i'j} = 1$ and $m_{ij'} = 1$. From the definition of M, $m_{i'j} = 1$ means that R_i' and C_j do not intersect. Since they do not intersect, $t(R_{i'})$,the clockwise endpoint of $R_{i'}$, must be counterclockwise from $s(C_j)$, the counterclockwise endpoint of C_j.(See Figure 3). Also, since $i < i'$, $t(R_i)$ must be counterclockwise from $t(R_{i'})$, and therefore R_i and C_j do not intersect on arc (p, q). Similarly we can show that $m_{ij'} = 1$ implies that R_i and C_j do not intersect on arc (q, p) either. Since R_i and C_j do not intersect anywhere on O, the corresponding matrix entry m_{ij} is 1. Therefore, M is γ-free by Lemma 1, and thus G is a 2-directional orthogonal ray graph, by Theorem 1. □

From Lemmas 2 and 4, we have the following

Theorem 2 A bipartite graph G is a 2-directional orthogonal ray graph if and only if its complement is a circular arc graph. □

Theorem 2 leads to some interesting consequences as follows. Since McConnell[10] showed a linear-time recognition algorithm for circular arc graphs, we have the following.

Theorem 3 It can be decided in $O(n^2)$ time whether an n-vertex graph is a 2-directional orthogonal ray graph. □

From Theorems 1 and 3, we have the following theorem which settles the open problem of recognizing γ-freeable matrices[6].

Theorem 4 It can be decided in $O((m+n)^2)$ time whether an $m \times n$ matrix is γ-freeable. □

Feder, Hell, and Huang[2] showed the following:

Theorem 5 A graph G which can be partitioned into two cliques is a circular arc graph if and only if the complement of G contains no induced cycles of length at least 6 and no edge-asteroids. □

From Theorems 2 and 5, we have

Corollary 2 A bipartite graph G is a 2-directional orthogonal ray graph if and only if G is chordal bipartite and contains no edge-asteroids. □

Since Hsu[5] showed that graph isomorphism can be solved in $O(mn)$ time for n-vertex m-edge circular arc graphs, we have the following.

Corollary 3 The graph isomorphism problem can be solved in $O(n^3)$ time for n-vertex 2-directional orthogonal ray graphs. □

On the other hand, Uehara, Toda, and Nagoya[19] showed that the isomorphism problem is GI-complete for chordal bipartite graphs. Thus the class of 2-directional orthogonal ray graphs provides a boundary case for the complexity of graph isomorphism. This is an improvement from the earlier boundary class, the interval bigraphs, which is a proper subset of the class of 2-directional orthogonal ray graphs, as we shall show in Section 4 .

3.4 Forbidden Subgraph Characterization

Trotter and Moore[18] showed the following.

Theorem 6 Let G be a graph which can be partitioned into two cliques and let G^c be its complement. Then G is a circular arc graph if and only if the interval dimension of the associated bipartite poset P_{G^c} is at most 2. □

From Theorems 2 and 8, we obtain the following.

Theorem 7 A graph G is a two-directional orthogonal ray graph if and only if the interval dimension of the associated bipartite poset P_G is at most 2. □

Trotter and Moore[18] provided the minimum list $\mathcal{P}$ of posets so that a bipartite poset has interval dimension at most two if and only if it does not contain a poset from $\mathcal{P}$ as a subposet. It is straigthforward to derive from $\mathcal{P}$ the minimal list of forbidden induced subgraphs for 2-directional orthogonal ray graphs. The list shown in Figure 4 contains 6 infinite families of graphs and 3 odd examples.

Theorem 8 A graph G is a two-directional orthogonal ray graph if and only if G does not contain any graph in Figure 4 as an induced subgraph. □

4. Class Hierarchy

In this section, we explore the relation among the classes of orthogonal ray graphs, 2-directional orthogonal ray graphs, and the graph classes mentioned in Section 2.

The following observation is implicit in a paper by Kostochka and Nesetril[7], and can be seen without difficulty.

Observation 1 A cycle C_{2n} of length $2n$ is an orthogonal ray graph if and only if $2 \leq n \leq 6$. □

Observation 2 The class of orthogonal ray graphs is a proper subset of the

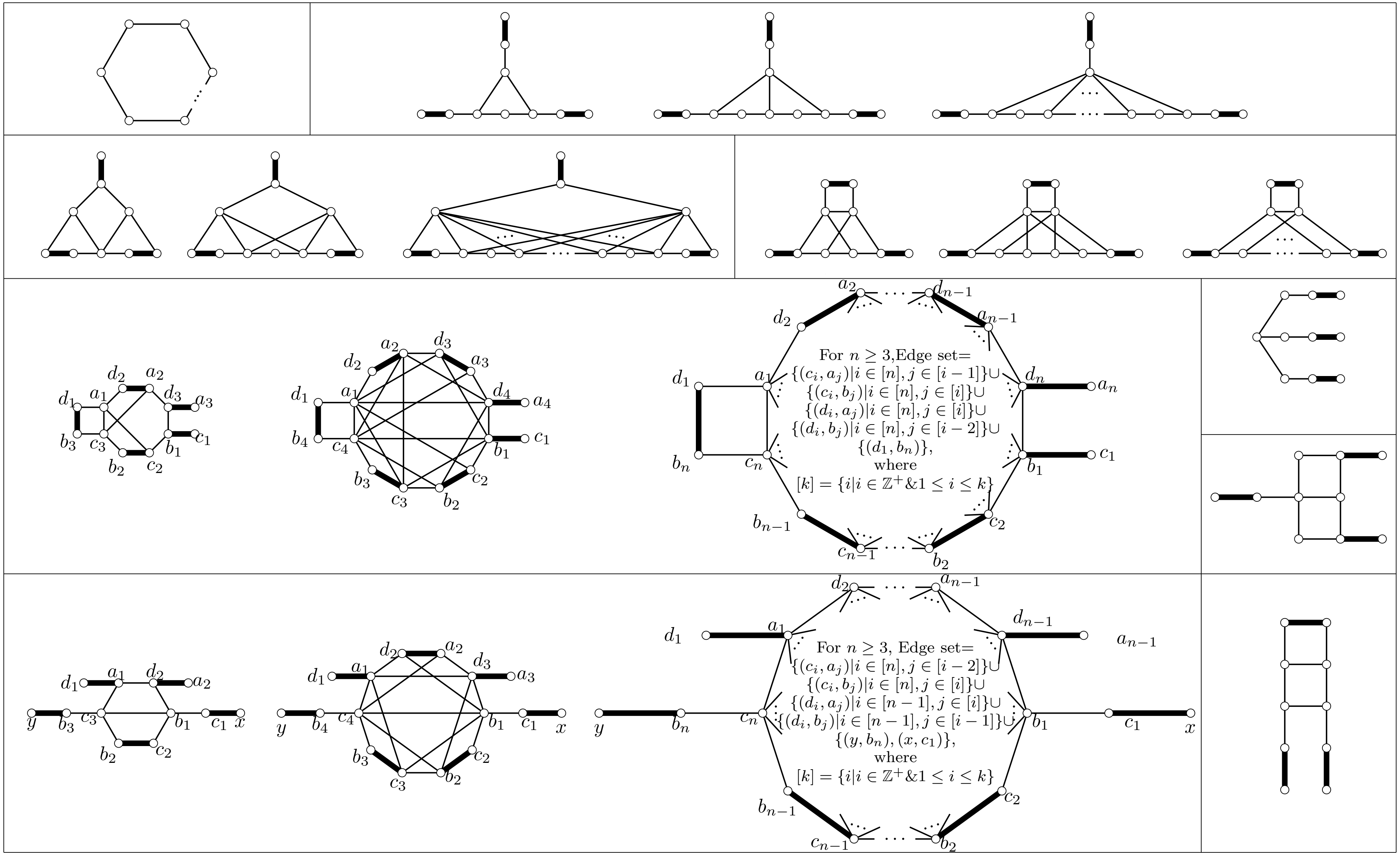

Fig. 4 Forbidden Subgraphs for 2-directional Orthogonal Ray Graphs (Bold edges constitute an edge-asteroid).

class of unit grid intersection graphs.

Proof: Let G be an orthogonal ray graph with bipartition (U, V). We can find a square S on the xy-plane with sides parallel to the x and y axes such that all the cross points of rays R_w, $w \in U \cup V$, lie inside S and such that each ray intersects with only one side of S. Let l be the length of a side of S. Let L_w be the line segment with one endpoint coinciding with the endpoint of R_w and the other endpoint on R_w at a distance l from the endpoint of R_w. We can easily see that G is a unit grid intersection graph for line segments L_w, $w \in U \cup V$. Thus the class of orthogonal ray graphs is a subset of the class of unit grid intersection graphs.

It is easy to see that C_{2n} is a unit grid intersection graph for any $n \geq 2$. Thus we conclude by Observation 1 that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersecion graphs. □

From Observation 1 and Corollary 2, we have the following.

Observation 3 The class of 2-directional orthogonal ray graphs is a proper subset of the class of orthogonal ray graphs. □

Otachi, Okamoto, and Yamazaki[12] showed that the class of graphs which have a γ-freeable bipartite adjancency matrix properly contains the class of interval bigraphs, and therefore we have the following.

Observation 4 The class of interval bigraphs is a proper subset of the class of 2-directional orthogonal ray graphs. □

The relationship between the various graph classes mentioned in this paper can be summarized as shown in Figure 5.

We conclude by noting that characterization and recognition of orthogonal ray graphs remain open.

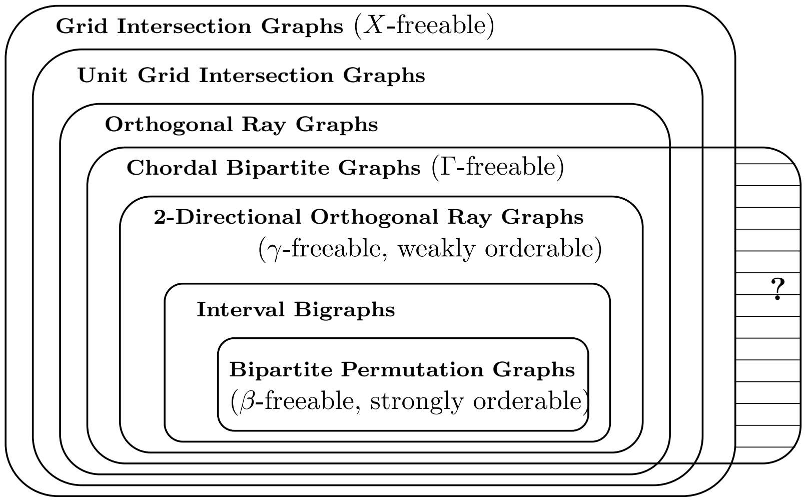

Fig. 5 Relationship between various graph classes.

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