Cheating Strategies
for the Gale-Shapley Algorithm
with Complete Preference Lists

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This paper deals with a strategic issue in the stable marriage model with complete preference lists (i.e., a preference list of an agent is a permutation of all the members of the opposite sex).

Given complete preference lists of all the men, a partial marriage, and complete preference lists of unmatched women, we consider the problem of finding preference lists of matched women such that the men-proposing Gale-Shapley algorithm applied to the lists produces a (perfect) marriage which is an extension of a given partial marriage. We propose a polynomial time algorithm for finding a desired set of preference lists, if theses exist.

We also deal with the case that complete preference lists of all the men and a partial marriage are given. In this case, we consider a problem of the existence of preference lists of all the women such that the men-proposing Gale-Shapley algorithm produces a marriage including a given partial marriage. We show NP-completeness of this problem.

1. Introduction

Given two sets of agents, men and women, Gale and Shapley[2] discussed a model, called the *stable marriage model*, in which each agent has preferences over agents of the opposite sex. A stable marriage is a one-to-one mapping between sets of men and women, such that there is no man-woman pair who would agree to leave their assigned mates in order to marry each other. Gale and Shapley showed that every set of preference lists admits at least one stable marriage by describing an algorithm, called the deferred acceptance algorithm (the Gale-Shapley algorithm), which *always* finds a stable marriage.

The Gale-Shapley algorithm is employed in a number of labor market clearinghouses and college admission systems. A notable version of the algorithm,

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called the men-proposing Gale-Shapley algorithm, works by having men make proposals to women and produces men-optimal marriage, which every man likes at least as well as any other stable marriage.

Since most applications of the Gale-Shapley algorithm involve the participation of independent agents, it is natural to ask whether agents can benefit by being dishonest about their preference lists. It is well-known that stating true preferences is a dominant strategy for the men in the men-proposing Gale-Shapley algorithm. In settings that allow incomplete preference lists, women on the other hand have strong incentives to submit false preferences. By contrast, little is known in the case of the stable marriage model with complete preference lists.

This paper deals with a strategic issue in the stable marriage model with complete preference lists (i.e., the preference list of an agent is a permutation of all the members of the opposite sex). In[6], the authors discussed the following problem; given complete preference lists of men and a marriage μ, find a set of complete preference lists of women such that the men-proposing Gale-Shapley algorithm applied to the lists produces μ. We established a simple necessary and sufficient condition for the existence of such a set of complete preference lists. The condition directly implies a polynomial time algorithm for the problem. In this paper, we extend our result and discuss cases that a given marriage is not perfect. More precisely, we consider the following two problems.

Given complete preference lists of all the men, a partial marriage, and complete preference lists of the unmatched women, we consider the problem of finding preference lists of matched women such that the men-proposing Gale-Shapley algorithm applied to the lists produces a marriage which is an extension of the given partial marriage. We propose a polynomial time algorithm for finding such a set of preference lists, if they exist.

We also deal with the case that complete preference lists of all the men and a partial marriage are given. In this case, we consider the problem of the existence of preference lists of all the women such that the men-proposing Gale-Shapley algorithm produces a (perfect) marriage including the given partial marriage. Surprisingly, this problem is shown to be NP-complete.

In the next section, we establish some terminology and give background. Section 3 gives definitions of the problems and our main result. In Section 4, we

describe a polynomial time algorithm for the first problem. In Section 5, we show that the second problem is NP-complete.

The issues of strategic manipulation in the stable marriage context are discussed in many papers (see books[5],[9] and the references therein, for example). Roth[8] showed that when the men-proposing algorithm is used, none of the men benefits by submitting a false preference list, regardless of how the other agents report their preferences. Dubins and Freedman[1] proved that no coalition of men could collectively manipulate their preferences in such a way as to strictly improve all of their mates in comparison to the men-optimal marriage. In settings that allow incomplete preference lists, women on the other hand have incentives to cheat in the men-proposing algorithm. Gale and Sotomayor[3] showed that a woman has an incentive to falsify her preference as long as she has at least two distinct stable mates. In fact, the women can force the women-optimal marriage μ by rejecting all the men except their mates in μ (see[4]).

A feature of this paper is that the agents are required to submit complete preference lists. Compared to the above results, little is known in the case of the stable marriage model with complete preference lists. Gusfield and Irving ([5], page 65) point to the absence of any general results in this setting. Teo, Sethuraman, and Tan[10] deal with the situation where there exists a specified woman w who is the only deceitful agent, and she knows the reported preferences of all the other agents. They proposed a polynomial time algorithm for constructing woman w's optimal cheating strategy. They also discussed the Singapore school-admissions problem, where the stable marriage model with 'complete' preference lists is a suitable representation of the problem.

2. Notations and Definitions

We denote two sets of agents by M and W, called men and women, both of size n. Each agent in $M \cup W$ has a *preference list* which is a totally ordered list of all the members of the opposite sex. Agent i prefers q to r if and only if, q precedes r on i's preference list. We consider the case with 'complete' preference lists, i.e., a preference list of an agent includes all the members of the opposite sex. We denote sets of preference lists of M and W by $\mathcal{L}^M$ and $\mathcal{L}^W$, respectively. A *partial marriage* is a mapping $\mu : (M \cup W) \to (M \cup W)$ satisfying

(1) $\forall m \in M,\ \mu(m) \in W \cup \{m\}$,

(2) $\forall w \in W,\ \mu(w) \in M \cup \{w\}$, and

(3) $w = \mu(m)$ if and only if $m = \mu(w)$.

If an agent $i \in M \cup W$ satisfies $\mu(i) \neq i$, we say that i is *matched* and $\mu(i)$ is the *mate* of i in partial marriage μ. We say that a pair of matched agents $\{m, \mu(m)\}$ (satisfying $m \neq \mu(m)$) is a *pair of mates*. Every agent with $\mu(i) = i$ is called *unmatched*. A (*perfect*) *marriage* is a partial marriage with no unmatched agent. We say a partial marriage μ' is an *extension* of μ, if every matched agent i in μ satisfies $\mu(i) = \mu'(i)$.

A pair $(m, w) \in M \times W$ is called a *blocking pair* for a perfect marriage μ, if m strictly prefers w to $\mu(m)$ and w strictly prefers m to $\mu(w)$. A perfect marriage with no blocking pair is called a *stable marriage*. Gale and Shapley[2] showed that a stable marriage always exists, and a simple algorithm called the *deferred acceptance algorithm* or the *Gale-Shapley algorithm* can find a stable marriage. Here we briefly describe a version of their algorithm in which men propose to women (these roles can naturally be reversed). In the following algorithm, we introduce an iteration number which will be used in a later section.

Men-Proposing Gale-Shapley Algorithm (GS-algorithm)

Step 0: Set the iteration number $q := 1$ and unmarried men $U := M$.
Initially, every woman has no current mate.

Step 1: If $U = \emptyset$, then output the current mate of every woman and stop.

Step 2: Choose a man $m \in U$. Let $w \in W$ be m's most preferred woman who hasn't yet rejected m.

Step 3: (Create a proposal from man m to woman w.)
If woman w has no mate, then update $U := U \setminus \{m\}$ and set m be the current mate of w.
Else if w prefers m to her current mate m', then w rejects m', update $U := U \setminus \{m\} \cup \{m'\}$ and set m to be the current mate of w.
Else, w rejects m.

Step 4: Update $q := q + 1$ and go to Step 1.

In the rest of this paper, the above algorithm is called the *GS-algorithm*. We denote the perfect marriage obtained by the GS-algorithm applied to a pair of lists $(\mathcal{L}^M, \mathcal{L}^W)$ by $GS(\mathcal{L}^M, \mathcal{L}^W)$.

It is known that the order of proposals (choice of $m \in U$ in Step 2) does not affect the output of the GS-algorithm[2),7]. If a man $m \in M$ proposed to a women $w \in W$ in an execution of the GS-algorithm, we say that m is a *suitor* of w. For any woman $w \in W$, it is easy to see that any possible execution of the GS-algorithm yields the same set of suitors of w. Conway showed that the set of stable marriages can be partially ordered as a lattice with the pair of extremal elements, called *men-optimal* and *women-optimal* marriages (see[7] for example). In fact, the GS-algorithm described above produces the men-optimal marriage[5),7].

We introduce two graphs which play an important role in this paper. Given a set of preference lists $\mathcal{L}^M$ of men over women and a partial marriage μ', $G(\mathcal{L}^M, \mu')$ denotes a directed bipartite graph, called a *suitor graph*, with a pair of vertex sets M, W and a set of directed edges A defined by

$$A = \{(w, \mu'(w)) \in W \times M \mid w \text{ is a matched woman in } \mu'\}$$
$$\cup \left\{ (m, w) \in M \times W \;\middle|\; \begin{array}{l} m \text{ is a matched man in } \mu' \text{ and} \\ m \text{ strictly prefers } w \text{ to } \mu'(m) \end{array} \right\}.$$

Here we note that if a man $m \in M$ is matched in μ', then vertex $m \in M$ has a unique incoming edge $(\mu'(m), m)$ in $G(\mathcal{L}^M, \mu')$; else, m is an isolated vertex. When a women $w \in W$ is matched in μ', vertex w has a unique outgoing edge $(w, \mu'(w))$. Let $\mu^* = \mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$ be the perfect marriage obtained by the GS-algorithm. Then, it is easy to see that a man-woman pair $\{m, w\}$ forms a directed edge (either (m, w) or (w, m)) in the suitor graph $G(\mathcal{L}^M, \mu^*)$, if and only if, m is the suitor of w.

Given the suitor graph $G(\mathcal{L}^M, \mu')$, we also define a directed graph $\overline{G}(\mathcal{L}^M, \mu')$, called a *rooted suitor graph*, as follows. We introduce an artificial vertex r, called the *root*, to $G(\mathcal{L}^M, \mu')$ and add directed edge (r, w) for each woman vertex $w \in W$ that has no incoming edge. When a vertex v of the (rooted) suitor graph corresponds to a man (woman), we say that v is a *man vertex* (a *woman vertex*), respectively.

When a given directed graph has a directed path from u to v, we say that v is *reachable from u*. An *outgoing spanning tree* of a rooted directed graph is a subset of directed edges T such that every vertex v except the root has a unique

incoming edge in T and a directed path $P_v \subseteq T$ from the root vertex to v. It is easy to see that a given rooted directed graph has at least one outgoing spanning tree if and only if every vertex is reachable from the root.

Lastly, we show a property of a rooted suitor graph defined by the stable marriage obtained by the GS-algorithm.

Lemma 2.1 Given a pair of sets of preference lists $\mathcal{L}^M$ and $\mathcal{L}^W$, the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu^*)$ corresponding to the perfect marriage $\mu^* = \mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$ has an outgoing spanning tree.

Proof. The definition of the GS-algorithm implies that for any woman $w \in W$, w has at least one suitor and her mate $\mu^*(w)$ is her most preferred suitor. For any women vertex $w \in W$, we define a parent vertex $\mathrm{prt}(w)$ of w as follows. If w has more than one suitor, the parent vertex of w is the second favorite with w in the set of suitors of w. If w has a unique suitor, the parent vertex of w is the root vertex r. For each man vertex $m \in M$, we define $\mathrm{prt}(m) = \mu^*(m)$. Let T be the subset of edges of the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu^*)$ defined by $T = \{(\mathrm{prt}(v), v) | v \in M \cup W\}$. We show that T forms an outgoing spanning tree in the rooted suitor graph.

Clearly from the definition, every vertex $v \in M \cup W$ has a unique incoming edge $(\mathrm{prt}(v), v)$ in T. Thus, we need to show that every vertex $v \in M \cup W$ has a directed path $P_v \subseteq T$ from the root r to v. Assume on the contrary that there exists a vertex $v \in M \cup W$ such that v is not reachable from the root in the directed graph induced by T. Since every vertex except the root has a unique incoming edge, we can find a directed cycle $C \subseteq T$ by traversing incoming edges in the opposite direction from vertex v. Let $V(C)$ be the set of vertices induced by C. For any woman vertex $w \in V(C)$, a man corresponding to her parent vertex $\mathrm{prt}(w)$ is rejected by w in an arbitrary execution of the GS-algorithm. Suppose that $(\mathrm{prt}(w), w)$ is a directed edge in C such that, the corresponding rejection was the first occasion, during an execution of the GS-algorithm, that a woman in $V(C)$ rejected a man corresponding to her parent vertex. This rejection took place because of a proposal of $\mu^*(w)$ to w. If w rejected $\mathrm{prt}(w)$ at the q-th iteration, man $\mu^*(w)$ proposed to w at the q-th or an earlier iteration. Let $w' \in V(C)$ be the woman with $\mathrm{prt}(w') = \mu^*(w)$. Then, man $\mu^*(w)$ prefers w' to w and thus w' rejected $\mu^*(w)$ at the q'-th iteration with $q' < q$. This contradicts

with the assumption that the rejection corresponding to $(\mathrm{prt}(w), w)$ was the first such rejection. $\qquad\square$

Given a pair of preference lists $(\mathcal{L}^M, \mathcal{L}^W)$, the outgoing spanning tree T of the rooted suitor graph defined in the above proof is called the *tree of second favorites*.

3. Problems and Main Result

In this paper, we consider the following four problems. The first problem is a fundamental problem discussed in our paper[6] and a special case of the other three problems. The second problem deals with the case that preference lists of some women are fixed, which plays an important role for solving the third problem. The third and fourth problems correspond to the two problems described in Section 1.

Problem Q1($\mathcal{L}^M, \mu$):

Input: A set of preference lists $\mathcal{L}^M$ of men M over women W and a perfect marriage μ.

Question: If there exists a set of preference lists $\mathcal{L}^W$ of women W over men M such that $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu$, then output $\mathcal{L}^W$. If not, say 'none exists.'

Next, we consider a case with a perfect marriage μ and a set of preference lists $\mathcal{L}^{W'}$ of a given subset W' of women.

Problem Q2($\mathcal{L}^M, \mu, \mathcal{L}^{W'}$):

Input: A set of preference lists $\mathcal{L}^M$ of men M over women W, a perfect marriage μ, and a set of preference lists $\mathcal{L}^{W'}$ of a (given) subset of women $W' \subseteq W$ over men M.

Question: If there exists a set of preference lists $\mathcal{L}^W$ of W over men M such that $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu$, then output $\mathcal{L}^W$. If not, say 'none exists.'

The above problem gives an insight for solving the third problem described below. We assume that preference lists of men, a partial marriage μ', and preference lists of unmatched women are given. We consider the problem of finding a cheating strategy for a coalition of matched women such that the GS-algorithm produces a perfect marriage with a particular mate in μ' for each matched woman.

Problem Q3($\mathcal{L}^M, \mu', \mathcal{L}^{W'}$):

Input: A set of preference lists $\mathcal{L}^M$ of men M over women W, a partial marriage μ' and a set of preference lists $\mathcal{L}^{W'}$ (over men M) of women W' unmatched in μ'.

Question: If there exists a set of preference lists $\mathcal{L}^W$ of W over men M such that $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and the perfect marriage $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$ is an extension of μ', then output $\mathcal{L}^W$. If not, say 'none exists.'

Lastly, we consider a problem of the existence of cheating strategy for coalition W to achieve a given partial marriage.

Problem Q4($\mathcal{L}^M, \mu'$):

Input: A set of preference lists $\mathcal{L}^M$ of men M over women W and a partial marriage μ'.

Question: Is there a set of preference lists $\mathcal{L}^W$ of W over men M such that the perfect marriage $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$ is an extension of μ'?

The aim of this paper is to show the following theorem.

Theorem 3.1 Problems Q1($\mathcal{L}^M, \mu$), Q2($\mathcal{L}^M, \mu, \mathcal{L}^{W'}$) and Q3($\mathcal{L}^M, \mu', \mathcal{L}^{W'}$) are polynomial time solvable. Problem Q4($\mathcal{L}^M, \mu$) is NP-complete.

4. Polynomial Time Solvability

In this section, we show that Problems Q1, Q2 and Q3 are polynomial time solvable.

4.1 Polynomial Time Solvability of Problem Q1

Recently, we discussed Problem Q1($\mathcal{L}^M, \mu$) in[6] and showed a necessary and sufficient condition for the existence of preference lists $\mathcal{L}^W$ of women with $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu$. The following theorem also gives a necessary and sufficient condition which is essentially the same as that obtained in[6].

Theorem 4.1 Given an instance of Problem Q1($\mathcal{L}^M, \mu$), there exists a set $\mathcal{L}^W$ of preference lists of W over M such that $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu$ if and only if the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu)$ has an outgoing spanning tree.

The proof is omitted.

The above theorem directly implies that we can check the existence of preference lists $\mathcal{L}^W$ such that $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu$ by searching the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu)$, which requires $\mathrm{O}(n^2)$ time[11]. Algorithm Q2 appearing in the next subsection also solves Problem Q1$(\mathcal{L}^M, \mu)$, since Problem Q1$(\mathcal{L}^M, \mu)$ is a special case of Problem Q2$(\mathcal{L}^M, \mu, \mathcal{L}^{W'})$.

4.2 Polynomial Time Solvability of Problem Q2

We propose a polynomial time algorithm for solving Problem Q2$(\mathcal{L}^M, \mu, \mathcal{L}^{W'})$ where $\mathcal{L}^{W'}$ is a set of preference lists of a (given) subset of women $W' \subseteq W$ over men M.

Algorithm Q2

Step 1: Construct the suitor graph $G(\mathcal{L}^M, \mu)$ and the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu)$.

Step 2: For each woman $w \in W$, construct a set $\delta(w)$ of incoming edges of woman vertex w in $G(\mathcal{L}^M, \mu)$.

Step 3: For each woman $w' \in W'$ with $\delta(w') \neq \emptyset$, let $\mathrm{prt}(w')$ be a man who is the top favorite with w' in the set $\{m \in M \mid (m, w) \in \delta(w')\}$ with respect to her preference list in $\mathcal{L}^{W'}$.

Step 4: For every woman $w' \in W'$ with $\delta(w') \neq \emptyset$, delete edges of $\overline{G}(\mathcal{L}^M, \mu)$ in $\delta(w') \setminus \{(\mathrm{prt}(w'), w')\}$ and obtain a directed graph G'.

Step 5: If G' does not have an outgoing spanning tree, then output 'none exists' and stop. Else, let T be an outgoing spanning tree in G'.

Step 6: For each woman $w \in W$, we set $\mathrm{prt}(w)$ to the man m such that (m, w) is the unique incoming edge in T. (Here we note that when $w' \in W'$ satisfies $\delta(w') \neq \emptyset$, $\mathrm{prt}(w')$ defined in this step coincides with that defined in Step 3, since w' has a unique incoming edge in G'.) Construct preference lists $\mathcal{L}^W$ of women as follows.

(1) For each woman $w' \in W'$, we use her preference list in $\mathcal{L}^{W'}$.

(2) For each woman $w \notin W'$ with $\mathrm{prt}(w) = r$, we construct a preference list (a total order of men M) such that the most preferred man is $\mu(w)$.

(3) For each woman $w \notin W'$ with $\mathrm{prt}(w) \neq r$, we construct a preference list (a total order of men M) of w such that woman w's first choice is $\mu(w)$ and

her second choice is $\mathrm{prt}(w)$.

Step 7: If the given perfect marriage μ is stable with respect to the pairs of preference lists $(\mathcal{L}^M, \mathcal{L}^W)$, then output the preference list $\mathcal{L}^W$. Else, output 'none exists.'

Theorem 4.2 Algorithm Q2 solves Problem Q2$(\mathcal{L}^M, \mu, \mathcal{L}^{W'})$ correctly.

Proof. Consider the case that Q2$(\mathcal{L}^M, \mu, \mathcal{L}^{W'})$ has a set of preference lists $\widetilde{\mathcal{L}^W}$ of women such that $\widetilde{\mathcal{L}^W}$ includes $\mathcal{L}^{W'}$ and $\mu = \mathrm{GS}(\mathcal{L}^M, \widetilde{\mathcal{L}^W})$. We show that Algorithm Q2 outputs such a set of preference lists of women.

Lemma 2.1 implies that the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu)$ has a tree T^* of second favorites (a specified outgoing spanning tree defined in the proof of Lemma 2.1). Clearly from the definition of G' obtained in Algorithm Q2, T^* is contained in the edge set of G' and thus Algorithm Q2 does not terminate at Step 5. Assume on the contrary that Algorithm Q2 terminates at Step 7, because perfect marriage μ is not stable with respect to preference lists $\mathcal{L}^M$ and $\mathcal{L}^W$ obtained in Step 6. Then μ has a blocking pair $(m^*, w^*) \in M \times W$. For any woman $w \notin W'$, her mate $\mu(w)$ is her top favorite in $\mathcal{L}^W$ and thus we have that $w^* \in W'$. For any woman $w' \in W'$, her preference list in $\mathcal{L}^W$ is the same as that in $\mathcal{L}^{W'}$. Since $\widetilde{\mathcal{L}^W}$ includes $\mathcal{L}^{W'}$, the preference list of $w' \in W'$ in $\mathcal{L}^W$ is also the same as that in $\widetilde{\mathcal{L}^W}$. Thus, the pair (m^*, w^*) is also a blocking pair of μ with respect to the pair of preference lists $(\mathcal{L}^M, \widetilde{\mathcal{L}^W})$. Contradiction.

Next, consider the case that Algorithm Q2 produced a set of preference lists $\mathcal{L}^W$. Then, we only need to show that the GS-algorithm applied to preference lists $\mathcal{L}^M$ and $\mathcal{L}^W$ produces the given marriage μ. Let μ^* be a perfect marriage obtained by the GS-algorithm applied to the lists in $\mathcal{L}^M$ and $\mathcal{L}^W$, i.e., we set $\mu^* = \mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$. In the following, we show that $\mu = \mu^*$.

Since Algorithm Q2 checked the stability of μ at Step 7, μ is a stable marriage with respect to lists $\mathcal{L}^M$ and $\mathcal{L}^W$. It is well-known that the GS-algorithm produces the men-optimal marriage. Thus, for every man $m \in M$, either $\mu^*(m) = \mu(m)$ holds or m prefers $\mu^*(m)$ to $\mu(m)$. This implies that, if man m proposed to woman w in an execution of the GS-algorithm, then the graph $G(\mathcal{L}^M, \mu)$ includes a directed edge ((m, w) or (w, m)) between m and w.

Let T be an outgoing spanning tree obtained in Step 5. In the rest of this proof, we show that $\mu = \mu^*$ by induction on heights of vertices defined by T. For

any vertex $i \in M \cup W$, $h(i)$ denotes the *height* of i in T, i.e., $h(i)$ is equal to the length of the unique path from r to i in the tree induced by T. We define $h(r) = 0$.

(1) Let $i \in M \cup W$ be an agent with $h(i) = 1$. This implies that i is a woman vertex satisfying $\mathrm{prt}(i) = r$. Clearly from the definition of the rooted suitor graph, vertex i does not have any incoming edge in the suitor graph $G(\mathcal{L}^M, \mu)$ and has a unique outgoing edge $(i, \mu(i))$. Since man $\mu^*(i)$ proposed to i in an execution of the GS-algorithm, the suitor graph $G(\mathcal{L}^M, \mu)$ has an edge between i and $\mu^*(i)$. From the above, equality $\mu(i) = \mu^*(i)$ is obtained.

(2) Assume that for any vertex $j \in M \cup W$, $h(j) = h'$ yields $\mu(j) = \mu^*(j)$. Let $i \in M \cup W$ be a vertex satisfying $h(i) = h' + 1 \geq 2$.

(2-1) If $h' + 1$ is an even number, i corresponds to a man, denoted by $m \in M$. Since man vertex m has a unique incoming edge $(\mu(m), m)$, $\mu(m)$ is the parent vertex of m, whose height is h'. The induction hypothesis yields that $\mu^*(\mu(m)) = \mu(\mu(m)) = m$. From the definition of marriage, $\mu^*(\mu(m)) = m$ implies that $\mu(m) = \mu^*(m)$.

(2-2) When $h' + 1$ is an odd number and i corresponds to a woman $w \notin W'$, we denote w's parent vertex $\mathrm{prt}(w)$ by $m \in M$, for simplicity. Since the rooted suitor graph includes directed edge (m, w), man m strictly prefers w to $\mu(m)$. From the induction hypothesis, $\mu^*(m) = \mu(m)$ and thus man m proposed to woman $\mu(m)$ in an execution of the GS-algorithm. Consequently, man m proposed to woman w and w rejected m in an execution of the GS-algorithm. In the preference lists $\mathcal{L}^W$, man m is w's second choice. Since w rejected her second choice m, w's mate obtained in an execution of the GS-algorithm is her first choice $\mu(w)$. Thus, we obtained the desired result that w's mate obtained in an execution of the GS-algorithm, denoted by $\mu^*(w)$, is $\mu(w)$.

(2-3) Lastly, consider the case that $h'+1$ is an odd number and i corresponds to a woman $w' \in W'$. We denote her parent vertex $\mathrm{prt}(w')$ by $m' \in M$, for simplicity. From induction hypothesis, $\mu^*(m') = \mu(m')$ and thus man m' proposed to woman w' and w' rejected m' in an execution of the GS-algorithm. From the definition of G', man m' is the top favorite of w' in the set $\{m \in M \mid (m, w') \in \delta(w')\}$ with respect to her preference list in $\mathcal{L}^{W'}$. The rejection of m' by w' implies

that $\mu^*(w)$ satisfies two conditions (1) w' strictly prefers $\mu^*(w')$ to m' and (2) $G(\mathcal{L}^M, \mu)$ includes a directed edge between $\mu^*(w')$ and w'. If $G(\mathcal{L}^M, \mu)$ includes directed edge $(\mu^*(w'), w')$, then $(\mu^*(w'), w') \in \delta(w')$ and thus w' strictly prefers m' to $\mu^*(w')$, which contradicts with condition (1). Consequently, $G(\mathcal{L}^M, \mu)$ includes directed edge $(w', \mu^*(w'))$ and thus we have $\mu^*(w') = \mu(w')$. □

Since the complexity of every step in Algorithm Q2 is bounded by $\mathrm{O}(n^2)$, the total computational time is also bounded by $\mathrm{O}(n^2)$.

When we solve Problem Q1$(\mathcal{L}^M, \mu)$, we do not need Steps 2-4 of Algorithm Q2, since G' is equivalent to the rooted suitor graph $\overline{G}(\mathcal{L}^M, \mu)$. At Step 7, we only need to output $\mathcal{L}^W$, since $\mu(w)$ is w's first choice in $\mathcal{L}^W$ and thus μ is always stable with respect to the pair $(\mathcal{L}^M, \mathcal{L}^W)$.

4.3 Polynomial Time Solvability of Problem Q3

Lastly, we show a technique to reduce Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$ to Problem Q2$(\mathcal{L}^M, \mu, \mathcal{L}^{W'})$ with a certain perfect marriage μ.

Given an instance of Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$, we construct a stable marriage instance obtained by restricting to unmatched agents in μ'. More precisely, construct the set M' of unmatched men and the set W' of unmatched women, first. For each man $m \in M'$, we construct a preference list of m from his list in $\mathcal{L}^M$ by deleting all the women not in W'. For every woman $w \in W'$, we also construct her preference list from the given preference list in $\mathcal{L}^{W'}$ similarly. We denote the obtained preference lists of agents M' and W' by $\mathcal{K}^{M'}$ and $\mathcal{K}^{W'}$, respectively. Let $\mathcal{N}$ be the set of all the stable marriages defined on $M' \cup W'$ with respect to the pair of preference lists $(\mathcal{K}^{M'}, \mathcal{K}^{W'})$. (We do not need to construct $\mathcal{N}$ explicitly.)

Consider the case that Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$ has a set of preference lists $\mathcal{L}^W$ of W over men M such that $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and the perfect marriage GS$(\mathcal{L}^M, \mathcal{L}^W)$ is an extension of μ'. Let ν^* be a perfect marriage on $M' \cup W'$ obtained from the perfect marriage GS$(\mathcal{L}^M, \mathcal{L}^W)$ by restricting to agents $M' \cup W'$. Then, it is obvious that ν^* is a stable marriage with respect to $(\mathcal{K}^{M'}, \mathcal{K}^{W'})$ i.e., $\nu^* \in \mathcal{N}$. In the following, we show that ν^* is easily obtained without a knowledge of preference lists in $\mathcal{L}^W$ of matched women $W \setminus W'$.

Let us consider the suitor graph $G(\mathcal{L}^M, \mu')$ defined by a given partial marriage

μ'. For each woman $w \in W$, $\delta'(w)$ denotes the set of incoming edges of w in $G(\mathcal{L}^M, \mu')$. We define a subset $\overline{\mathcal{N}}$ of stable marriages $\mathcal{N}$ by

$$\overline{\mathcal{N}} = \{\nu \in \mathcal{N} \mid \forall w' \in W', \forall (m, w') \in \delta'(w'), w' \text{ strictly prefers } \nu(w') \text{ to } m\}.$$

We show that $\nu^* \in \overline{\mathcal{N}}$. For any woman $w' \in W'$, a man m with $(m, w') \in \delta'(w')$ is rejected by w', when we apply the GS-algorithm to the pair of lists $(\mathcal{L}^M, \mathcal{L}^W)$. Thus, for any pair $(m, w') \in \delta'(w')$ with $w' \in W'$, woman w' strictly prefers $\nu^*(w')$ to m. Consequently, $\nu^* \in \overline{\mathcal{N}}$ holds. Next, we show that $\overline{\mathcal{N}}$ has a unique minimal element with respect to a dominance relation defined by $\mathcal{K}^{W'}$. A stable marriage $\nu_1 \in \mathcal{N}$ is said to dominate a stable marriage $\nu_2 \in \mathcal{N}$, written $\nu_1 \succeq \nu_2$, if every woman in W' has at least as good a mate in ν_1 as she has in ν_2 with respect to $\mathcal{K}^{W'}$, i.e., every woman in W' prefers ν_1 to ν_2 or is indifferent between them. A stable marriage $\nu \in \overline{\mathcal{N}}$ is called *minimal*, if $\{\nu' \in \overline{\mathcal{N}} \mid \nu \succeq \nu'\} = \{\nu\}$ holds.

Lemma 4.3 The poset of stable marriages $(\overline{\mathcal{N}}, \succeq)$ has a unique minimal element.

Proof. Assume that the poset $(\overline{\mathcal{N}}, \succeq)$ has a mutually distinct pair of minimal stable marriages ν_1 and ν_2. It is well-known that $\mathcal{N}$ has a lattice structure and there exists a stable marriage, denoted by $\nu_3 \in \mathcal{N}$, in which each woman in W' receives the poorer of her mates in ν_1 and ν_2. Obviously, $\nu_3 \in \overline{\mathcal{N}}$ holds. The dominance relation $\nu_1 \succeq \nu_3$ and minimality of ν_1 implies $\nu_1 = \nu_3$. Similarly, we have $\nu_2 = \nu_3$. Thus, $\nu_1 = \nu_2$ holds giving a contradiction. $\square$

The following theorem uniquely determines stable marriage ν^*.

Theorem 4.4 Assume that Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$ has a set of preference lists $\mathcal{L}^W$ of W over men M such that $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and the perfect marriage $\mu^* = \text{GS}(\mathcal{L}^M, \mathcal{L}^W)$ is an extension of μ'. Let ν^* be a marriage obtained from μ^* by restricting to unmatched agents of μ'. Then ν^* is the unique minimal element of a poset $(\overline{\mathcal{N}}, \succeq)$.

Proof. Obviously, ν^* is a stable marriage in $\overline{\mathcal{N}}$. We denote a unique minimal element of the poset $(\overline{\mathcal{N}}, \succeq)$ by $\nu^{\min}$ and show that $\nu^{\min} = \nu^*$. Let $\mu^{\min}$ be the perfect marriage on $M \cup W$ obtained by forming the union of $\nu^{\min}$ and μ'. Then, we only need to show that $\mu^{\min} = \mu^*$.

Recall that M' and W' denote the sets of unmatched men and women of μ', respectively. First, we show that $\mu^{\min}$ is a stable marriage with respect to the pair of preference lists $(\mathcal{L}^M, \mathcal{L}^W)$. Assume on the contrary that $\mu^{\min}$ is unstable. Then $\mu^{\min}$ has a blocking pair $(m^*, w^*) \in M \times W$.

(1) Since $\nu^{\min}$ is a stable marriage with respect to $(\mathcal{K}^{M'}, \mathcal{K}^{W'})$, it is obvious that $(m^*, w^*) \notin M' \times W'$.

(2) Both μ^* and $\mu^{\min}$ are extensions of μ' and thus we have $(m^*, w^*) \notin (M \setminus M') \times (W \setminus W')$.

(3) Consider the case that $(m^*, w^*) \in M' \times (W \setminus W')$. Every man $m' \in M'$ prefers $\nu^{\min}(m')$ to $\nu^*(m')$ or is indifferent between them, otherwise $(m', \nu^*(m'))$ becomes a blocking pair of $\nu^{\min}$. Thus, $m^* \in M'$ prefers $\nu^{\min}(m^*)$ to $\nu^*(m^*) = \mu^*(m^*)$. Since $\mu^*(w^*) = \mu^{\min}(w^*)$, the pair (m^*, w^*) is also a blocking pair of μ^* giving a contradiction.

(4) Consider the remaining case that $(m^*, w^*) \in (M \setminus M') \times W'$. Since $m^* \notin M'$ prefers w^* to $\mu^{\min}(m^*) = \mu'(m^*)$, the suitor graph $G(\mathcal{L}^M, \mu')$ has a directed edge (m^*, w^*) and thus $(m^*, w^*) \in \delta'(w^*)$. Then, the definition of $\overline{\mathcal{N}}$ implies that w^* prefers $\nu^{\min}(w^*) = \mu^{\min}(w^*)$ to m^*, which contradicts the assumption that (m^*, w^*) is a blocking pair of $\mu^{\min}$.

From the above discussion, $\mu^{\min}$ is a stable marriage with respect to $(\mathcal{L}^M, \mathcal{L}^W)$.

If we assume $\mu^{\min} \neq \mu^*$, there exists a man $m' \in M'$ satisfying $\mu^{\min}(m') \neq \mu^*(m')$. Since every woman in W' prefers ν^* to $\nu^{\min}$ or is indifferent between them, m' strictly prefers $\nu^{\min}(m') = \mu^{\min}(m')$ to $\nu^*(m') = \mu^*(m')$ (otherwise $(m', \nu^*(m'))$ becomes a blocking pair of $\nu^{\min}$). Then, μ^* is not the men-optimal marriage with respect to $(\mathcal{L}^M, \mathcal{L}^W)$. This contradicts with the fact that the GS-algorithm produces the men-optimal marriage. $\square$

It is well-known that the poset $(\mathcal{N}, \succeq)$ forms a distributive lattice, defined by a 'rotation poset' (see[5] for example). By constructing the rotation poset, we can find the unique minimal marriage of $(\overline{\mathcal{N}}, \succeq)$ in polynomial time. The required computational effort is bounded by $\text{O}(n^4)$. In fact, we only need the set of all the rotations and thus we can reduce the time complexity to $\text{O}(n^2)$ (details are omitted).

Then we can solve Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$ as follows.

Algorithm Q3

Step 1: Construct the suitor graph $G(\mathcal{L}^M, \mu')$.

Step 2: Construct the sets of preference lists $(\mathcal{K}^{M'}, \mathcal{K}^{W'})$ from $(\mathcal{L}^M, \mathcal{L}^{W'})$ by restricting to unmatched agents of μ'.

Step 3: If $\overline{\mathcal{N}} = \emptyset$, then output 'none exists' and stop. Else, find the unique minimal element $\nu^{\min}$ of $\overline{\mathcal{N}}$.

Step 4: Construct the perfect marriage $\mu^{\min}$ on $M \cup W$ by forming the union of μ' and $\nu^{\min}$.

Step 5: Solve Problem Q2$(\mathcal{L}^M, \mu^{\min}, \mathcal{L}^{W'})$ by Algorithm Q2.

Theorem 4.5 Algorithm Q3 correctly solves Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$.

Proof. Assume that Problem Q3$(\mathcal{L}^M, \mu', \mathcal{L}^{W'})$ has a set of preference lists $\mathcal{L}^W$ of W over men M such that $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and the perfect marriage $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W)$ is an extension of μ'. Theorem 4.4 implies that $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu^{\min}$. Thus, Algorithm Q2 applied to Problem Q2 $(\mathcal{L}^M, \mu^{\min}, \mathcal{L}^{W'})$ at Step 5 outputs a desired solution.

The inverse implication is easy. If Algorithm Q2, executed at Step 5, produces a set of preference lists $\mathcal{L}^W$ of W over men M, then $\mathcal{L}^W$ includes $\mathcal{L}^{W'}$ and $\mathrm{GS}(\mathcal{L}^M, \mathcal{L}^W) = \mu^{\min}$ is an extension of μ'. $\qquad\square$

5. NP-completeness of Problem Q4

In this section, we discuss the intractability of Problem Q4$(\mathcal{L}^M, \mu')$.

Theorem 5.1 Problem Q4$(\mathcal{L}^M, \mu')$ is NP-complete.

Here we describe the outline of our proof. The proof is by reduction from the satisfiability problem (SAT) described as follows.

Satisfiability problem (SAT):

Input: A finite set X of Boolean variables and a collection $\mathcal{C} \subseteq 2^X$ of clauses.

Question: Is there a satisfying truth assignment to X for $\mathcal{C}$?

In the proof, we define sets of agents consisting of $2|\mathcal{C}|+3|X|$ men and $2|\mathcal{C}|+3|X|$ women, a set of preference lists $\mathcal{L}^M$ of men over women, and a partial marriage μ' with $2|\mathcal{C}| + |X|$ pairs of mates, algorithmically.

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