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Head-Needed Strategy of Higher-Order Rewrite Systems and Its Decidable Classes

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The present paper discusses a head-needed strategy and its decidable classes of higher-order rewrite systems (HRSs), which is an extension of the head-needed strategy of term rewriting systems (TRSs). We discuss strong sequential and NV-sequential classes having the following three properties, which are mandatory for practical use: (1) the strategy reducing a head-needed redex is head normalizing (2) whether a redex is head-needed is decidable, and (3) whether an HRS belongs to the class is decidable. The main difficulty in realizing (1) is caused by the β-reductions induced from the higher-order reductions. Since β-reduction changes the structure of higher-order terms, the definition of descendants for HRSs becomes complicated. In order to overcome this difficulty, we introduce a function, PV, to follow occurrences moved by β-reductions. We present a concrete definition of descendants for HRSs by using PV and then show property (1) for orthogonal systems. We also show properties (2) and (3) using tree automata techniques, a ground tree transducer (GTT), and recognizability of redexes.

1. Introduction

Higher-order rewrite systems (HRSs)[14], an extension of term rewriting systems (TRSs) obtained by introducing higher-order facility, are used in functional programming, logic programming, and theorem proving as a model that contains the notion of λ-calculus. Properties of HRSs such as termination and confluence have been investigated[7),8),11),14)–16]. On the other hand, there have been several studies on reduction strategies of TRSs, which are related lazy evaluation or strict analysis of programs. Huet and Lévy presented the following theorem on the optimal normalizing strategy of an orthogonal TRS[6]. They stated that a reducible term having a normal form contains at least one needed redex to be reduced in every reduction sequence to a normal form. They also showed the normalization property, whereby the normal form of a given term can always be obtained by repeated reduction of the needed redexes. Middeldorp generalized this result for head-needed reduction, which computes the head normal forms of terms[12].

We discuss strong sequential[6] and NV-sequential classes[18] having the following three properties, which are mandatory for practical use: (1) the strategy reducing a head-needed redex is head normalizing (2) whether a redex is head-needed is decidable, and (3) whether an HRS belongs to the class is decidable.

Arguing the head normalization property requires the concept of descendants of redex in a given reduction sequence. Oostrom showed a definition of a descendant of a Pattern Rewriting System[16], but the definition was abstract. The main difficulty for (1) is caused by the β-reductions induced from the higher-order reductions. Since β-reduction changes the structure of the higher-order term, the definition of a descendant of an HRS becomes complicated. Origin tracking[3] has made it possible to represent descendants for HRSs, but this requires rather complicated steps. In the present paper, we introduce a function, PV, to follow occurrence moves caused by β-reduction sequences. More precisely, given a term t, a substitution σ, a variable F, and a position p in $F\sigma$, PV computes positions in a β-normal form of $t\sigma$ corresponding to the position p. The PV function allows us to treat a β-reduction sequence rather easily. That is, we can use a concrete procedure to calculate descendants of higher-order reductions by using PV. We also define developments of HRS, corresponding to parallel reduction of TRS, and give a concrete proof of the diamond property for the developments. Based on the results of a previous study[10], we also show properties (2) and (3) using tree automata techniques, a ground tree transducer (GTT), and recognizability of redexes.

Oostrom shows the normalizing property of outer-most fair reduction[17]. Although, its result strongly relates to the normalizing property of needed reduction, it is difficult to show the head normalizing property of head-needed reduction by Oostrom's result.

Contributions of the present paper are following.

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- We propose a variant of position system that ignores λ-position. The system has a benefit that position movement caused by substitution into patterns can be treated first-order case.
- We also introduced a recursive definition of patterns. This enables us to prove properties related patterns formally.
- We give a simple definition of descendant relation by using function PV that characterizes position movement caused by β-reduction sequences directly. Unless using our position system and PV, we must compute reverse of β-reductions, replacement by rewrite rule, and β-reductions in sequence for descendants.
- We show diamond property for HRSs, in which redexes that must be reduced are presented explicitly in each arrow in the diagram of the property.
- We defined top-down decomposition of development sequences and define an order on them. This order is useful for various proofs dealing with development sequence.

Many of the results presented in the present paper were first reported in Reference [9]. Since a number of proofs in FLOPS2002 turned out to be incomplete, we fixed bugs of proofs and redesigned the order of development. Moreover, we clarified the decidable classes.

2. Preliminaries

Let S be a set of basic types. The set τ_s of types is generated from S by the function space constructor $\rightarrow$ as follows:

$\tau_s \supseteq S$

$\tau_s \supseteq \{\alpha \rightarrow \alpha' | \alpha, \alpha' \in \tau_s\}$

Let $\mathcal{X}_\alpha$ be a set of variables of type α, and let $\mathcal{F}_\alpha$ be a set of function symbols of type α. The set of all variables is denoted as $\mathcal{X} = \bigcup_{\alpha \in \tau_s} \mathcal{X}_\alpha$, and the set of all function symbols is denoted as $\mathcal{F} = \bigcup_{\alpha \in \tau_s} \mathcal{F}_\alpha$. *Simply typed λ-terms* are defined by the following inference rules:

$$\frac{x \in \mathcal{X}_\alpha}{x : \alpha} \quad \frac{f \in \mathcal{F}_\alpha}{f : \alpha} \quad \frac{s : \alpha \rightarrow \alpha' \quad t : \alpha}{(s\ t) : \alpha'} \quad \frac{x : \alpha \quad s : \alpha'}{(\lambda x.s) : \alpha \rightarrow \alpha'}$$

If $t : \alpha$ is inferred from the rules, then t is a simply typed λ-term of type α. The set of all simply typed λ-terms is denoted as $\mathcal{T}$. A simply typed λ-term is called a *higher-order term* or a *term*. We use the concepts of bound variables and free variables. The sets of bound and free variables occurring in a term t are denoted as $BV(t)$ and $FV(t)$, respectively. The set $FV(t) \cup BV(t)$ is denoted as $Var(t)$. A higher-order term without free variables is called a *ground term*. If a term s is generated by renaming bound variables in a term t, then s and t are equivalent and are denoted as $s \equiv t$. We use F, G, X, Y, and Z for free variables, and x, y, and z for bound variables unless it is known to be free or bound from other conditions. We sometimes write $\vec{x}$ for a sequence $x_1 x_2 \cdots x_m$ ($m \geq 0$). We also use c, d, f, g, and h for function symbols and a for a variable or a function symbol.

β-reduction is the operation that replaces $(\lambda x.s)t$ in a term by $s\{x \mapsto t\}$, where $(\lambda x.s)t$ is called a *β-redex*. Let s be a term of type $\alpha \rightarrow \alpha'$, and let $x \notin Var(s)$ be a variable of type α. Then, *η-expansion* is the operation that replaces s in a term by $\lambda x.(sx)$ if the term produces no new β-redex. A term is said to be *η-long*, if the term is in normal form with respect to η-expansion. In addition, a term is said to be *normalized* if the term is in $\beta\eta$-long normal form. A normalized term of t is denoted as $t\!\downarrow$. Each higher-order term has a unique normalized term [1].

A mapping $\sigma : \mathcal{X} \mapsto \mathcal{T}$ from variables to higher-order terms is called a *substitution* if $\sigma(X)$ has the same type of X and the domain $Dom(\sigma) = \{X \mid X \not\equiv \sigma(X)\}$ is finite. If $Dom(\sigma) = \{X_1, \ldots, X_n\}$ and $\sigma(X_i) \equiv t_i$, we also write the mapping as $\sigma = \{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$. Let W be a set of variables, and let σ be a substitution. We write $\sigma|_W$ for the substitution obtained by restricting the domain of σ to $Dom(\sigma) \cap W$ and write $\sigma|_{\overline{W}}$ for that obtained by restricting the domain of σ to$Dom(\sigma) \cap (\mathcal{X} - W)$. For a substitution σ, the set of free variables in the range of σ is defined by $VRan(\sigma) = \bigcup_{X \in Dom(\sigma)} FV(\sigma(X))$.

A substitution $\sigma = \{X_1 \mapsto t_1, \ldots, X_n \mapsto t_n\}$ is extended to a mapping $\tilde{\sigma}$ from higher-order terms to higher-order terms as follows:

$\tilde{\sigma}(t) = ((\lambda X_1 \cdots X_n.t)t_1 \cdots t_n)\!\downarrow_\beta$

Generally, when we extend a substitution σ to $\tilde{\sigma}$, we need the condition whereby the domain and range of σ do not contain any bound variables of a term to which the substitution $\tilde{\sigma}$ is applied. Here, note that when we adopt the above definition of $\tilde{\sigma}$obtained using β-reduction, we need not mention the condition explicitly. The above condition can always be satisfied by appropriately renaming the bound variables.

In the following, we will write simply σ for $\tilde{\sigma}$ and $t\sigma$ for $\sigma(t)$. A substitution σ is said to be normalized if $\sigma(X)$ is normalized for all $X \in Dom(\sigma)$.

Each normalized term can be represented by the form $\lambda x_1 \cdots x_m.(\cdots(at_1)\cdots t_n)$, where $m, n \geq 0$, $a \in \mathcal{F} \cup \mathcal{X}$, and$(\cdots(at_1)\cdots t_n)$ is a basic type. In the present paper, we denote the term t by $\lambda x_1 \cdots x_m.a(t_1, \ldots, t_n)$. The *top symbol* of t is defined as $top(t) \equiv a$.

We define the notion of positions in normalized terms based on the form of $\lambda x_1 \cdots x_m.a(t_1, \ldots, t_n)$. In order to simplify the definition of descendants given in the following section, we ignore lambda bindings in assigning positions to terms. A *position* of a normalized term is a sequence of natural numbers. The set of positions in $t \equiv \lambda\vec{x}.a(t_1, \ldots, t_n)$ is defined by $Pos(t) = \{\varepsilon\} \cup \{ip \mid 1 \leq i \leq n, p \in Pos(t_i)\}$. Let p and r be positions. We write $p \preceq r$ if $pq = r$ for some position q. Moreover, if $q \neq \varepsilon$, that is, if $p \neq r$, we write $p \prec r$. If $p \not\preceq r$ and $p \not\succeq r$, we write $p|r$. The subterm $t|_p$ of t at p is defined as follows:

$$(\lambda\vec{x}.a(t_1, \ldots, t_n))|_p \equiv \begin{cases} a(t_1, \ldots, t_n) & \text{if } p = \varepsilon \\ t_i|_q & \text{if } p = iq \end{cases}$$

$Pos_{FV}(t)$ indicates the set of all positions $p \in Pos(t)$ such that $top(t|_p)$ is a free variable in a normalized term t. $t[u]_p$ represents the term obtained by replacing $t|_p$ in a normalized term t by normalized term u having the same basic type as $t|_p$. This is defined as follows:

$$(\lambda\vec{x}.a(t_1, \ldots, t_n))[u]_p \equiv \begin{cases} \lambda\vec{x}.u & \text{if } p = \varepsilon \\ \lambda\vec{x}.a(\ldots, t_i[u]_q, \ldots) & \text{if } p = iq \end{cases}$$

Let t be a normalized term such that $top(t) \in \mathcal{F}$, and Let $u{\downarrow_\eta}$ denote the η-normal form of u [13]. Here, t is said to be a *pattern* if $u_1{\downarrow_\eta}, \ldots, u_n{\downarrow_\eta}$ are different bound variables for the arguments u_i of each free variable F in $t \equiv C[F(u_1, \ldots, u_n)]$. Moreover, t is said to be fully-extended if $u_1{\downarrow_\eta} \cdots u_n{\downarrow_\eta}$ is a sequence of all bound variables in the scope of $C[\,]$. The recursive definition of patterns is based on the concept of the B-pattern. Let B be a set of variables. Then, the set of B-patterns is defined as follows:

(1) $F(t_1, \ldots, t_n)$ is a B-pattern if $F \notin B$ and $t_1, \ldots, t_n$ are η-long normal forms of pairwise distinct variables in B,

(2) $a(t_1, \ldots, t_n)$ is a B-pattern if $a \in \mathcal{F} \cup B$ and $t_1, \ldots, t_n$ are B-patterns,

(3) $\lambda x_1 \cdots x_n.t$ is a B-pattern if t is a $(B \cup \{x_1, \ldots, x_n\})$-pattern.

Patterns are defined using the concept of B-patterns as follows: t is a pattern if and only if t is a $\emptyset$-pattern and $top(t) \in \mathcal{F}$. Furthermore, a pattern t is fully-extended, if t satisfies (1') rather than (1), where

(1') $F(t_1, \ldots, t_n)$ is a B-pattern if $F \notin B$, $\{t_1, \ldots, t_n\} = \{x{\downarrow} \mid x \in B\}$, and $t_i \not\equiv t_j$ for $i \neq j$.

Let α be a basic type, let $l : \alpha$ be a pattern, and let $r : \alpha$ be a normalized term such that $FV(l) \supseteq FV(r)$. Then, $l \blacktriangleright r : \alpha$ is called a higher-order rewrite rule of type α. A *higher-order rewrite system* (HRS) is a set of higher-order rewrite rules. Let $\mathcal{R}$ be an HRS, let $l \blacktriangleright r$ be a rewrite rule of $\mathcal{R}$, and let σ be a substitution. Then, $l\sigma{\downarrow}$ is said to be a *redex*. If p is a position such that $s \equiv s[l\sigma\downarrow]_p$ and $t \equiv s[r\sigma\downarrow]_p$, then s can be reduced to t, which is denoted as $s \xrightarrow{p}_R t$, or simply $s \xrightarrow{p} t$, $s \to_R t$, or $s \to t$. For the case in which $p \neq \varepsilon$, the reduction of s to t is denoted as $s \xrightarrow{\succ\varepsilon} t$. Since all rewrite rules are of basic type, t is normalized if s is so [15].

The reflexive transitive closure of the reduction relation $\to$ is denoted as $\to^*$. If there exists an infinite reduction sequence $t \equiv t_0 \to t_1 \to \cdots$ from t, t *is said to have an infinite reduction sequence*. If there exists no term that has an infinite reduction sequence, $\to$ is said to be *terminating*. If $\to_\mathcal{R}$ is terminating, HRS is also said to be $\mathcal{R}$ terminating. We sometimes refer to a reduction sequence $A : t_0 \to t_1 \to \cdots \to t_n$ by the label A. We sometimes denote the i-th reduction $t_{i-1} \to t_i$ as A_i. A is also denoted as $A_1; A_2; \cdots; A_n$, where $A; B$ denotes the concatenation of sequences A and B.

Let $BV_p(t)$ denote the set of variables that appears as lambda abstractions in the path from the position ε to p in normalized term t. $BV_p(t)$ is defined as follows:

$$BV_p(\lambda x_1 \cdots x_m.a(t_1, \ldots, t_n)) \equiv \begin{cases} \{x_1, \ldots, x_m\} & \text{if } p = \varepsilon \\ \{x_1, \ldots, x_m\} \cup BV_q(t_i) & \text{if } p = iq \end{cases}$$

Let $l \blacktriangleright r$ and $l' \blacktriangleright r'$ be rewrite rules. If there exist substitutions σ and σ' and a position $p \notin Pos_{FV}(l')$ such that $l\sigma\downarrow \equiv l'|_p(\sigma'|_{\overline{BV_p(l')}})\downarrow$, then these rewrite rules are said to *overlap* ⋆1. If HRS $\mathcal{R}$ has overlapping rules, $\mathcal{R}$ is said to be overlapping. When $\mathcal{R}$ is not overlapping and every rule of $\mathcal{R}$ is left-linear, $\mathcal{R}$ is said to be *orthogonal*.

3. Head Normalizing Strategy

3.1 Descendant

Considering a reduction $s \to t$, the term t is obtained by replacing a redex in s by a term. Since the other redexes in s may appear in different positions in t, we must take note of the redex positions in order to discuss the needed redex. Thus, the concept of descendants was proposed [5),6)]. In the sequel, we extend the definition of descendants of TRSs to that of HRSs.

In TRSs, it is easy to track descendants of redexes. However, this is not easy in HRSs because the positions of redexes move considerably by β-reductions taken in a reduction.

Example 1 Consider the following HRS $\mathcal{R}_1$,

$$\mathcal{R}_1 = \begin{cases} apply(\lambda x.F(x), X) \blacktriangleright F(X) \\ \qquad\qquad c \blacktriangleright d, \end{cases}$$

and a reduction $A_1 : s \equiv apply(\lambda x.f(g(x), x), c) \to f(g(c), c) \equiv t$. Descendants of a redex c that occurs at position 2 of s are positions 2 and 11 of t. As shown in **Fig. 1**, there is no descendant of redex position ε because it is reduced.

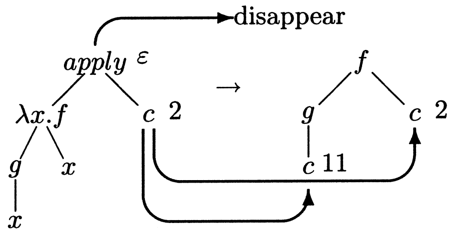

Fig. 1 Descendants.

In order to follow the positions of redexes correctly, we present mutually recursive functions PV and PT, each of which returns a set of positions. The function PV that takes a term t, a substitution σ, a variable F, and a position p as arguments computes the set of positions in $t\sigma\downarrow$ that originate from $(F\sigma)|_p$. The function PT that takes a term t, a substitution σ, and a position p as arguments computes the set of positions in $t\sigma\downarrow$ that originate from $t|_p$. In Example 1, we have $PV(F(X), \{F \mapsto \lambda x.f(g(x), x), X \mapsto a\}, X, \varepsilon) = \{11, 2\}$.

Definition 1 (PV) Let t be a normalized term, let σ be a normalized substitution, and let F be a variable. The function PV is defined as follows for a position $p \in Pos(F\sigma)$.

$PV(F, \sigma, F, p) = \{p\}$ (PV1)

$PV(a(t_1, \ldots, t_n), \sigma, F, p) = \bigcup_i \{iq \mid q \in PV(t_i, \sigma, F, p)\}$
  if $n > 0$, and $a \in \mathcal{F} \cup \overline{Dom(\sigma)}$ (PV2)

$PV(\lambda x_1 \cdots x_n.t', \sigma, F, p) = PV(t', \sigma|_{\overline{\{x_1,\ldots,x_n\}}}, F, p)$
  if $n > 0$, and $F \notin \{x_1, \ldots, x_n\}$ (PV3)

$PV(G(t_1, \ldots, t_n), \sigma, F, p) = \bigcup_i PV(t', \sigma', y_i, PV(t_i, \sigma, F, p))$
  if $n > 0, G \in Dom(\sigma), G \neq F$
  where $G\sigma \equiv \lambda y_1 \ldots y_n.t'$ and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$ (PV4)

$PV(F(t_1, \ldots, t_n), \sigma, F, p) = (\bigcup_i PV(t', \sigma', y_i, PV(t_i, \sigma, F, p))) \cup PT(t', \sigma', p)$
  if $n > 0, F \in Dom(\sigma)$
  where $F\sigma \equiv \lambda y_1 \ldots y_n.t'$ and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$ (PV5)

$PV(t, \sigma, F, p) = \emptyset$
  if $t \equiv G \neq F$ or $t \in \mathcal{F}$ (PV6)

Here, $PV(t, \sigma, F, P)$ denotes $\bigcup_{p \in P} PV(t, \sigma, F, p)$ for a set P of positions.

Definition 2 (PT) Let t be a normalized term, and let σ be a normalized substitution. The function PT is defined as follows for a position $p \in Pos(t)$.

For $p = \varepsilon$,

$PT(t, \sigma, p) = \{\varepsilon\}$ (PT1)

For $p \neq \varepsilon$, (let $p = ip'$)

$PT(a(t_1, \ldots, t_n), \sigma, p) = \{iq \mid q \in PT(t_i, \sigma, p')\}$
  if $t \equiv, n > 0$, and $a \in \mathcal{F} \cup \overline{Dom(\sigma)}$ (PT2)

$PT(\lambda x_1 \cdots x_n.t', \sigma, p) = PT(t', \sigma|_{\overline{\{x_1,\ldots,x_n\}}}, p)$
  if $n > 0$ (PT3)

⋆1 The original definition of overlapping [14)] is formal but complicated because the concept of the lifter is used to prohibit the substitution to free variables in a subterm that is bound in the original term.

$PT(G(t_1,\ldots,t_n),\sigma,p) = PV(t',\sigma',y_i,PT(t_i,\sigma,p'))$
if $n>0, G \in Dom(\sigma)$
where $G\sigma \equiv \lambda y_1 \ldots y_n.t'$ and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow,\ldots,y_n \mapsto t_n\sigma\downarrow\}$. (PT4)

Here, we sometimes write $PT(t,\sigma,P)$ for $\bigcup_{p\in P} PT(t,\sigma,p)$ where P is a set of positions.

Example 2 The following are examples of PV and PT. Let $\sigma_1 = \{F \mapsto \lambda y.f(y)\}$ and $\sigma_2 = \{y \mapsto g(\lambda x.h(f(x)))\}$.

(1) For any substitution σ, $PT(f(y),\sigma,\varepsilon) = \{\varepsilon\}$ by (PT1).
(2) For any substitution σ, $PV(y,\sigma,y,11)\} = \{11\}$ by (PV1).
(3) For any substitution σ, $PV(f(y),\sigma,y,11) = \{111\}$ by (PV2) and (2).
(4) For any substitution σ, $PV(f(y),\sigma,y,\emptyset) = \bigcup_{p\in\emptyset} PV(f(y),\sigma,y,p) = \emptyset$.
(5) $PV(x,\sigma_1,F,\varepsilon) = \emptyset$ by (PV6).
(6) $PV(F(x),\sigma_1,F,\varepsilon) = \{\varepsilon\}$ by (PV5), (5), (4), and (1).
(7) $PV(h(F(x)),\sigma_1,F,\varepsilon) = \{1\}$ by (PV2) and (6).
(8) $PV(\lambda x.h(F(x)),\sigma_1,F,\varepsilon) = \{1\}$ by (PV3) and (7).
(9) $PV(g(\lambda x.h(F(x))),\sigma_1,F,\varepsilon) = \{11\}$ by (PV2) and (8).
(10) $PV(f(y),\sigma_2,y,11) = \{111\}$ by (PV2) and (2).
(11) $PV(F(g(\lambda x.h(F(x)))),\sigma_1,F,\varepsilon) = \{111,\varepsilon\}$ by (PV5), (10), (9), and (1).

Let us demonstrate that PV and PT are well-defined. For this purpose, we introduce a well-founded order $>_{\rhd\beta}$ on pairs of a term and a substitution:

$\langle s,\theta\rangle >_{\rhd\beta} \langle t,\sigma\rangle \Leftrightarrow s\theta \to_\beta^+ t\sigma$ or $s\theta \rhd t\sigma$,

where $\rhd$ is the proper subterm relation of $\trianglerighteq$ defined as follows:

$$s \trianglerighteq t \Leftrightarrow \begin{cases} s \equiv t, \\ s \equiv \lambda x_1 \cdots x_n.s' \wedge s' \trianglerighteq t, \text{ or} \\ s \equiv s_1\ s_2 \wedge \exists i\ s_i \trianglerighteq t. \end{cases}$$

In the remainder of the present paper, the well-founded order $>_{\rhd\beta}$ will play an important role in proving claims in the form of $\forall t, \forall\sigma\ P(t,\sigma)$. These proofs will use Noetherian induction over the set of pairs $\langle t,\sigma\rangle$ with the order $>_{\rhd\beta}$.

Lemma 1 *PV and PT are well-defined.*

Proof. First, we show the termination of the recursive calls in the definitions of PV and PT by induction on $\langle t,\sigma\rangle$ with $>_{\rhd\beta}$. This is trivial for the cases of Definition 1 and 2, except for (PV4), (PV5), and (PT4). Consider the case of (PV4), where we have two recursive calls of PV. Let $t \equiv G(t_1,\ldots,t_n)$, $G\sigma \equiv \lambda y_1 \cdots y_n.t'$, and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow,\ldots,y_n \mapsto t_n\sigma\downarrow\}$. Since $t\sigma \equiv (G(t_1,\ldots,t_n))\sigma \equiv (\lambda y_1 \cdots y_n.t')(t_1\sigma,\ldots,t_n\sigma) \to_\beta t'\sigma'$, we have $\langle t,\sigma\rangle >_{\rhd\beta} \langle t',\sigma'\rangle$. We also have $t\sigma \equiv (\lambda y_1 \cdots y_n.t')(t_1\sigma,\ldots,t_n\sigma) \rhd t_i\sigma$. Thus, we know the termination of computing PV for the case of (PV4). The proofs for the cases of (PV5) and (PT4) can be performed in a manner similar to that described above.

Next, we show the following two claims:

$PV(t,\sigma,F,p) \subseteq Pos(t\sigma\downarrow)$ for $p \in Pos(F\sigma)$
$PT(t,\sigma,p) \subseteq Pos(t\sigma\downarrow)$ for $p \in Pos(t)$.

This can be shown by simultaneous induction on the well-founded order $>_{\rhd\beta}$ over the pairs $\langle t,\sigma\rangle$. Here, we give the proof only for the case (PV4). Let $p \in Pos(F\sigma)$, $t \equiv G(t_1,\ldots,t_n)$, $G\sigma \equiv \lambda y_1 \cdots y_n.t'$, and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow,\ldots,y_n \mapsto t_n\sigma\downarrow\}$. Since $\langle t,\sigma\rangle >_{\rhd\beta} \langle t_i,\sigma\rangle$, we have $PV(t_i,\sigma,F,p) \subseteq Pos(t_i\sigma\downarrow)$ by induction. Let $q \in PV(t_i,\sigma,F,p)$. Then, $PV(t',\sigma',y_i,q) \subseteq Pos(t'\sigma'\downarrow)$ follows from induction, because $\langle t,\sigma\rangle >_{\rhd\beta} \langle t',\sigma'\rangle$. The claim follows from $t\sigma\downarrow \equiv t'\sigma'\downarrow$. The claims for the cases (PV5) and (PT4) can be shown in a similar manner. □

Thanks to the position system that ignores λ-positions, we have the following lemma on PV.

Lemma 2 *If l is a pattern, then $PV(l,\sigma,F,p) = \{p'p \mid top(l|_{p'}) = F\}$ for every $F \in FV(l)$.* □

The proof of Lemma 2 is found in Appendix A.1.

Next, we present a definition of descendants.

Definition 3 (descendants of HRSs) Let $A : s[l\sigma\downarrow]_u \to_{l \blacktriangleright r} s[r\sigma\downarrow]_u$ be a reduction with a rewrite rule $l \blacktriangleright r \in \mathcal{R}$, a substitution σ, a term s, and a position u in s. The set of descendants of a position v in $Pos(s[l\sigma\downarrow]_u)$ by A is then defined as follows:

$$v\backslash A = \begin{cases} \{v\} & \text{if } v \mid u \text{ or } v \prec u \\ \{up_3 \mid p_3 \in PV(r,\sigma,F,p_2')\} & \\ & \text{if } v = up' \text{ and } p' \in PV(l,\sigma,F,p_2) \\ & \text{for some } F \in FV(l) \text{ and } p_2 \in Pos(F\sigma) \\ \emptyset & \text{otherwise.} \end{cases}$$

This definition of descendants is rather general, because we have not considered that l is a pattern. Since the present paper assumes that l is a pattern, this definition is simplified as follows by Lemma 2:

$$v\backslash A = \begin{cases} \{v\} & \text{if } v \mid u \text{ or } v \prec u \\ \{up_3 \mid p_3 \in PV(r, \sigma, top(l|_{p_1}), p_2)\} & \\ \quad \text{if } v = up_1p_2 \text{ and } p_1 \in Pos_{FV}(l) & \\ \emptyset & \text{otherwise.} \end{cases}$$

For a set D of positions, $D\backslash A$ denotes the set $\bigcup_{v \in D} v\backslash A$. For a reduction sequence $B : s \to^* t$, $D\backslash B$ is naturally defined.

Example 3 Consider the HRS $\mathcal{R}_1$ and the reduction sequence A_1 in Example 1. The descendants of a redex position 2 in s by the reduction A_1 are as follows:

$2\backslash A_1 = PV(F(X), \sigma, X, \varepsilon) = \{11, 2\}$

where $\sigma = \{F \mapsto \lambda x.f(g(x), x), X \mapsto c\}$.

Example 4 Consider the following HRS $\mathcal{R}_2$, a substitution σ, and a reduction sequence A:

$$\begin{array}{lcl} \mathcal{R}_2 & = & \{f(g(\lambda x.F(x))) \blacktriangleright F(g(\lambda x.h(F(x))))\}, \\ \sigma & = & \{F \mapsto \lambda y.f(y)\}, \\ A & : & f(g(\lambda x.f(x))) \equiv f(g(\lambda x.F(x)))\sigma{\downarrow} \\ & & \to F(g(\lambda x.h(F(x))))\sigma{\downarrow} \equiv f(g(\lambda x.h(f(x)))). \end{array}$$

The descendants of position 11 by the reduction sequence A are $11\backslash A = \{111, \varepsilon\}$ because $PV(F(g(\lambda x.h(F(x)))), \sigma, F, \varepsilon) = \{111, \varepsilon\}$ from Example 2.

In the following, we are only interested in descendants of redex positions. For convenience, we identify redex positions with redexes. We show the property whereby descendants of a redex are redexes.

Theorem 1 (redex preservation of descendants) Let $\mathcal{R}$ be an orthogonal HRS, and let $A : s \xrightarrow{u} t$ be a reduction. If $s|_v$ is a redex, then $t|_p$ are also redexes for all $p \in v\backslash A$. □

Note that if $u \prec v$ then $t|_p$ is an instance of $s|_v$. On the other hand, $t|_p = s|_v$ for TRSs. The proof of Theorem 1 is found in Appendix A.2.

3.2 Development and Its Properties

Middeldorp [12] discussed the head normalization of TRS using the concept of parallel reduction $\mathrel{+\!\!\!\!\!\!\rightarrow}$. He used the property whereby descendants of redexes on parallel positions are redexes on parallel positions in order to show that the head-needed strategy is normalizing. However, the property does not hold in HRSs because of β-reduction. Thus, we cannot discuss the head normalization of HRS by directly following the discussions of Middeldorp. Thus, we introduce the concept of *development* of HRS.

Now, we formalize the development with annotated redex positions and show the diamond property. The development $\mathrel{\circ\!\!\!\rightarrow}^D$ of normalized terms is defined by the following inference rules:

$$\frac{A_i : s_i \mathrel{\circ\!\!\!\rightarrow}^{D_i} t_i \ (i=1,\ldots,n)}{a(s_1,\ldots,s_n) \mathrel{\circ\!\!\!\rightarrow}^{D} a(t_1,\ldots,t_n)} \quad D = \bigcup_i \{ip \mid p \in D_i\} \qquad (A)$$

$$\frac{A' : s' \mathrel{\circ\!\!\!\rightarrow}^{D} t'}{\lambda x_1 \cdots x_n.s' \mathrel{\circ\!\!\!\rightarrow}^{D} \lambda x_1 \cdots x_n.t'} \qquad (L)$$

$$\frac{A_i : s_i \mathrel{\circ\!\!\!\rightarrow}^{D_i} t_i \ (i=1,\ldots,n) \quad f(s_1,\ldots,s_n) \equiv l\theta'{\downarrow} \quad f(t_1,\ldots,t_n) \equiv l\theta{\downarrow} \quad (l \blacktriangleright r) \in R}{f(s_1,\ldots,s_n) \mathrel{\circ\!\!\!\rightarrow}^{D} r\theta{\downarrow} \quad D = \{\varepsilon\} \cup \bigcup_i \{ip \mid p \in D_i\}} \qquad (R)$$

where D is a set of redex positions. Sometimes it is convenient to use the following (R') for (R).

$$\frac{A'' : l\theta'{\downarrow} \mathrel{\circ\!\!\!\rightarrow}^{D'} l\theta{\downarrow} \quad (l \blacktriangleright r) \in R}{l\theta'{\downarrow} \mathrel{\circ\!\!\!\rightarrow}^{D} r\theta{\downarrow}} \quad \varepsilon \notin D' \wedge D = D' \cup \{\varepsilon\} \qquad (R')$$

The development $\mathrel{\circ\!\!\!\rightarrow}^D$ is also simply denoted as $\mathrel{\circ\!\!\!\rightarrow}$.

The descendants of developments are defined as follows.

Definition 4 (descendants of developments) Let p be a position in s, and let $A : s \mathrel{\circ\!\!\!\rightarrow}^D t$ be a development. Then, the set of descendants of p by A is defined as follows:

$$p\backslash A = \begin{cases} \{\varepsilon\} & \text{in case of } (A) \text{ and } p = \varepsilon \\ \{ip'' \mid p'' \in p'\backslash A_i\} & \text{in case of } (A) \text{ and } p = ip' \\ p\backslash A' & \text{in case of } (L) \\ (p\backslash A'')\backslash A''' & \text{in case of } (R') \text{ where } A''' : l\theta{\downarrow} \to r\theta{\downarrow} \end{cases}$$

where A_i, A', and A'' are reductions shown in the definition of developments.

Let A and B be developments such that $A : s \mathrel{\circ\!\!\!\rightarrow}^D t_1$ and $B : s \mathrel{\circ\!\!\!\rightarrow} t_2$. The development starting from t_2, in which all redexes at positions $D\backslash B = \{p\backslash B \mid p \in D\}$ are contracted, is denoted as $A\backslash B$. Let A and B be development sequences such that $A : s_1 \twoheadrightarrow^* s_2$ and $B : t_1 \twoheadrightarrow^* t_2$. Here, A and B are said to be *permutation equivalent*, which is denoted as $A \simeq B$, if $s_1 \equiv t_1$, $s_2 \equiv t_2$ and $p\backslash A = p\backslash B$ for every redex position p in s_1. The following lemma corresponds to Lemma 2.4 in Reference [6), which shows the diamond property [15),16). Raamsdonk also reported the diamond property of development [19). However, our lemma shows not only the existence of reduction sequences, but also that the descendants of a redex in s with respect to two development sequences from s to u are the same.

Lemma 3 (diamond property) Let $\mathcal{R}$ be an orthogonal HRS. If A and B are developments starting from the same term, then $A;(B\backslash A) \simeq B;(A\backslash B)$. □

The proof of Lemma 3 is found in Appendix A.3.

3.3 Head-Needed Redex

We extend the notion of head normalization of TRSs to that of HRSs. The remainder of this section assumes the orthogonality of HRSs $\mathcal{R}$.

Definition 5 (head normal form) Let $\mathcal{R}$ be an HRS. A term that cannot be reduced to any redex is said to be in *head normal form*.

Lemma 4 (Reference 9)) Let t be in head normal form. If there exists a reduction sequence $s \xrightarrow{\succ\varepsilon} {}^*t$, s is in head normal form. □

Definition 6 (head-needed redex) A redex r in a term t is *head-needed* if a descendant of r is reduced in every reduction sequence from t to a head normal form.

Lemma 5 Let t be in non-head-normal form. Then, the pattern of the first redex, which appears in every reduction sequence from t to a redex, is unique.

Proof. Similar to the proof of Lemma 4.2 in Ref. 12), the lemma is proved by Theorem 1 and orthogonality. □

Proposition 1 Let $\mathcal{R}$ be fully extended. Assume that we rewrite a term s at position p by some rewrite rule and obtain the term t. If $t \equiv l\sigma\!\downarrow$ for a substitution σ and a fully-extended pattern l such that $p \notin Pos_{\mathcal{F}}(l)$, then there exists some substitution σ' such that $s \equiv l\sigma'\!\downarrow$.

Proof. We prove the more general claim.

Claim Let B be a set of variables, and let l be a fully extended linear B-pattern. If $s \xrightarrow{p} l\sigma\!\downarrow$, $top(l|_p) \notin B \cup \mathcal{F}$, and $(Dom(\sigma) \cup VRan(\sigma)) \cap B = \emptyset$, then there exists a substitution σ' such that $s \equiv l\sigma'\!\downarrow$ and $(Dom(\sigma') \cup VRan(\sigma')) \cap B = \emptyset$.

Our proposition is the claim for the case in which $B = \emptyset$. The claim can be proved by induction on the structure of l.

(1) Let $l \equiv F(x_1\!\downarrow, \ldots, x_n\!\downarrow)$, $B = \{x_1, \ldots, x_n\}$, and $F \notin B$. Consider the substitution $\sigma' = \{F \mapsto \lambda x_1 \cdots x_n.s\}$. Then, $l\sigma'\!\downarrow \equiv (\lambda x_1 \cdots x_n.s)(x_1\!\downarrow, \ldots, x_n\!\downarrow)\!\downarrow \equiv s$, i.e., $l\sigma'\!\downarrow \equiv s$.

(2) Let $l \equiv a(t_1, \ldots, t_n)$ for $a \in \mathcal{F} \cup B$ and $n > 0$. Since $(Dom(\sigma) \cup VRan(\sigma)) \cap B = \emptyset$, the rewriting $s \xrightarrow{p} l\sigma\!\downarrow$ means $s \equiv a(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n$, and $l\sigma\!\downarrow \equiv a(t_1\sigma\!\downarrow, \ldots, t_n\sigma\!\downarrow)$, where $s_j \xrightarrow{p'} t_j\sigma\!\downarrow$ for j such that $p = jp'$, and for any $i \neq j$ $t_i\sigma\!\downarrow \equiv s_i$. By induction, there exists a substitution σ'' such that $t_j\sigma''\!\downarrow \equiv s_j$ and $(Dom(\sigma'') \cup VRan(\sigma'')) \cap B = \emptyset$. Hence, $s \equiv a(s_1, \ldots, t_j\sigma''\!\downarrow, \ldots, s_n) \equiv a(t_1\sigma\!\downarrow, \ldots, t_j\sigma''\!\downarrow, \ldots, t_n\sigma\!\downarrow)$. Consider the substitution σ' such that $\sigma' = \{x \mapsto \sigma''(x) \mid x \in Var(t_j)\} \cup \{x \mapsto \sigma(x) \mid x \notin Var(t_j) \wedge x \in Var(t_i)$ for some $i \neq j\}$. From the linearity of l, σ' holds $(Dom(\sigma') \cup VRan(\sigma')) \cap B = \emptyset$ and $s \equiv a(t_1\sigma'\!\downarrow, \ldots, t_n\sigma'\!\downarrow) \equiv l\sigma'\!\downarrow$.

(3) Let $l \equiv \lambda x_1 \cdots x_n.t$, where t is a $(B \cup \{x_1, \ldots, x_n\})$-pattern. Let $s \equiv \lambda x_1 \cdots x_n.s'$, and let $\sigma'' = \sigma|_{\overline{\{x_1,\ldots,x_n\}}}$. Then, $(Dom(\sigma'') \cup VRan(\sigma'')) \cap (B \cup \{x_1, \ldots, x_n\}) = \emptyset$ and $s' \xrightarrow{p} t\sigma''\!\downarrow$. By induction hypothesis, there exists a substitution σ' such that $s' \equiv t\sigma'$ and $(Dom(\sigma') \cup VRan(\sigma')) \cap (B \cup \{x_1, \ldots, x_n\}) = \emptyset$. Thus, $s \equiv \lambda x_1 \cdots x_n.s' \equiv \lambda x_1 \cdots x_n.(t\sigma'\!\downarrow) \equiv l\sigma'\!\downarrow$. □

Theorem 2 Let $\mathcal{R}$ be a fully-extended orthogonal HRS. Every term that is not in head normal form contains a head-needed redex.

Proof. Similar to the proof of Theorem 4.3 in Ref. 12), the theorem is proved by Lemma 4, Lemma 5, and Proposition 1. □

We cannot remove the fully-extended condition from this theorem because of the following counterexample.

Counterexample 1 Consider the following orthogonal HRS:

$$\mathcal{R} = \begin{cases} f(\lambda x.z) & \blacktriangleright \; c \\ g(z) & \blacktriangleright \; c, \end{cases}$$

where f, g, and c are function symbols and z and x are variables. The term $f(\lambda x.g(g(x)))$ is not in head normal form and contains no head-needed redex. The first rule cannot be applied to the term because the free variable z does not match $g(g(x))$, which contains a bound variable x. The second rule can be applied to two redexes: $g(x)$ and $g(g(x))$. However, neither redex is head-needed. By a reduction sequence $f(\lambda x.g(g(x))) \to f(\lambda x.g(c)) \to c$, the redex $g(g(x))$ is not head-needed. By another reduction sequence $f(\lambda x.g(g(x))) \to f(\lambda x.c) \to c$, the redex $g(x)$ is not head-needed.

In the proof of Theorem 2, the properties shown in Proposition 1 are necessary, whereas in left-linear TRSs, the property holds trivially.

3.4 Top-down Decomposition of Development

To prove the main theorem of the present paper, we must introduce the cost of development. First, we define top-down decomposition.

Definition 7 (top-down decomposition) Let $A : s \mathrel{\circ\!\!\!\rightarrow} t$ be a development. If there exists a sequence of positions $p_1, p_2, \ldots, p_n$ such that $s \equiv s_0 \xrightarrow{p_1} s_1 \xrightarrow{p_2} s_2 \xrightarrow{p_3} \cdots \xrightarrow{p_n} s_n \equiv t$ and $p_i \not\succ p_j$ for any $i < j$ $(1 \leq i < j \leq n)$, then the rewrite sequence is called a *top-down decomposition* of the development A. If the length of a top-down decomposition is minimal in top-down decompositions of the descendant A, we call the decomposition a minimal top-down decomposition.

Here, we introduce the concept of the *top-down property* of a development, which is recursively defined as follows: Consider a development $s \mathrel{\circ\!\!\!\rightarrow} t$, and let $t \equiv \lambda x_1 \cdots x_n.a(t_1, \ldots, t_m)$, where $a \in \mathcal{F} \cup \mathcal{X}$ and $n \geq 0$. The development $s \mathrel{\circ\!\!\!\rightarrow} t$ has the top-down property, (1) if $s \mathrel{\circ\!\!\!\rightarrow}^{\emptyset} t$, i.e. $s \equiv t$, or (2) if, for some $k \geq 0$, there exists a unique set of positions $D(\varepsilon \notin D)$, and the term $u \equiv \lambda x_1 \cdots x_n.a(u_1, \ldots, u_m)$, then $s \xrightarrow{\varepsilon}{}^{k} u \mathrel{\circ\!\!\!\rightarrow}^{D} t$ and $u_i \mathrel{\circ\!\!\!\rightarrow} t_i$ has the top-down property for all $i = 1, \ldots, m$. We often write $\xrightarrow{\succ\varepsilon}\!\!\circ$ for $\mathrel{\circ\!\!\!\rightarrow}^{D}$ when $\varepsilon \notin D$.

Lemma 6 Any development has a minimal top-down decomposition, the length of which is uniquely determined. □

The proof of Lemma 6 is found in Appendix A.4.

From Lemma 6, we can define the *cost* of a development $A : s \mathrel{\circ\!\!\!\rightarrow} t$ by the length of the minimal top-down decomposition of A, which is denoted as $|s \mathrel{\circ\!\!\!\rightarrow} t|$.

3.5 Head Normalizing Strategy

Middeldorp introduced $\mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\nabla}$ and $\mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\Delta}$ in order to divide a parallel reduction $\mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{D}$ into two parts with respect to a given set B of parallel redex positions [12]. He used the property $\mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\nabla} \cdot \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\Delta} \subseteq \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\Delta} \cdot \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\nabla}$ to prove his main theorem. In this section, we prove the properties of development of an HRS corresponding to those of parallel reduction of a TRS. This allows us to follow Middeldorp's example.

Definition 8 (∇ and Δ) Let D be a set of positions, and let B be another set of positions. When the set D satisfies the condition $\forall p \in D, \exists q \in B, q \prec p$, we write the set D by $D_{\nabla B}$. In contrast, when $\forall p \in D, \forall q \in B, q \not\preceq p$, we write the set D by $D_{\Delta B}$.
We sometimes write D_{∇} for $D_{\nabla B}$ and ∇ for $D_{\nabla}$and D_{Δ} for $D_{\Delta B}$ and Δ for D_{Δ},when D and B are interpreted as trivial.

Here, we prove the following Lemma 7, which means $\mathrel{\circ\!\!\!\rightarrow}^{\nabla} \cdot \mathrel{\circ\!\!\!\rightarrow}^{\Delta} \subseteq \mathrel{\circ\!\!\!\rightarrow}^{\Delta} \cdot \mathrel{\circ\!\!\!\rightarrow}^{\nabla}$. This corresponds to $\mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\nabla} \cdot \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\Delta} \subseteq \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\Delta} \cdot \mathrel{+\!\!\!\!\!\!\parallel\!\!\rightarrow}^{\nabla}$ in Ref. 12).

Lemma 7 Let B be a set of redex positions of a term t, and let D and D' be sets of the redex positions that can be written by $D_{\nabla B}$ and $D'_{\Delta B}$, respectively. Let $A_1 : t \mathrel{\circ\!\!\!\rightarrow}^{D} t_1$ and $A_2 : t_1 \mathrel{\circ\!\!\!\rightarrow}^{D'} t_2$ be developments. Then, there exist developments $A_3 : t \mathrel{\circ\!\!\!\rightarrow}^{D'} t_3$ and $A_4 : t_3 \mathrel{\circ\!\!\!\rightarrow}^{D''} t_2$ such that $D'' = D \backslash A_3$ can be written as $(D \backslash A_3)_{\nabla(B \backslash A_3)}$ for some t_3. □

To prove Lemma 7, we must follow the moves of redexes, which complicates the proof. Thus, for the purpose of readability, we give he proof in Appendix A.5. Here, note that $|A_2| = |A_3|$ holds. In other words, the costs of developments A_2 and A_3 are equal in Lemma 7 because the reduced positions in A_1 are strictly below D' or are disjoint from D'.

Now we are at the position to show the main result of this paper. The proof proceeds in a similar way to that of the main theorem in Ref. 12).

Proposition 2 If a development $s \mathrel{\circ\!\!\!\rightarrow} t$ is divided into $s \mathrel{\circ\!\!\!\rightarrow}^{D} s'$ and $s' \mathrel{\circ\!\!\!\rightarrow}^{D'} t$, where D_{Δ} and D'_{∇}, then $|s \mathrel{\circ\!\!\!\rightarrow} t| \geq |s \mathrel{\circ\!\!\!\rightarrow} s'|$.

Proof. The sequence obtained by concatenating the decomposition of $s \mathrel{\circ\!\!\!\!\twoheadrightarrow}^{D} s'$ and the sequence of $s' \mathrel{\circ\!\!\!\!\twoheadrightarrow}^{D'} t$ is a decomposition of $s \mathrel{\circ\!\!\!\!\twoheadrightarrow} t$.Thus, the proposition holds. □

Definition 9 Let $A = A_1; A_2; \cdots; A_n$ and $B = B_1; B_2; \cdots; B_n$ be development sequences of length n. We write $A > B$ if there exists an $i \in \{1, \ldots, n\}$ such that $|A_i| > |B_i|$ and $|A_j| = |B_j|$ for every $i < j \leq n$. We also write $A \geq B$ if $A > B$ or $|A_j| = |B_j|$ for every $1 \leq j \leq n$.

Definition 10 Let A be a development sequence and B be a development starting from the same term. We write $B \perp A$ if any descendant of redexes reduced in B is not reduced in A.

The following two lemmas correspond to Lemma 5.4 and 5.5 in Ref. 12). These lemmas can be proved in the similar way to Ref. 12).

Lemma 8 Let $A : s \mathrel{\circ\!\!\!\!\twoheadrightarrow}^* s_n$ and $B : s \mathrel{\circ\!\!\!\!\twoheadrightarrow} t$ be such that $B \perp A$. If s_n is in head normal form then there exists a development sequence $C : t \mathrel{\circ\!\!\!\!\twoheadrightarrow}^* t_n$such that $A \geq C$ and t_n is in head normal form. □

This lemma is proved using Lemma 3, Lemma 7, and Proposition 2.

Lemma 9 Let $A : s \mathrel{\circ\!\!\!\!\twoheadrightarrow}^* s_n$ and $B : s \mathrel{\circ\!\!\!\!\twoheadrightarrow} t$, such that $B \not\perp A$. If s_n is in head normal form, then there exists a development sequence $C : t \mathrel{\circ\!\!\!\!\twoheadrightarrow}^* t_n$ such that $A > C$ and t_n is in head normal form. □

This lemma is proved using Lemma 3 and Lemma 8.

Theorem 3 Let $\mathcal{R}$ be an orthogonal HRS. Let t be a term that has a head normalizing reduction. There is no development sequence starting from t that contains infinitely many head-needed rewriting steps. □

Using Lemma 8 and Lemma 9, this theorem is proved in the same manner as Ref. 12).

From Theorem 2 and Theorem 3, the head-needed reduction is a head normalizing strategy in fully-extended orthogonal HRS. In other words, we obtain a head normal form by reducing head-needed redexes, if the head normal form exists.

4. Decidable Classes of Higher-Order Rewrite Systems

Since rewrite relations $\to^*_{\mathcal{R}}$ of HRSs are generally undecidable, in the same manner as TRSs, the neededness of a reduction is undecidable. Therefore, we give a sufficient condition with approximations of reductions by a manner similar to that described in Ref. 4). We have shown that $\to^*_{\mathcal{R}} [L]$ is recognizable by Theorem 14 of Ref. 10), which uses a Ground Tree Transducer (GTT) [2).]

4.1 Approximations

Let $\mathcal{R}$ be an HRS over the signature $\mathcal{F}$. Let $\bullet_\alpha$ be a fresh constant of basic type α. $\mathcal{R}$ is extended to an HRS over $\mathcal{F}_\bullet = \mathcal{F} \cup \{\bullet_\alpha \mid \alpha \in S\}$. We denote the set of all normal forms with respect to $\mathcal{R}$ over $\mathcal{T}(\mathcal{F}_\bullet, \emptyset)$ as $NF_{\mathcal{R}}$. Let $\mathcal{R}_\bullet$ be $\mathcal{R} \cup \{\bullet_\alpha \to \bullet_\alpha \mid \alpha \in S\}$. Thus, $NF_{\mathcal{R}_\bullet}$ is $NF_{\mathcal{R}} \cap \mathcal{T}(\mathcal{F}, \emptyset)$. The type of $\bullet$ will be omitted if it is explicit from the context.

Lemma 10 Let $\mathcal{R}$ be an HRS in which both sides are patterns and share no variables. The set $(\to^*_{\mathcal{R}})[NF_{\mathcal{R}_\bullet}]$ is recognizable.

Proof. This lemma follows from Theorem 14 of Ref. 10). □

Here, we define approximations on HRS $\mathcal{R}_s$ and $\mathcal{R}_{nv}$, which correspond to strong sequential rewriting and NV-sequential rewriting on a TRS [4)], respectively.

Definition 11 (approximation) Let $\mathcal{R}$ and $\mathcal{S}$ be HRSs over the same signature. If $\to^*_{\mathcal{R}} \subseteq \to^*_{\mathcal{S}}$ and $NF_{\mathcal{R}} = NF_{\mathcal{S}}$, $\mathcal{S}$ is said to approximate $\mathcal{R}$.

Definition 12 ($\mathcal{R}$-needed) Let $\mathcal{R}$ be an HRS over a signature $\mathcal{F}$. Let Δ be a redex of type α in $C[\Delta] \in \mathcal{T}(\mathcal{F}, \emptyset)$. Δ is $\mathcal{R}$-needed if and only if there is no $t \in NF_{\mathcal{R}_\bullet}$ such that $C[\bullet_\alpha] \to^*_{\mathcal{R}} t$

Lemma 11 Let $\mathcal{S}$ be an approximation of an HRS $\mathcal{R}$. Each $\mathcal{S}$-needed redex is $\mathcal{R}$-needed.

Proof. Each redex of $\mathcal{S}$ is also a redex of $\mathcal{R}$ from Definition 11. Each reduction relation of $\mathcal{R}$ is also a reduction relation of $\mathcal{S}$. Thus, if a redex is $\mathcal{S}$-needed, then it is also $\mathcal{R}$-needed. □

Definition 13 An approximation $\mathcal{R}_s$ is an HRS obtained from $\mathcal{R}$ by replacing the right-hand side of each rewrite rule by new free variables.

Definition 14 An approximation $\mathcal{R}_{nv}$ is an HRS obtained from $\mathcal{R}$ by replacing any subterms in the right-hand sides of rewrite rules in which the top is a free variable by new fully-extended free disjoint variables.

$\mathcal{R}_s$ and $\mathcal{R}_{nv}$ satisfy conditions of approximations in Definition 11. Both sides of the rewrite rules of $\mathcal{R}_s$ and $\mathcal{R}_{nv}$ are patterns because the right-hand side of $\mathcal{R}_s$ is a free variable and there are no nesting free variables in the right-hand side of $\mathcal{R}_{nv}$. If we can find an $\mathcal{R}_s$(or $\mathcal{R}_{nv}$)-needed redex, the redex is also $\mathcal{R}$-needed from Lemma 11. Therefore, we can obtain head normal form on $\mathcal{R}$ by rewriting $\mathcal{R}_s$ (or $\mathcal{R}_{nv}$)-needed redexes repeatedly.

Example 5 Approximations $\mathcal{R}_s$ and $\mathcal{R}_{nv}$ are as follows:

$$\mathcal{R} = \{map(\lambda x.F(x), cons(X,Y)) \rhd cons(F(X), map(\lambda x.F(x), Y))\},$$
$$\mathcal{R}_s = \{map(\lambda x.F(x), cons(X,Y)) \rhd Z\},$$
$$\mathcal{R}_{nv} = \{map(\lambda x.F(x), cons(X,Y)) \rhd cons(Z, map(\lambda x.G(x), W))\}.$$

4.2 Needed Reductions

Next, we discuss the properties of needed reductions on approximations of HRSs.

Definition 15 ($\mathcal{R}$-NEEDED) Let $\mathcal{R}$ be an HRS, and let $M_\bullet \in \mathcal{T}(\mathcal{F}_\bullet, \emptyset)$ be a set of all terms that contain exactly one occurrence of $\bullet$. If $C[\bullet] \in M_\bullet$ such that there is no $t \in NF_{\mathcal{R}_\bullet}$ with $C[\bullet] \to^*_{\mathcal{R}} t$, then the set of all terms that satisfies $C[\bullet]$ is said to be $\mathcal{R}$-NEEDED.

The following theorem for HRS holds in a manner similar to The TRS version of this theorem (Theorem 15 in Ref. 4)).

Theorem 4 Let $\mathcal{R}$ be an HRS. If $(\to^*_{\mathcal{R}})[NF_{\mathcal{R}_\bullet}]$ is recognizable, then $\mathcal{R}$-NEEDED is recognizable. □

Lemma 12 Let $\mathcal{R}$ be an HRS. Relations $\to^*_{\mathcal{R}_s}$ and $\to^*_{\mathcal{R}_{nv}}$ are recognizable.

Proof. For each approximation $\mathcal{S} \in \{\mathcal{R}_s, \mathcal{R}_{nv}\}$, there is a GTT that recognizes $\to_{\mathcal{S}}$. Thus, the relation $\to^*_{\mathcal{S}}$ is recognizable Theorem 14 of Ref. 10) □

The following theorem follows from Lemma 10, Lemma 12, and Theorem 4.

Theorem 5 Let $\mathcal{R}$ be a left-linear HRS over signature $\mathcal{F}$. For each approximation $\mathcal{S} \in \{\mathcal{R}_s, \mathcal{R}_{nv}\}$, whether a redex of a term in $\mathcal{T}(\mathcal{F}, \emptyset)$ is $\mathcal{S}$-needed is decidable. □

From this theorem, we can decide needed reductions of approximation of HRSs. Therefore, the theorem is a sufficient condition of the decision problem of HRSs.

For example, the following HRS $\mathcal{R}$ causes infinite reduction when using the inner-most strategy or the left-most outer-most strategy. However, needed reduction leads its result.

Example 6

$$\begin{array}{rrcl} \mathcal{R} = \{ & if\ (X,Y,True()) & \rhd & X, \\ & if\ (X,Y,False()) & \rhd & Y, \\ & isZero(Zero) & \rhd & True(), \\ & isZero(Succ(X)) & \rhd & False(), \\ & apply(\lambda x.F(x), X) & \rhd & F(X), \\ & fromn(X) & \rhd & cons(X, fromn(Succ(X)))\ \ \} \end{array}$$

$$\begin{array}{ll} & if(fromn(Succ(Zero)), cons(Succ(Zero), []), \\ & \qquad \underline{apply(\lambda x.isZero(x), Succ(Zero))}) \\ \to & if(fromn(Succ(Zero)), cons(Succ(Zero), []), \underline{isZero(Succ(Zero))}) \\ \to & \underline{if(fromn(Succ(Zero)), cons(Succ(Zero), []), False())} \\ \to & cons(Succ(Zero), []) \end{array}$$

In this example, a function *fromn* makes an infinite list in the inner-most strategy or the left-most outer-most strategy. However, the needed reduction strategy reduces the underlined parts, the functions *apply* and *is Zero* of which are not left-most, but rather outer-most, and obtains its result

The following are approximations $\mathcal{R}_s$ and $\mathcal{R}_{nv}$ of the example $\mathcal{R}$. Using a GTT made from the following rules, whether a redex of $\mathcal{R}$ is needed is decidable.

$$\begin{array}{rrcl} \mathcal{R}_s = \{ & if(X,Y,True()) & \rhd & Z, \\ & if(X,Y,False()) & \rhd & Z, \\ & isZero(Zero) & \rhd & X, \\ & isZero(Succ(X)) & \rhd & Y, \\ & apply(\lambda x.F(x), X) & \rhd & Y, \\ & fromn(X) & \rhd & Y\ \ \} \end{array}$$

$$\begin{array}{rrcl}\mathcal{R}_{nv} = \{ & \mathit{if}(X,Y,\mathit{True}()) & \triangleright & Z, \\ & \mathit{if}(X,Y,\mathit{False}()) & \triangleright & Z, \\ & \mathit{isZero}(\mathit{Zero}) & \triangleright & \mathit{True}(), \\ & \mathit{isZero}(\mathit{Succ}(X)) & \triangleright & \mathit{False}(), \\ & \mathit{apply}(\lambda x.F(x), X) & \triangleright & Y, \\ & \mathit{fromn}(X) & \triangleright & \mathit{cons}(Y, \mathit{fromn}(\mathit{Succ}(Z))) \}\end{array}$$

4.3 Head-Needed Reductions

In the remainder of the present paper, we generalize the above-mentioned research on needed reduction to relate to head-needed reduction. We can show the following theorem in the same manner as in the case of a TRS in Ref. 4).

Definition 16 Let $\mathcal{F}$ be a signature. Let $\mathcal{F}_\circ = \mathcal{F} \cup \{f_\circ \mid f \in \mathcal{F}\}$, where every $f_\circ$ has the same arity as f. Let $\mathcal{R}$ be an orthogonal HRS over the signature $\mathcal{F}$. Let $\Delta \in \mathcal{T}(\mathcal{F}, \emptyset)$ be a redex. We write t° for the term that is obtained from t by marking its head symbol. $\mathcal{R}_\circ$ denotes the HRS $\mathcal{R} \cup \{l^\circ \triangleright r \mid l \triangleright r \in \mathcal{R}\}$.
Let $\mathcal{R}$ and $\mathcal{S}$ be HRSs over the same signature $\mathcal{F}$. Redex Δ in $C[\Delta] \in \mathcal{T}(\mathcal{F}, \emptyset)$ is said to be $(\mathcal{R}, \mathcal{S})$-head-needed if there is no term $t \in \mathit{HNF}_{\mathcal{S}_\circ}$ such that $C[\Delta^\circ] \to^*_{\mathcal{R}} t$. The set of all terms $C[\Delta^\circ]$ such that there is no term $t \in \mathit{HNF}_{\mathcal{S}_\circ}$ with $C[\Delta^\circ] \to^*_{\mathcal{R}} t$ is denoted as $(\mathcal{R}, \mathcal{S})$-HEAD-NEEDED.

Theorem 6 Let $\mathcal{R}$ and $\mathcal{S}$ be HRSs over the same signature $\mathcal{F}$. If $(\to^*_{\mathcal{R}})[\mathit{HNF}_{\mathcal{S}_\circ}]$ is recognizable and $\mathcal{R}$ is left-linear, then $(\mathcal{R}, \mathcal{S})$- HEAD-NEEDED is recognizable.

Proof. Theorem 37 in Ref. 4), which holds on TRSs, also holds on HRSs. □

Lemma 13 Let $\mathcal{R}$ be a left-linear HRS. The set $(\to^*_{\mathcal{R}_\alpha})[\mathit{HNF}_{(\mathcal{R}_\beta)_\circ}]$ is recognizable for $\alpha, \beta \in \{s, nv\}$.

Proof. For $\beta \in \{s, nv\}$, the set of reducible terms on HRS $\mathcal{R}_\beta$ is recognizable from Ref. 10). Therefore, $[\mathit{HNF}_{(\mathcal{R}_\beta)_\circ}]$ is also recognizable. For $\alpha \in \{s, nv\}$, the relation $\to^*_{\mathcal{R}_\alpha}$ is recognizable by Lemma 12. Thus, $(\to^*_{\mathcal{R}_\alpha})[\mathit{HNF}_{(\mathcal{R}_\beta)_\circ}]$ is recognizable for $\alpha, \beta \in \{s, nv\}$, i.e., $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$-HEAD-NEEDED is recognizable. □

From Theorem 6 and Lemma 13, the head-neededness of a redex is decidable for s and nv approximations as follows.

Corollary 1 Let $\mathcal{R}$ be a left-linear HRS over a signature $\mathcal{F}$. Whether a redex in a term is $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$-head-needed for $\alpha, \beta \in \{s, nv\}$ is decidable.

4.4 Decidability of Membership Problem

The decidability of the membership problem on a TRS is discussed using an abstracted model[4]. Therefore, we can also discuss that on an HRS using the same method, i.e., the following theorems hold.

Definition 17 Let α and β be approximation mappings. The class of HRSs $\mathcal{R}$ such that each term in non-$\mathcal{R}_\beta$-head normal form has an $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$-head-needed redex is denoted as CBN-HNF$_{\alpha,\beta}$.

Theorem 7 Let $\mathcal{R}$ and $\mathcal{S}$ be HRSs such that $(\mathcal{R}, \mathcal{S})$-HEAD-NEEDED is recognizable. The set of terms that have an $(\mathcal{R}, \mathcal{S})$-head-needed redex is recognizable.

Proof. Theorem 41 in Ref. 4) holds on HRSs. □

Theorem 8 Let $\mathcal{R}$ be an HRS, and let α and β be an approximation mapping such that HNF$_{\mathcal{R}_\beta}$ and $(\mathcal{R}_\alpha, \mathcal{R}_\beta)$-HEAD-NEEDED are recognizable. Whether $\mathcal{R} \in$ CBN-HNF$_{\alpha,\beta}$ holds is decidable.

Proof. Theorem 42 in Ref. 4) holds on HRSs. □

5. Conclusions

We have introduced the function PV to follow the moves of an occurrence caused by β-reduction sequences and have given a concrete procedure to calculate the descendants for developments of an HRS using PV. The proposed position system identifies occurrences of $\lambda x.t$ and t in $C[\lambda x.t]$, which has an advantage in that the behavior of the movement of positions is the same as that in the first-order case in applying a substitution to a pattern. We believe that PV is a useful tool to prove a number of properties related to HRSs. This function has helped us to prove the permutation equivalence of the diamond property. In addition, we have shown that the head-needed reduction is a normalizing strategy for orthogonal HRSs. Thus, we can derive a normal form of a term by repeated reduction of head-needed redexes. Using a GTT and recognizability of redexes, we have shown that we can find head-needed redex and execute head-needed reduction. In addition, we have shown that whether the HRS belongs to a class of head-needed reduction for a given HRS is decidable.

Oostrom showed the (head-)normalizing property of outer-most fair reduction [17]⋆1. since the result on this reduction is strongly related to the (head-)normalizing property of (head-)needed reduction, one may think that the latter result are derived from the former or vice versa. However, these are difficult. Based on results for both head-needed reduction and outer-most fair reduction, the following relationship between head-needed reduction and outer-most fair reduction is obtained.

(1) *Outer-most fair reduction starting from a term having a head normal form is hyper head-needed reduction.*
Since outer-most fair reduction from a term is normalizing [17], it is also hyper head-needed reduction ⋆2.

(2) *Head-needed reduction starting from a term having a head normal form is outer-most fair reduction.*
Since head-needed reduction is head normalizing, if a term has a head normal form, head-needed reduction from the term is also outer-most fair reduction.

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⋆1 Outer-most fair reduction is a reduction whereby each outer-most redex is reduced or erased in the reduction.
⋆2 Hyper head-needed reduction is a reduction whereby each head-needed redex is reduced or erased in the reduction.

Appendix

A.1 Proof of Lemma 2

In order to prove Lemma 2, we must first describe a number of properties.

Proposition 3 Let t be a term, let σ be a substitution, let F be a variable, and Let p be a position.

(a) If $F \notin FV(t)$, then $PV(t,\sigma,F,p) = \emptyset$.

(b) If $t = G(X_1\downarrow,\dots,X_n\downarrow)$, $p \in Pos(X_i\sigma)$, and $G \notin Dom(\sigma)$, then $PV(t,\sigma,X_i,p) = \{ip\}$.

Proof. From the definition of PV, this proposition is trivial. □

Lemma 14 Let t be a term, and let σ be a substitution for $\forall X \in Dom(\sigma)$, $X\sigma = Y\downarrow$ for some variable Y. If $p \in Pos(t)$, then $PT(t,\sigma,p) = \{p\}$.

Proof. We prove the claim by induction on t. We have four cases from the definition of PT.

(PT1) Since $p = \varepsilon$, we have $PT(t,\sigma,p) = \{p\}$.

(PT2) Let $t \equiv a(t_1,\dots t_n)$ and $p = ip'$. Then, by induction, $PT(t_i,\sigma,p') = \{p\}$. Hence, $PT(t,\sigma,p) = \{iq \mid q \in PT(t_i,\sigma,p')\} = \{ip'\} = \{p\}$.

(PT3) Let $t \equiv \lambda x_1\cdots x_n.t'$. Then, by induction, we have $PT(t,\sigma,p) = PT(t',\sigma|_{\overline{\{x_1,\dots,x_n\}}},p) = \{p\}$.

(PT4) Let $t \equiv G(t_1,\dots,t_n)$, $p = ip'$, $G\sigma = Y\downarrow = \lambda y_1\cdots y_n.Y(y_1\downarrow,\dots,y_n\downarrow)$, and $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow,\dots,y_n \mapsto t_n\sigma\downarrow\}$. Then, $PT(t_i,\sigma,p') = \{p'\} \subseteq Pos(t_i\sigma\downarrow)$ by induction and Lemma 1. We have $PT(t,\sigma,p) = PV(Y(y_1\downarrow,\dots,y_n\downarrow),\sigma',y_i,p') = \{ip'\}$ by Proposition 3 (b), because $p' \in Pos(t_i\sigma\downarrow) = Pos(y_i\sigma')$. □

Lemma 15 Let t be a term such that $t = F(X_1\downarrow,\dots,X_n\downarrow)$. If $p \in Pos(F\sigma)$, $F \in Dom(\sigma)$, and $X_i \notin Dom(\sigma)$ for all i, then $PV(t,\sigma,F,p) = \{p\}$.

Proof. From Proposition 3 (a), $PV(X_i\downarrow,\sigma,F,p) = \emptyset$. Let $F\sigma = \lambda y_1\cdots y_n.t'$. Thus, (PV5) shows $PV(F(X_1\downarrow,\dots,X_n\downarrow),\sigma,F,p) = PT(t',\sigma',p)$, where $\sigma' = \{y_1 \mapsto X_1\downarrow\ \sigma\downarrow,\dots,y_n \mapsto X_n\downarrow\ \sigma\downarrow\} = \{y_1 \mapsto X_1\downarrow\ \dots y_n \mapsto X_n\downarrow\}$. Since $p \in Pos(t') = Pos(F\sigma)$, we have $PT(t',\sigma',p) = \{p\}$ by Lemma 14. □

Based on the above lemmas, we obtain the following lemma.

Lemma 2 If l is a pattern, then $PV(l,\sigma,F,p) = \{p'p \mid top(l|_{p'}) = F\}$ for every $F \in FV(l)$.

Proof. We show by induction on the structure of l that $PV(l,\sigma,F,p) = \{p'p \mid top(l|_{p'}) = F\}$for every $F \in (FV(l)-B)$ and B-pattern l such that $Dom(\sigma)\cap B = \emptyset$. From the definition of PV, we have six cases.

(PV1) We have $PV(F,\sigma,F,p) = \{p\} = \{p'p \mid top(F|_{p'}) = F\}$.

(PV2) Let $l \equiv a(t_1,\dots,t_n)$. We have $PV(t_i,\sigma,F,p) = \{p''p \mid top(t_i|_{p''}) = F\}$ by induction. Thus, $PV(l,\sigma,F,p) = \bigcup_i\{ip''p \mid top(t_i|_{p''}) = F\}$. Hence, $PV(l,\sigma,F,p) = \{p'p \mid top(l|_{p'}) = F\}$.

(PV3) Let $l \equiv \lambda x_1\cdots x_n.t$. Then, by induction, $PV(t,\sigma|_{\overline{\{x_1,\dots,x_n\}}},F,p) = \{p'p \mid top(t|_{p'}) = F\}$, because $Dom(\sigma|_{\overline{\{x_1,\dots,x_n\}}}) \cap (B \cup \{x_1,\dots,x_n\}) = \emptyset$.

(PV4) Let $l \equiv G(t_1,\dots,t_n)$. Then, $G \notin B$ and $t_i = x_i\downarrow$, where x_i is a pairwise distinct variable in B. Hence, the claim holds by Lemma 15.

(PV5) Same as for (PV4).

(PV6) Trivial. □

This lemma means that a position $p \in Pos(F\sigma)$ moves to $p'p$ for the position p' of F in pattern l, which is the same behavior as that observed in the case of the first order.

A.2 Proof of Theorem 1

In order to prove this theorem, the following lemma must be prepared.

Lemma 16 Let t be a normalized term, let σ be a normalized substitution, and let F be a variable.

(a) Let $p \in Pos(F\sigma)$ be a position. Then, for any $q \in PV(t,\sigma,F,p)$ there exists a substitution θ such that $(t\sigma\downarrow)|_q \equiv ((F\sigma)|_p)\theta\downarrow$.

(b) Let $p \in Pos(t)$ be a position. Then, for any $q \in PT(t,\sigma,p)$ there exists a substitution θ such that $(t\sigma\downarrow)|_q \equiv (t|_p)\theta\downarrow$.

Proof. We prove (a) and (b) simultaneously by induction on $\langle t,\sigma\rangle$ with $>_{\rhd\beta}$.

First, we show the proof of (a). Let $p \in Pos(F\sigma)$ and $q \in PV(t,\sigma,F,p)$. We have six cases from the definition of PV.

(PV1) We have $q = p$ from $t \equiv F$. Hence, the claim holds.

(PV2) Let $t \equiv a(t_1,\dots,t_n)$. Since $a \in \mathcal{F} \cup \overline{Dom(\sigma)}$ and $q \in PV(t,\sigma,F,p)$, there exist q' and i such that $q = iq'$ and $q' \in PV(t_i,\sigma,F,p)$. Since $\langle t,\sigma\rangle >_{\rhd\beta} \langle t_i,\sigma\rangle$, there exists θ' such that $(t_i\sigma\downarrow)|_{q'} \equiv ((F\sigma)|_p)\theta'\downarrow$ by induction. Thus, $(t\sigma\downarrow)|_q \equiv (a(t_1,\dots,t_n)\sigma\downarrow)|_{i\cdot q'} \equiv (t_i\sigma\downarrow)|_{q'} \equiv ((F\sigma)|_p)\theta'\downarrow$ holds.

(PV3) Let $t \equiv \lambda x_1\cdots x_n.t'$, and let $\sigma' = \sigma|_{\overline{\{x_1,\dots,x_n\}}}$. Then, $q \in PV(t',\sigma',F,p)$. Since $\langle t,\sigma\rangle >_{\rhd\beta} \langle t',\sigma'\rangle$, there exists θ' such that $(t'\sigma'\downarrow)|_q \equiv ((F\sigma')|_p)\theta'\downarrow$ by induction. Then, $(t\sigma\downarrow)|_q \equiv ((F\sigma)|_p)\theta'\downarrow$ holds from $(t\sigma\downarrow)|_q \equiv (t'\sigma'\downarrow)|_q$ and $F \notin \{x_1, \ldots, x_n\}$.

(PV4) Let $t \equiv G(t_1, \ldots, t_n)$, $F \neq G$, and $G\sigma \equiv \lambda y_1 \cdots y_n.t'$. Then, there exist q' and i such that $q' \in PV(t_i, \sigma, F, p)$ and $q \in PV(t', \sigma', y_i, q')$, where $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, there exists θ' such that $(t'\sigma'\downarrow)|_q \equiv ((y_i\sigma')|_{q'})\theta'\downarrow$ by induction. Thus, $(t\sigma\downarrow)|_q \equiv (t'\sigma'\downarrow)|_q \equiv ((y_i\sigma')|_{q'})\theta'\downarrow \equiv ((t_i\sigma\downarrow)|_{q'})\theta'\downarrow$ holds. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$, there exists θ'' such that $(t_i\sigma\downarrow)|_{q'} \equiv ((F\sigma)|_p)\theta''\downarrow$ by induction. Therefore, $(t\sigma\downarrow)|_q \equiv ((F\sigma)|_p)\theta''\downarrow\theta'\downarrow \equiv ((F\sigma)|_p)\theta''\theta'\downarrow$ holds.

(PV5) Let $t \equiv F(t_1, \ldots, t_n)$ and $F\sigma \equiv \lambda y_1 \cdots y_n.t'$. Then, there exists i such that $q \in PV(t', \sigma', y_i, PV(t_i, \sigma, F, p))$ or $q \in PT(t', \sigma', p)$, where $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$. The former case can be shown in the same manner as (PV4). In the latter case, since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, there exists θ such that $(t'\sigma'\downarrow)|_q \equiv (t'|_p)\theta\downarrow$ by induction. Thus, $(t\sigma\downarrow)|_q \equiv (t'\sigma'\downarrow)|_q \equiv (t'|_p)\theta\downarrow \equiv ((\lambda y_1 \cdots y_n.t')|_p)\theta\downarrow \equiv ((F\sigma)|_p)\theta\downarrow$ holds.

(PV6) (PV6) is obvious because $PV(t, \sigma, F, p) = \emptyset$.

Next, we show the proof of (b). Let $t|_p$ be a redex, and let $q \in PT(t, \sigma, p)$. We have four cases from the definition of PT.

(PT1) We have $q = \varepsilon$ from $p = \varepsilon$. Thus, we have $(t\sigma\downarrow)|_q \equiv t\sigma\downarrow \equiv (t|_p)\sigma\downarrow$.

(PT2) Let $p = ip'$ and $t \equiv a(t_1, \ldots, t_n)$. There exists q' such that $q = iq'$ and $q' \in PT(t_i, \sigma, p')$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$ and $t|_p \equiv t_i|_{p'}$, there exists θ such that $(t_i\sigma\downarrow)|_{q'} \equiv (t_i|_{p'})\theta\downarrow$ by induction. From $a \in \mathcal{F} \cup \overline{Dom(\sigma)}$, we have $(t\sigma\downarrow)|_q \equiv a(t_1\sigma\downarrow, \ldots, t_n\sigma\downarrow)|_{iq'} \equiv (t_i\sigma\downarrow)|_{q'} \equiv (t_i|_{p'})\theta\downarrow \equiv (a(t_1, \ldots, t_n)|_{ip'})\theta\downarrow \equiv (t|_p)\theta\downarrow$.

(PT3) Let $t \equiv \lambda x_1 \cdots x_n.t'$, and let $\sigma' = \sigma|_{\overline{\{x_1, \ldots, x_n\}}}$. Then, $q \in PT(t', \sigma', p)$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, and $t|_p \equiv t'|_p$, there exists θ' such that $(t'\sigma'\downarrow)|_q \equiv (t'|_p)\theta'\downarrow$ by induction. Then, $(t\sigma\downarrow)|_q \equiv (t|_p)\theta'\downarrow$ holds from $(t\sigma\downarrow)|_q \equiv (t'\sigma'\downarrow)|_q$.

(PT4) Let $p = ip'$, $t \equiv G(t_1, \ldots, t_n)$, and $G\sigma \equiv \lambda y_1 \cdots y_n.t'$. Then, there exists q' such that $q' \in PT(t_i, \sigma, p')$ and $q \in PV(t', \sigma', y_i, q')$, where $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, there exists θ' such that $(t'\sigma'\downarrow)|_q \equiv ((y_i\sigma')|_{q'})\theta'\downarrow$ by induction. Thus, $(t\sigma\downarrow)|_q \equiv (t'\sigma'\downarrow)|_q \equiv ((y_i\sigma')|_{q'})\theta'\downarrow \equiv ((t_i\sigma\downarrow)|_{q'})\theta'\downarrow$ holds. Moreover, since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$, there exists a position θ'' such that $(t_i\sigma\downarrow)|_{q'} \equiv ((t_i|_{p'})\theta''\downarrow)$. Therefore, $(t\sigma\downarrow)|_q \equiv ((t_i|_{p'})\theta''\downarrow)\theta'\downarrow \equiv (G(t_1, \ldots, t_n)|_{ip'})\theta''\theta'\downarrow \equiv (t|_p)\theta''\theta'\downarrow$ holds. □

Theorem 1 Let $\mathcal{R}$ be an orthogonal HRS, and let $A: s \to t$ be a reduction. If $s|_v$ is a redex, then $t|_p$ are also redexes for all $p \in v \backslash A$.

Proof. Let $s \equiv s[l\sigma\downarrow]_u \to s[r\sigma]_u \equiv t$. In case of $v|u$ or $v \prec u$, the theorem follows from orthogonality. In case of $v = up_1p_2$ for $p_1 \in Pos_{FV}(l)$, the theorem follows from Lemma 16 and the fact that instances of redex are redexes. □

A.3 Proof of Lemma 3

In this section, we assume orthogonality.

Proposition 4 Let s and t be normalized terms. Let $s \mathrel{\circ\!\!\!\twoheadrightarrow}^{D} t$ be a development, and let θ and σ be normalized substitutions. If for each $F \in Dom(\theta) \cap FV(s)$ we have some D_F such that $F\theta \mathrel{\circ\!\!\!\twoheadrightarrow}^{D_F} F\sigma$, then $s\theta\downarrow \mathrel{\circ\!\!\!\twoheadrightarrow}^{D'} t\sigma\downarrow$, where $D' = \bigcup_F PV(s, \theta, F, D_F) \cup PT(s, \theta, D)$.

Proof. By Noetherian induction on $\langle s, \theta\rangle$ with $>_{\rhd\beta}$, we prove the claim $P(s, \theta)$ defined as follows:

If $s \mathrel{\circ\!\!\!\twoheadrightarrow}^{D} t$ and $F\theta \mathrel{\circ\!\!\!\twoheadrightarrow}^{D_F} F\sigma$ for any $F \in Dom(\theta) \cap FV(s)$, there exists a development $s\theta\downarrow \mathrel{\circ\!\!\!\twoheadrightarrow}^{D'} t\sigma\downarrow$, where $D' = \bigcup_F PV(s, \theta, F, D_F) \cup PT(s, \theta, D)$.

From the definition of developments, we have several cases.

(A) First, we consider the case in which $s \mathrel{\circ\!\!\!\twoheadrightarrow}^{D} t$ is derived by the inference rule (A) of the definition of developments. We have two subcases:

(1) If $top(s) \in \mathcal{F} \cup \overline{Dom(\theta)}$, then we have $s \equiv a(s_1, \ldots, s_n)$, $t \equiv a(t_1, \ldots, t_n)$, $s_i \mathrel{\circ\!\!\!\twoheadrightarrow}^{D_i} t_i$, and $D = \bigcup_i \{iq \mid q \in D_i\}$. Since $\langle s, \theta\rangle >_{\rhd\beta} \langle s_i, \theta\rangle$, the induction hypothesis asserts that $s_i\theta\downarrow \mathrel{\circ\!\!\!\twoheadrightarrow}^{D'_i} t_i\sigma\downarrow$ for $D'_i = \bigcup_F PV(s_i, \theta, F, D_F) \cup PT(s_i, \theta, D_i)$. Thus, by the definition of developments, we have $s\theta\downarrow \equiv a(s_1\theta\downarrow, \ldots, s_n\theta\downarrow) \mathrel{\circ\!\!\!\twoheadrightarrow}^{D'} a(t_1\sigma\downarrow, \ldots, t_n\sigma\downarrow) \equiv t\sigma\downarrow$, where $D' = \bigcup_i \{iq \mid q \in D'_i\}$. Furthermore, we can calculate D' from D'_i:

$$\begin{aligned} D' &= \textstyle\bigcup_i \{iq \mid q \in \bigcup_F PV(s_i, \theta, F, D_F) \cup PT(s_i, \theta, D_i)\} \\ &= \textstyle\bigcup_F \bigcup_i \{iq \mid q \in PV(s_i, \theta, F, D_F)\} \cup \bigcup_i \{iq \mid q \in PT(s_i, \theta, D_i)\} \\ &= \textstyle\bigcup_F PV(a(s_1, \ldots, s_n), \theta, F, D_F) \cup PT(a(s_1, \ldots, s_n), \theta, \{iq \mid q \in D_i\}) \\ &= \textstyle\bigcup_F PV(s, \theta, F, D_F) \cup PT(s, \theta, D). \end{aligned}$$

(2) If $top(s) \in Dom(\theta)$, we have $s \equiv G(s_1, \ldots, s_n)$, $t \equiv G(t_1, \ldots, t_n)$, $s_i \mathrel{\circ\!\!\!\twoheadrightarrow}^{D_i} t_i$, and $D = \bigcup_i \{iq \mid q \in D_i\}$. Since $\langle s, \theta\rangle >_{\rhd\beta} \langle s_i, \theta\rangle$, as in the case above, we

can assert $s_i\theta{\downarrow} \twoheadrightarrow^{D'_i} t_i\sigma{\downarrow}$ for $D'_i = \bigcup_F PV(s_i,\theta,F,D_F) \cup PT(s_i,\theta,D_i)$ by induction. Let $G\theta \equiv \lambda y_1 \cdots y_n.u$, and let $G\sigma \equiv \lambda y_1 \cdots y_n.u'$. Then, we have $s\theta{\downarrow} \equiv (G\theta)(s_1\theta,\dots,s_n\theta){\downarrow} \equiv u\theta'{\downarrow}$, where $\theta' = \{y_1 \mapsto s_1\theta{\downarrow},\dots,y_n \mapsto s_n\theta{\downarrow}\}$, and $t\sigma{\downarrow} \equiv (G\sigma)(t_1\sigma,\dots,t_n\sigma){\downarrow} \equiv u'\sigma'{\downarrow}$, where $\sigma' = \{y_1 \mapsto t_1\sigma{\downarrow},\dots,y_n \mapsto t_n\sigma{\downarrow}\}$. Thus, we have $y_i\theta' \equiv s_i\theta{\downarrow} \twoheadrightarrow^{D'_i} t_i\sigma{\downarrow} \equiv y_i\sigma'$, where $D'_i = \bigcup_F PV(s_i,\theta,F,D_F) \cup PT(s_i,\theta,D_i)$. Moreover, note that $G\theta \twoheadrightarrow^{D_G} G\sigma$, and $u \twoheadrightarrow^{D_G} u'$ follows from$\lambda y_1 \cdots y_n.u \equiv G\theta \twoheadrightarrow^{D_G} G\sigma \equiv \lambda y_1 \cdots y_n.u'$. Since $s\theta{\downarrow} \equiv u\theta'{\downarrow}$, i.e., $s\theta \to_\beta^+ u\theta'$, we have $\langle s,\theta\rangle >_{\rhd\beta} \langle u,\theta'\rangle$. Thus, we have $u\theta'{\downarrow} \twoheadrightarrow^{D'} u'\sigma'{\downarrow}$ for $D' = \bigcup_i PV(u,\theta',y_i,D'_i) \cup PT(u,\theta',D_G)$ by induction. Hence, $s\theta{\downarrow} \equiv u\theta'{\downarrow} \twoheadrightarrow^{D'} u'\sigma'{\downarrow} \equiv t\sigma{\downarrow}$. Here, we can calculate D' as follows: Moreover, we have

$$\begin{aligned} D' &= \textstyle\bigcup_i PV(u,\theta',y_i,(\bigcup_F PV(s_i,\theta,F,D_F) \cup PT(s_i,\theta,D_i))) \\ &\quad \cup PT(u,\theta',D_G) \\ &= \textstyle\bigcup_i PV(u,\theta',y_i,\bigcup_F PV(s_i,\theta,F,D_F)) \\ &\quad \textstyle\cup \bigcup_i PV(u,\theta',y_i,PT(s_i,\theta,D_i)) \cup PT(u,\theta',D_G) \\ &= \textstyle\bigcup_F \bigcup_i PV(u,\theta',y_i,PV(s_i,\theta,F,D_F)) \\ &\quad \textstyle\cup PT(u,\theta',D_G) \cup \bigcup_i PV(u,\theta',y_i,PT(s_i,\theta,D_i)) \\ &= \textstyle\bigcup_F PV(G(s_1,\dots,s_n),\theta,F,D_F) \cup PT(G(s_1,\dots,s_n),\theta,D). \end{aligned}$$

Thus, we have $D' = \bigcup_F PV(s,\theta,F,D_F) \cup PT(s,\theta,D)$.

(L) Next, we consider the case of applying rule (L), that is, $s \equiv \lambda x_1 \dots x_n.s' \twoheadrightarrow^D \lambda x_1 \dots x_n.t' \equiv t$. Here, since $s\theta \rhd s'\theta$, we have $\langle s,\theta\rangle >_{\rhd\beta} \langle s',\theta\rangle$. Thus, by induction, we have $s'\theta{\downarrow} \twoheadrightarrow^{D'} t'\sigma{\downarrow}$ for $D' = \bigcup_F PV(s',\theta,F,D_F) \cup PT(s',\theta,D)$. Applying rule (L) to development, we have $s\theta{\downarrow} \equiv \lambda x_1 \cdots x_n.(s'\theta{\downarrow}) \twoheadrightarrow^{D'} \lambda x_1 \cdots x_n.(t'\sigma{\downarrow}) \equiv t\sigma{\downarrow}$. Note that the condition $Dom(\theta) \cap \{x_1,\dots,x_n\} = \emptyset$ is required, but this problem can be solved trivially.

(R) Let $s \equiv f(s_1,\dots,s_n) = l\theta'{\downarrow}$, $t = r\sigma'{\downarrow}$, $f(t_1,\dots,t_n) = l\sigma'{\downarrow}$, $l \blacktriangleright r \in \mathcal{R}$, $s_i \twoheadrightarrow^{D_i} t_i$, and $D = \{\varepsilon\} \cup \bigcup_i \{iq \mid q \in D_i\}$. Since $\langle s,\theta\rangle >_{\rhd\beta} \langle s_i,\theta\rangle$, we have $s_i\theta{\downarrow} \twoheadrightarrow^{D'_i} t_i\sigma{\downarrow}$ by induction, where $D'_i = \bigcup_F PV(s_i,\theta,F,D_F) \cup PT(s_i,\theta,D_i)$. We have $s\theta{\downarrow} \equiv f(s_1\theta{\downarrow},\dots,s_n\theta{\downarrow}) \equiv (l\theta'{\downarrow})\theta{\downarrow} \equiv l\theta''{\downarrow}$ and $f(t_1\sigma{\downarrow},\dots,t_n\sigma{\downarrow}) \equiv (l\sigma'{\downarrow})\sigma{\downarrow} \equiv l\sigma''{\downarrow}$ for some θ'' and σ''. By rule (R) to development, $f(s_1\theta{\downarrow},\dots,s_n\theta{\downarrow}) \twoheadrightarrow r\sigma''{\downarrow}$. Note that $(l\sigma'{\downarrow})\sigma{\downarrow} \equiv l\sigma''{\downarrow}$ implies that $t\sigma{\downarrow} \equiv (r\sigma'{\downarrow})\sigma{\downarrow} \equiv r\sigma''{\downarrow}$ because $FV(l) \supseteq FV(r)$. Therefore, we have $s\theta{\downarrow} \twoheadrightarrow^{D'} t\sigma{\downarrow}$, where $D' = \{\varepsilon\} \cup \bigcup_i \{iq \mid q \in D'_i\}$. Here, we have

$$\begin{aligned} D' &= \textstyle\{\varepsilon\} \cup \bigcup_F \bigcup_i \{iq \mid q \in PV(s_i,\theta,F,D_F)\} \cup \bigcup_i \{iq \mid q \in PT(s_i,\theta,D_i)\} \\ &= \textstyle PT(f(s_1,\dots,s_n),\theta,\{\varepsilon\}) \cup \bigcup_F PV(f(s_1,\dots,s_n),\theta,F,D_F) \\ &\quad \textstyle\cup \bigcup_i PT(f(s_1,\dots,s_n),\theta,\{iq \mid q \in D_i\}) \\ &= \textstyle\bigcup_F PV(s,\theta,F,D_F) \cup PT(s,\theta,D). \end{aligned}$$

□

As a special case of Proposition 4, the following corollary holds.

Corollary 2 Let s be a term, and let θ and σ be normalized substitutions. Let $F\theta \twoheadrightarrow^{D_F} F\sigma$ for each $F \in Dom(\theta) \cap FV(s)$. Then, $s\theta{\downarrow} \twoheadrightarrow^{D'} s\sigma{\downarrow}$, where (1) $D' = \bigcup_F PV(s,\theta,F,D_F)$. In particular, (2) if s is a pattern, then $D' = \bigcup_F \{p'p \mid top(s|_{p'}) = F, p \in D_F\}$. □

From the definition of descendants, the following proposition is trivial.

Proposition 5 Let s_1 and t_1 be terms, and let A and A' be developments such that $A : \lambda x.s_1 \twoheadrightarrow^{D_A} \lambda x.t_1$ and $A' : s_1 \twoheadrightarrow^{D_A} t_1$. Their descendants hold $\{q\}\backslash A' = \{q\}\backslash A$ for any position q. □

Next, we present the following lemma in order to prove the main lemma of this section.

Lemma 17 Let $A_F : F\theta \twoheadrightarrow^{D_F} F\sigma$ be a development, for every $F \in Dom(\theta)$. In addition, let $A : s \twoheadrightarrow^D t$ and $A' : s\theta{\downarrow} \twoheadrightarrow^{D'} t\sigma{\downarrow}$ be developments.

(a) $PV(s,\theta,F,p)\backslash A' = PV(t,\sigma,F,p\backslash A_F)$ for any $F, p \in Occ(F\theta)$ is a redex on s, and $F \in Dom(\theta)$.

(b) $PT(s,\theta,p)\backslash A' = PT(t,\sigma,p\backslash A)$ for any $p \in Occ(s)$, which is a redex on s.

Proof. We prove (a) and (b) simultaneously by induction on $s \twoheadrightarrow^D t$. First, we show (a).

(A) Consider the case $s \equiv a(s_1,\dots,s_n) \twoheadrightarrow^D a(t_1,\dots,t_n) \equiv t$ and $a \in \mathcal{F} \cup \mathcal{X}$. Let $D_i = \{p'' \mid ip'' \in D\}$ for each i, and let $A_i : s_i \twoheadrightarrow^{D_i} t_i$. Let $A'_i : s_i\theta{\downarrow} \twoheadrightarrow^{D'_i} t_i\sigma{\downarrow}$.

$$\begin{array}{ll} PV(s_i,\theta,F,p')\backslash A'_i = PV(t_i,\sigma,F,p'\backslash A_F) & \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s_i,\theta,p')\backslash A'_i = PT(t_i,\sigma,p'\backslash A_i) & \text{if } \exists p' \in Occ(s_i) \end{array}$$

holds by induction.

(A-PV2) Consider the case in which $p' \in Occ(F\theta)$ and $a \in \mathcal{F} \cup \overline{Dom(\theta)}$.

$$\begin{array}{lr} PV(s,\theta,F,p')\backslash A' & \\ = \bigcup_i \{iq \mid q \in PV(s_i,\theta,F,p')\}\backslash (a(s_1,\ldots,s_n)\theta{\downarrow} \mathrel{\circ\!\!\!\rightarrow} a(t_1,\ldots,t_n)\sigma{\downarrow}) & \\ & \text{by PV2} \\ = \bigcup_i \{iq \mid q \in PV(s_i,\theta,F,p')\backslash A'_i & \text{by Def. 4(A)} \\ = \bigcup_i \{iq \mid q \in PV(t_i,\sigma,F,p'\backslash A_F)\} & \text{from I.H.} \\ = PV(t,\sigma,F,p'\backslash A_F) & \text{by PV2} \end{array}$$

(A-PV4) Consider the case in which $a = G \in Dom(\theta)$. Let $G\theta \equiv \lambda y_1 \cdots y_n.s'$, $s \equiv G(s_1,\ldots,s_n)\theta{\downarrow}$, $t \equiv G(t_1,\ldots,t_n)\sigma{\downarrow}$, and $\theta' = \sigma'$.

$$\begin{array}{lr} PV(s,\theta,F,p')\backslash A' & \\ = \bigcup_i PV(s',\theta',y_i,PV(s_i,\theta,F,p'))\backslash (G(s_1,\ldots,s_n)\theta{\downarrow} \mathrel{\circ\!\!\!\rightarrow} G(t_1,\ldots,t_n)\sigma{\downarrow}) & \\ & \text{by def. of PV4} \\ = \bigcup_i PV(s',\theta',y_i,P_i)\backslash (s'\theta'{\downarrow} \mathrel{\circ\!\!\!\rightarrow} t'\sigma'{\downarrow}) & \\ = \bigcup_i PV(t',\sigma',y_i,P_i\backslash A_{y_i}) & \text{from I.H.} \\ = \bigcup_i PV(t',\sigma',y_i,PV(t_i,\sigma,F,p'\backslash A_F)) & \\ = PV(t,\sigma,F,p'\backslash A_F) & \text{by def. of PV4.} \end{array}$$

(A-PV5) Consider the case in which $a = F \in Dom(\theta)$. Let $F\theta \equiv \lambda y_1 \cdots y_n.s'$, $s \equiv F(s_1,\ldots,s_n)\theta{\downarrow}$, $F\sigma \equiv \lambda y_1 \cdots y_n.t'$, and $t \equiv F(t_1,\ldots,t_n)\sigma{\downarrow}$. Let $\theta' = \{y_1 \mapsto s_1\theta{\downarrow},\ldots,y_n \mapsto s_n\theta{\downarrow}\}$, $\sigma' = \{y_1 \mapsto t_1\sigma{\downarrow},\ldots,y_n \mapsto t_n\sigma{\downarrow}\}$, and $s\theta{\downarrow} = s'\theta'{\downarrow}$. Let $P_i = PV(s_i\theta,F,p')$.

$$\begin{array}{lr} PV(s,\theta,F,p')\backslash A' & \\ = (\bigcup_i PV(s',\theta',y_i,PV(s_i,\theta,F,p')) \cup PT(s',\theta',p'))\backslash A' & \text{by def. of PV5} \\ = (\bigcup_i PV(s',\theta',y_i,P_i) \cup PT(s',\theta',p'))\backslash (s'\theta'{\downarrow} \mathrel{\circ\!\!\!\rightarrow} t'\sigma'{\downarrow}) & \\ & \text{by } s\theta = s'\theta' \text{ and } t\sigma = t'\sigma' \\ = \bigcup_i PV(t',\sigma',y_i,P_i\backslash A_{y_i}) \cup PT(t'\sigma',p'\backslash A_F) & \text{from I.H. of (a) and (b)} \\ = \bigcup_i PV(t',\sigma',y_i,PV(t_i,\sigma,F,p'\backslash A_F)) \cup PT(t'\sigma',p'\backslash A_F) & \\ = PV(t,\sigma,F,p'\backslash A_F) & \text{by def. of PV5.} \end{array}$$

(L) Consider the case in which $s \equiv \lambda x_1 \cdots x_n.s' \mathrel{\circ\!\!\!\rightarrow} \lambda x_1 \cdots x_n.t' \equiv t$ and A_1 : $s' \mathrel{\circ\!\!\!\rightarrow}^{D_1} t'$. Let $A'_1 : s'\theta{\downarrow} \mathrel{\circ\!\!\!\rightarrow}^{D'_1} t'\sigma{\downarrow}$. For developments A_F and A_1,

$$\begin{array}{ll} PV(s',\theta,F,p')\backslash A'_1 = PV(t',\sigma,F,p'\backslash A_F) & \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s',\theta,p')\backslash A'_1 = PT(t',\sigma,p'\backslash A_1) & \text{if } \exists p' \in Occ(s') \end{array}$$

holds from induction.

(L-PV3) Consider the case in which $p' \in Occ(F\theta)$. Let $s \equiv \lambda x_1 \cdots x_n.s'$, $t \equiv \lambda x_1 \cdots x_n.t'$, $s\theta{\downarrow} = (\lambda x_1 \cdots x_n.s')\theta{\downarrow} = \lambda x_1 \cdots x_n.(s'\theta|_{\overline{\{x_1,\ldots,x_n\}}}){\downarrow}$, and $t\sigma{\downarrow} = (\lambda x_1 \cdots x_n.t')\sigma{\downarrow} = \lambda x_1 \cdots x_n.(t'\sigma|_{\overline{\{x_1,\ldots,x_n\}}}){\downarrow}$.

$$\begin{array}{lr} PV(s,\theta,F,p')\backslash A' & \\ = PV(s',\theta|_{\overline{\{x_1,\ldots,x_n\}}},F,p')\backslash A' & \text{by def. of PV3} \\ = PV(s',\theta|_{\overline{\{x_1,\ldots,x_n\}}},F,p')\backslash ((\lambda x_1 \cdots x_n.s')\theta|_{\overline{\{x_1,\ldots,x_n\}}}{\downarrow} & \\ \quad \mathrel{\circ\!\!\!\rightarrow} (\lambda x_1 \cdots x_n.t')\sigma|_{\overline{\{x_1,\ldots,x_n\}}}{\downarrow}) & \\ = PV(s',\theta|_{\overline{\{x_1,\ldots,x_n\}}},F,p')\backslash (s'\theta|_{\overline{\{x_1,\ldots,x_n\}}}{\downarrow} \mathrel{\circ\!\!\!\rightarrow} t'\sigma|_{\overline{\{x_1,\ldots,x_n\}}}{\downarrow}) & \\ = PV(t',\sigma|_{\overline{\{x_1,\ldots,x_n\}}},F,p'\backslash A_F) & \text{from I.H.} \\ = PV(t,\sigma,F,p'\backslash A_F) & \text{by PV3} \end{array}$$

(R) Consider the case in which $s \equiv f(s_1,\ldots,s_n) \mathrel{\circ\!\!\!\rightarrow} f(t_1,\ldots,t_n) \mathrel{\circ\!\!\!\rightarrow} r\theta'' \equiv t$ and $A_i : s_i \mathrel{\circ\!\!\!\rightarrow}^{D_i} t_i$. Let $A'_i : s_i\theta{\downarrow} \mathrel{\circ\!\!\!\rightarrow}^{D'_i} t_i\sigma{\downarrow}$. For developments A_F and A_i, where $p_i \in Occ(s_i\theta{\downarrow})$,

$$\begin{array}{ll} PV(s_i,\theta,F,p')\backslash A'_i = PV(t_i,\sigma,F,p'\backslash A_F) & \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s_i,\theta,p'_i)\backslash A'_i = PT(t_i,\sigma,p'_i\backslash A_i) & \text{if } \exists p'_i \in Occ(s_i) \end{array}$$

holds from induction.

(R-PV-1) Consider the case in which $p' \in Occ(F\theta)$, we have $PV(s,\theta,F,p')\backslash (s\theta{\downarrow} \equiv f(s_1,\ldots,s_n)\theta{\downarrow} \mathrel{\circ\!\!\!\rightarrow} f(t_1,\ldots,t_n)\sigma{\downarrow} \mathrel{\circ\!\!\!\rightarrow} t\sigma{\downarrow}) = \{iq \mid q \in PV(t_i,\sigma,F,p'\backslash A_F)\} \backslash (f(t_1,\ldots,t_n)\sigma{\downarrow} \mathrel{\circ\!\!\!\rightarrow} t\sigma{\downarrow})$ from the case (A). Since we can denote, $f(t_1,\ldots,t_n)\sigma{\downarrow} = l\theta''\sigma, t\sigma{\downarrow} = r\theta''\sigma$, letting $\theta'' = \theta, p'\backslash A_F = p$, we have

$$\begin{array}{lr} PV(l\theta,\sigma,F,p)\backslash ((l\theta)\sigma \mathrel{\circ\!\!\!\rightarrow} (r\theta)\sigma) & \\ = \bigcup_{y \in Var(l)} \{p'p \mid top(l|_{p'}) = y, \quad p \in PV(y\theta,\sigma,F,p)\}\backslash ((l\theta)\sigma \mathrel{\circ\!\!\!\rightarrow} (r\theta)\sigma) & \\ = \bigcup_y PV(r,\theta\sigma,y,PV(y\theta,\sigma,F,p)) & \text{from Def. 3} \\ = PV(r\theta,\sigma,F,p) & \\ = PV(t,\sigma,F,p'\backslash A_F) & \end{array}$$

(R-PV-2) Consider the case in which $p' \in Occ(F\theta)$, $p \in PV(s, \theta, F, p')$.

$$\begin{aligned} & p\backslash(s\theta\!\downarrow \equiv f(s_1,\ldots,s_n)\theta\!\downarrow \twoheadrightarrow t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \\ &= p\backslash(f(s_1,\ldots,s_n)\theta\!\downarrow \twoheadrightarrow f(t_1,\ldots,t_n)\theta\!\downarrow \underset{l\blacktriangleright r}{\twoheadrightarrow} t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \\ &= PV(t,\theta,F,p')\backslash(t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \end{aligned}$$

Next, we show (b).

(A) Consider the case in which $s \equiv a(s_1,\ldots,s_n) \twoheadrightarrow^{D} a(t_1,\ldots,t_n) \equiv t$ and $a \in \mathcal{F} \cup \mathcal{X}$. Let $D_i = \{p'' | ip'' \in D\}$ for each i, and let $A_i : s_i \twoheadrightarrow^{D_i} t_i$. Let $A'_i : s_i\theta\!\downarrow \twoheadrightarrow^{D'_i} t_i\sigma\!\downarrow$.

$$\begin{aligned} PV(s_i,\theta,F,p')\backslash A'_i &= PV(t_i,\sigma,F,p'\backslash A_F) && \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s_i,\theta,p'_i)\backslash A'_i &= PT(t_i,\sigma,p'\backslash A_i) && \text{if } \exists p'_i \in Occ(s_i) \end{aligned}$$

holds from induction.

(A-PT2) Consider the case in which $p' \in Occ(s)$ and $a \in \mathcal{F} \cup \overline{Dom(\theta)}$,

$$\begin{aligned} & PT(s,\theta,p')\backslash A' \\ &= \{iq | q \in PT(s_i,\theta,p'_i), p' = ip'_i\}\backslash A' && \text{from PT2} \\ &= \{iq | q \in PT(s_i,\theta,p'_i)\backslash A'_i, p' = ip'_i\} \\ &= \{iq | q \in PT(t_i,\sigma,p'\backslash A_i), p' = ip'_i\} && \text{from I.H.} \\ &= PT(t,\sigma,p'\backslash A) && \text{from PT2} \end{aligned}$$

(A-PT4) Consider the case in which $p' \in Occ(s)$ and $a \in Dom(\theta)$,

$$\begin{aligned} & PT(s,\theta,p')\backslash A' \\ &= PV(s',\theta',y_i,PT(s_i,\theta,p'_i))\backslash A' && \text{where } p' = ip'_i \text{ from PT4} \\ & \text{since } \langle s,\theta\rangle >_{\rhd\beta} \langle s',\theta'\rangle, \text{ we use I.H. of (a), then} \\ &= PV(t',\sigma',y_i,PT(s_i,\theta,p'_i)\backslash A'_i) \\ &= PV(t',\sigma',y_i,PT(t_i,\theta,p'_i\backslash A'_i)) && \text{from I.H. of (b)} \\ &= PT(t,\sigma,p'\backslash A) && \text{from PT4} \end{aligned}$$

(L) Consider the case in which $s \equiv \lambda x_1\cdots x_n.s' \twoheadrightarrow \lambda x_1\cdots x_n.t' \equiv t$ and $A_1 : s' \twoheadrightarrow^{D_1} t'$. Let $A'_1 : s'\theta\!\downarrow \twoheadrightarrow^{D'_1} t'\sigma\!\downarrow$. For developments A_F and A_1,

$$\begin{aligned} PV(s',\theta,F,p')\backslash A'_1 &= PV(t',\sigma,F,p'\backslash A_F) && \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s',\theta,p')\backslash A'_1 &= PT(t',\sigma,p'\backslash A_1) && \text{if } \exists p' \in Occ(s') \end{aligned}$$

holds from induction.

(L-PT3) Consider the case in which $p' \in Occ(s)$,

$$\begin{aligned} & PT(s,\theta,p')\backslash A' \\ &= PT(s',\theta|_{\overline{\{x_1,\ldots,x_n\}}},p')\backslash A' && \text{from PT3} \\ &= PT(s',\theta|_{\overline{\{x_1,\ldots,x_n\}}},p')\backslash A'_1 \\ &= PT(t',\sigma|_{\overline{\{x_1,\ldots,x_n\}}},p'\backslash A) && \text{from I.H.} \\ &= PT(t,\sigma,p'\backslash A) && \text{from PT3} \end{aligned}$$

(R) Consider the case in which $s \equiv f(s_1,\ldots,s_n) \twoheadrightarrow f(t_1,\ldots,t_n) \twoheadrightarrow r\theta'' \equiv t$ and $A_i : s_i \twoheadrightarrow^{D_i} t_i$. Let $A'_i : s_i\theta\!\downarrow \twoheadrightarrow^{D'_i} t_i\sigma\!\downarrow$. For developments A_F and A_i, where $p_i \in Occ(s_i\theta\!\downarrow)$,

$$\begin{aligned} PV(s_i,\theta,F,p')\backslash A'_i &= PV(t_i,\sigma,F,p'\backslash A_F) && \text{if } \exists F, p' \in Occ(F\theta) \\ PT(s_i,\theta,p'_i)\backslash A'_i &= PT(t_i,\sigma,p'_i\backslash A_i) && \text{if } \exists p'_i \in Occ(s_i) \end{aligned}$$

holds by induction.

(R-PT-1) Consider the case in which $p' \in Occ(s)$,

$$\begin{aligned} & PT(s,\theta,p')\backslash(s\theta\!\downarrow \equiv f(s_1,\ldots,s_n)\theta\!\downarrow \twoheadrightarrow \\ & f(s_1,\ldots,s_n)\sigma\!\downarrow \twoheadrightarrow f(t_1,\ldots,t_n)\sigma\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \\ &= \{iq | q \in PT(t_i,\sigma,p'_i\backslash A_i)\}\backslash(f(t_1,\ldots,t_n)\sigma\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) && \text{from the case (A)} \\ &= PT(t,\sigma,p'\backslash A) \quad \text{where } p' = ip',\ D = \{iq | q \in D_i\} \end{aligned}$$

(R-PT-2) Consider the case in which $p' \in Occ(s)$, $p \in PT(s,\theta,p')$.

$$\begin{aligned} & p\backslash(s\theta\!\downarrow \equiv f(s_1,\ldots,s_n)\theta\!\downarrow \twoheadrightarrow t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \\ &= p\backslash(f(s_1,\ldots,s_n)\theta\!\downarrow \twoheadrightarrow f(t_1,\ldots,t_n)\theta\!\downarrow \underset{l\blacktriangleright r}{\twoheadrightarrow} t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) \\ &= PT(t,\theta,p'\backslash A)\backslash(t\theta\!\downarrow \twoheadrightarrow t\sigma\!\downarrow) && \text{from def. of PT} \\ &= PT(t,\sigma,p'\backslash A) \end{aligned}$$

□

Lemma 18 Let $A : s \twoheadrightarrow^{D} t$ and $B : s \xrightarrow{\varepsilon} t'$ such that $\varepsilon \notin D$. Then, $A; B\backslash A \simeq B; A\backslash B$.

Proof. Let $s \equiv l\theta\!\downarrow$, $t \equiv l\sigma\!\downarrow$, and let $t' \equiv r\theta\!\downarrow$. First, we show that $r\theta\!\downarrow \twoheadrightarrow^{D\backslash B} r\sigma\!\downarrow$, because $l\sigma\!\downarrow \xrightarrow{\varepsilon} r\sigma\!\downarrow$ holds trivially. Let $F \in Dom(\theta) \cap FV(r)$. Then, from orthogonality, $F\theta\!\downarrow \twoheadrightarrow^{D_F} F\sigma\!\downarrow$ for some set D_F of positions. We have $r\theta\!\downarrow \twoheadrightarrow^{D'} r\sigma\!\downarrow$ by Corollary 2 (1), where $D' = \bigcup_F PV(r,\theta,F,D_F)$. We also have $D = \bigcup_F \{p'p \mid top(l|_{p'}) = F, p \in D_F\}$ by Corollary 2 (2). On the other hand, $D\backslash B = \bigcup_{v\in D}\{p_3 \mid$

$p_3 \in PV(r,\theta,top(l|_{p_1}),p_2), v = p_1p_2, p_1 \in Pos_{FV}(l)\}$.

$$\begin{aligned} p \in D\backslash B &\Leftrightarrow \exists p_1,p_2\ p \in PV(r,\theta,top(l|_{p_1}),p_2) \\ &\quad\ \text{where } p_1p_2 \in D, p_1 \in Pos_{FV}(l) \\ &\Leftrightarrow \exists F \in FV(l)\ p \in PV(r,\theta,F,p_2) \\ &\quad\ \text{where } p_2 \in D_F \\ &\Leftrightarrow p \in \textstyle\bigcup_{F\in FV(l)} PV(r,\theta,F,D_F) = D' \end{aligned}$$

Second, we show $q\backslash(A;B\backslash A) = q\backslash(B;A\backslash B)$ for each redex position q in s. In the case of $q = \varepsilon$, this is trivial. Consider the case of $q \neq \varepsilon$. From orthogonality, we assume $q = p_1p_2$ and $top(l|_{p_1}) = F$. Let $A_F : F\theta \twoheadrightarrow^{D_F} F\sigma$. Since $q\backslash A = \{p_1p_3 \mid p_3 \in p_2\backslash A_F\}$, we have

$$\begin{aligned} q\backslash(A;B\backslash A) &= \textstyle\bigcup_{p_3\in p_2\backslash A_F} p_1p_3\backslash(B\backslash A) \\ &= \textstyle\bigcup_{p_3\in p_2\backslash A_F} PV(r,\sigma,F,p_3) \\ &= PV(r,\sigma,F,p_2\backslash A_F). \end{aligned}$$

Since $q\backslash B = PV(r,\theta,F,p_2)$, we have

$$\begin{aligned} q\backslash(B;A\backslash B) &= PV(r,\theta,F,p_2)\backslash(A\backslash B) \\ &= PV(r,\sigma,F,p_2\backslash A_F) \end{aligned}$$

from Lemma 17 (a). Therefore, the claim holds. □

Lemma 3. Let $\mathcal{R}$ be an orthogonal HRS. If A and B are developments starting from the same term, then $A;(B\backslash A) \simeq B;(A\backslash B)$.

Proof. Let A and B be developments such that $A : s \twoheadrightarrow^{D_A} t$ and $B : s \twoheadrightarrow^{D_B} t'$, respectively. Then, we show the following:
(1) there exists u such that $t \twoheadrightarrow^{D_B\backslash A} u$ and $t' \twoheadrightarrow^{D_A\backslash B} u$, and
(2) $\forall p \in redex(s)$, $p\backslash(A;B\backslash A) = p\backslash(B;A\backslash B)$
by induction on the structure of s. We have several cases, according to the inference rules of development applied for A and B.

(1) Consider the case in which inference rule (L) is used for both A and B. Let $s \equiv \lambda x_1\cdots x_n.s_1$. Since $A_1 : s_1 \twoheadrightarrow^{D_A} t_1$ and $B_1 : s_1 \twoheadrightarrow^{D_B} t'_1$, where $t \equiv \lambda x_1\cdots x_n.t_1$ and $t' \equiv \lambda x_1\cdots x_n.t'_1$, it follows from induction that $t_1 \twoheadrightarrow^{D_B\backslash A_1} u_1$ and $t'_1 \twoheadrightarrow^{D_A\backslash B_1} u_1$ for some u_1, and $p\backslash(A_1;(B_1\backslash A_1)) = p\backslash(B_1;(A_1\backslash B_1))$. Since $D_B\backslash A_1 = D_B\backslash A$ and $D_A\backslash B_1 = D_A\backslash B$ from Definition 4, we have $\lambda x_1\cdots x_n.t_1 \twoheadrightarrow^{D_B\backslash A} \lambda x_1\cdots x_n.u_1$ and $\lambda x_1\cdots x_n.t'_1 \twoheadrightarrow^{D_A\backslash B} \lambda x_1\cdots x_n.u_1$. On the other hand, $p\backslash(A;(B\backslash A)) = p\backslash(B;(A\backslash B))$ holds from Proposition 5 and $p\backslash(A_1;(B_1\backslash A_1)) = p\backslash(B_1;(A_1\backslash B_1))$.

(2) Consider the case in which rule (A) is used for both A and B. Let $s \equiv a(s_1,\ldots,s_n)$. We have $A_i : s_i \twoheadrightarrow^{D_{A_i}} t_i$ and $B_i : s_i \twoheadrightarrow^{D_{B_i}} t'_i$, where $t = a(t_1,\ldots,t_n)$, $t' = a(t'_1,\ldots,t'_n)$, $D_{A_i} = \{p \mid ip \in D_A\}$, and $D_{B_i} = \{p \mid ip \in D_B\}$. Hence, by induction, we have $t_i \twoheadrightarrow^{D_{B_i}\backslash A_i} u_i$ and $t'_i \twoheadrightarrow^{D_{A_i}\backslash B_i} u_i$. Thus, we have $a(t_1,\ldots,t_n) \twoheadrightarrow^{D'} a(u_1,\ldots,u_n)$ and $a(t'_1,\ldots,t'_n) \twoheadrightarrow^{D''} a(u_1,\ldots,u_n)$, where $D' = \bigcup_i\{ip \mid p \in D_{B_i}\backslash A_i\}$ and $D'' = \bigcup_i\{ip \mid p \in D_{A_i}\backslash B_i\}$. Here,

$$\begin{aligned} D_B\backslash A &= \textstyle\bigcup_i\{ip'' \mid ip' \in D_B, p'' \in p'\backslash A_i\} \\ &= \textstyle\bigcup_i\{ip'' \mid p' \in D_{B_i}, p'' \in p'\backslash A_i\} \\ &= D' \end{aligned}$$

We also have $D_A\backslash B = D''$, which proves (1). On the other hand, $p\backslash(A;(B\backslash A)) = \{\varepsilon\} = p\backslash(B;(A\backslash B))$ if $p = \varepsilon$. For the case in which $p = iq$, we have $q\backslash(A_i;(B_i\backslash A_i)) = q\backslash(B_i;(A_i\backslash B_i))$ by induction. Since $iq\backslash A = \{iq' \mid q' \in q\backslash A_i\}$ and $iq\backslash B = \{iq' \mid q' \in q\backslash B_i\}$, $p\backslash(A;(B\backslash A)) = p\backslash(B;(A\backslash B))$.

(3) Consider the case in which (A) is used for A and (R) is used for B. Let $s \equiv a(s_1,\ldots,s_n)$. We have $s \equiv a(s_1,\ldots,s_n) \twoheadrightarrow^{D_B-\{\varepsilon\}} a(t'_1,\ldots,t'_n) \equiv t'' \xrightarrow{\varepsilon} t'$ for some t''. From case 2, there exist developments $B' : t \twoheadrightarrow^{D'_B} u''$ and $A' : t'' \twoheadrightarrow^{D'_A} u''$, where $\varepsilon \notin D'_A \cup D'_B$, and $p\backslash(A;(B''\backslash A)) = p\backslash(B'';(A\backslash B''))$, where $B' = B''\backslash A$ and $A' = A\backslash B''$. Therefore, there exist developments $t'' \twoheadrightarrow u'' \twoheadrightarrow u$ and $t'' \twoheadrightarrow t' \twoheadrightarrow u$ from Lemma 18. For any $p \in redex(t'')$, $p\backslash(A';(B'''\backslash A')) = p\backslash(B''';(A'\backslash B'''))$, where $B''' : t'' \xrightarrow{\varepsilon} t'$, from Lemma 18. Thus, the claim holds (See **Fig. 2**).

(4) Consider the case in which (R) is used for both of A and B. Let $s \equiv a(s_1,\ldots,s_n)$. We have $s \equiv a(s_1,\ldots,s_n) \twoheadrightarrow^{D_B-\{\varepsilon\}} a(t'_1,\ldots,t'_n) \equiv t'' \xrightarrow{\varepsilon} t'$ and $s \equiv a(s_1,\ldots,s_n) \twoheadrightarrow^{D_A-\{\varepsilon\}} a(t_1,\ldots,t_n) \equiv t''' \xrightarrow{\varepsilon} t$ for some t'' and t'''.

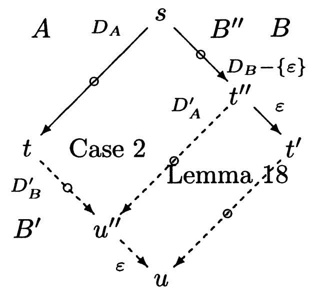

Fig. 2 Case 3 of the proof of Lemma 3.

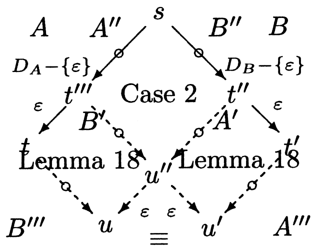

Fig. 3 Case 4 of the proof of Lemma 3.

From case 2, there exist developments $B' : t''' \circ\!\!\!\!\to^{D'_B} u''$ and $A' : t'' \circ\!\!\!\!\to^{D'_A} u''$, where $\varepsilon \notin D'_A \cup D'_B$. Therefore, there exist developments $t'' \circ\!\!\!\!\to u'' \circ\!\!\!\!\to u'$ and $t'' \circ\!\!\!\!\to t' \circ\!\!\!\!\to u$ from Lemma 18. There also exist developments $t''' \circ\!\!\!\!\to u'' \circ\!\!\!\!\to u$ and $t''' \circ\!\!\!\!\to t \circ\!\!\!\!\to u$ from Lemma 18. Since developments $u'' \circ\!\!\!\!\to u$ and $u'' \circ\!\!\!\!\to u'$ are contracted at the same position ε, u and u' are the same. We also have $p\backslash(A;(B\backslash A)) = p\backslash(B;(A\backslash B))$ from Case 2 and Lemma 18. (See **Fig. 3**) □

A.4 Proof of Lemma 6

In order to prove Lemma 6, we need to describe a number of properties.

Proposition 6 For any terms s and t, if $s \circ\!\!\!\!\to t$ satisfies the top-down property then if $s \circ\!\!\!\!\to t$ has minimal top-down decomposition, and vice versa.

Proof. We prove this proposition by induction on the structure of t. Since $s \circ\!\!\!\!\to t$ has the top-down property, there exists a development $s \overset{\varepsilon}{\to}{}^k u \equiv \lambda x_1 \cdots x_n.a(u_1,\ldots,u_m) \overset{\succ\varepsilon}{\circ\!\!\!\!\to}{}^D \lambda x_1 \cdots x_n.a(t_1,\ldots,t_m) \equiv t$ for some k, where u and D are uniquely determined and $u_i \circ\!\!\!\!\to t_i$ has the top-down property for all $i = 1,\ldots,m$. By induction, $u_i \circ\!\!\!\!\to t_i$ has minimal top-down decomposition, the length of which is k_i. Let k_0 be the minimal k such that $s \overset{\varepsilon}{\to}{}^k u$. Thus, we can make a minimal top-down decomposition such that $s \overset{\varepsilon}{\to}{}^{k_0} \lambda x_1 \cdots x_n.a(u_1,\ldots,u_m) \to^{k_1} \cdots \to^{k_m} \lambda x_1 \cdots x_n.a(t_1,\ldots,t_m)$ the length of which is $k_0 + k_1 + \cdots + k_m$. The reverse is trivial. □

Lemma 19 Let s and t be normalized terms, and let θ and σ be normalized substitutions. If $s \circ\!\!\!\!\to t$ satisfies the top-down property and $F\theta \circ\!\!\!\!\to F\sigma$ satisfies the top-down property for any $F \in FV(t) \cap Dom(\sigma)$, then $s\theta{\downarrow} \circ\!\!\!\!\to t\sigma{\downarrow}$ satisfies the top-down property.

Proof. Proposition 4 asserts that there exists the development $s\theta{\downarrow} \circ\!\!\!\!\to t\sigma{\downarrow}$. This development can be proved to have the top-down property as follows.

Since $s \circ\!\!\!\!\to t$ has the top-down property, we know that there exists a term u such that $s \overset{\varepsilon}{\to}{}^* u \overset{\succ\varepsilon}{\circ\!\!\!\!\to} t$ and $u \overset{\succ\varepsilon}{\circ\!\!\!\!\to} t$ has the top-down property.

First, based on the fact that $s \overset{\varepsilon}{\to}{}^* u$, we can easily prove that

(1) $s\theta{\downarrow} \overset{\varepsilon}{\to}{}^* u\theta{\downarrow}$.

Second, the fact that $u \overset{\succ\varepsilon}{\circ\!\!\!\!\to} t$ has the top-down property allows us to prove that

(2) $u\theta{\downarrow} \circ\!\!\!\!\to t\sigma{\downarrow}$ has top-down property.

Combining (1) and (2), we know that $s\theta{\downarrow} \circ\!\!\!\!\to t\sigma{\downarrow}$ has the top-down property.

The above fact (2) can be proved by induction on $\langle t,\sigma\rangle$ with $>_{\rhd\beta}$:

(A) Consider the case in which the development $u \circ\!\!\!\!\to t$ is generated from definition rule (A). Let $u \equiv a(u_1,\ldots,u_n)$ and $t \equiv a(t_1,\ldots,t_n)$.

(1) If $a \in \mathcal{F} \cup \overline{Dom(\sigma)}$, then $u_i \circ\!\!\!\!\to t_i$ satisfies the top-down property for each i, because $u \circ\!\!\!\!\to t$ satisfies the top-down property. It follows from $\langle t,\sigma\rangle >_{\rhd\beta} \langle t_i,\sigma\rangle$ that $u_i\theta{\downarrow} \circ\!\!\!\!\to t_i\sigma{\downarrow}$ satisfies the top-down property by induction. Thus, $u\theta{\downarrow} \equiv a(u_1\theta{\downarrow},\ldots,u_n\theta{\downarrow}) \overset{\succ\varepsilon}{\circ\!\!\!\!\to} a(t_1\sigma{\downarrow},\ldots,t_n\sigma{\downarrow}) \equiv t\sigma{\downarrow}$ satisfies the top-down property.

(2) Consider the subcase in which $a \in Dom(\sigma)$. We write G for a. Let $G\theta \equiv \lambda x_1 \cdots x_n.u'$, $G\sigma \equiv \lambda x_1 \cdots x_n.t'$, $\theta' = \{x_1 \mapsto u_1\theta{\downarrow},$

$\ldots, x_n \mapsto u_n\theta\downarrow\}$, and $\sigma' = \{x_1 \mapsto t_1\sigma\downarrow, \ldots, x_n \mapsto t_n\sigma\downarrow\}$. Then, $u\theta\downarrow \equiv (\lambda x_1 \cdots x_n.u')(u_1\theta, \ldots, u_n\theta)\downarrow \equiv u'\theta'\downarrow$, and $t\sigma\downarrow \equiv (\lambda x_1 \cdots x_n.t')(t_1\sigma, \ldots, t_n\sigma)\downarrow \equiv t'\sigma'\downarrow$. Since $G\theta\downarrow \twoheadrightarrow G\sigma\downarrow$ satisfies the top-down property, $u' \twoheadrightarrow t'$ satisfies the top-down property. Thus, $x_i\theta' \equiv u_i\theta\downarrow \twoheadrightarrow t_i\sigma\downarrow \equiv x_i\sigma'$ satisfies the top-down property by induction. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, the development $u'\theta'\downarrow \twoheadrightarrow t'\sigma'\downarrow$ satisfies the top-down property by induction.

(L) Since $u \equiv \lambda x_1 \cdots x_n.u' \twoheadrightarrow \lambda x_1 \cdots x_n.t' \equiv t$ satisfies the top-down property, the claim is easily shown by induction.

Hence, $u\theta\downarrow \xrightarrow{\varepsilon}{}^{*} u' \xrightarrow{\succ\varepsilon} t\sigma\downarrow$ and $u'|_i \twoheadrightarrow t\sigma\downarrow|_i$ satisfies the top-down property for any $i \in Pos(u')$. Therefore, $s\theta\downarrow \twoheadrightarrow t\sigma\downarrow$ satisfies the top-down property. □

Lemma 6 Any development has minimal top-down decomposition, the length of which is uniquely determined.

Proof. We show that any $s \twoheadrightarrow t$ has the top-down property. This can be proved inductively with respect to the definition of developments. Thus, Proposition 6 asserts that there exists a minimal top-down decomposition of the development for which the length is uniquely determined.

(A) Case in which $s \equiv a(s_1, \ldots, s_n) \twoheadrightarrow a(t_1, \ldots, t_n) \equiv t$ and $s_i \twoheadrightarrow t_i$: Developments $s_i \twoheadrightarrow t_i$ have the top-down property by induction. Since we can write the development $s \twoheadrightarrow t$ as $s \xrightarrow{\varepsilon}{}^{0} s \xrightarrow{\succ\varepsilon} t$, the development $s \twoheadrightarrow t$ has the top-down property.

(L) The case in which $s \equiv \lambda x_1 \cdots x_n.s' \twoheadrightarrow \lambda x_1 \cdots x_n.t' \equiv t$: By induction, development $s' \twoheadrightarrow t'$ has the top-down property. Thus, development $s \twoheadrightarrow t$ also has the top-down property.

(R') The case in which $s \equiv l\theta'\downarrow \twoheadrightarrow^{D'} l\theta\downarrow \xrightarrow{\varepsilon} r\theta\downarrow \equiv t$ where $\varepsilon \notin D'$: For any $F \in FV(l)$, the orthogonality of the HRS under consideration asserts that there exist developments $F\theta' \twoheadrightarrow F\theta$, which have the top-down property, by induction. A development $s \twoheadrightarrow r$ also has the top-down property because we can write $s \xrightarrow{\varepsilon} t$. Thus, by Lemma 19, the development $l\theta'\downarrow \twoheadrightarrow r\theta\downarrow$ has the top-down property. Therefore, the development has top-down decomposition by Proposition 6. □

A.5 Proof of Lemma 7

In order to prove Lemma 7, we must prepare three lemmas.

Lemma 20 Let F be a variable, let t be a term, let σ be a substitution, and let v and v' be positions such that $v' \prec v$. Then, the following hold:

(a) If $F\sigma|_{v'}$ is a redex, $\forall p \in PV(t, \sigma, F, v)$, $\exists p' \in PV(t, \sigma, F, v')$, $p' \prec p$.

(b) If $t|_{v'}$ is a redex, then $\forall p \in PT(t, \sigma, v)$, $\exists p' \in PT(t, \sigma, v')$, $p' \prec p$.

Proof. We prove (a) and (b) simultaneously by induction on $\langle t, \sigma\rangle$ with $>_{\rhd\beta}$. First, we consider (a). Let $P = PV(t, \sigma, F, v)$, and let $P' = PV(t, \sigma, F, v')$. We have six cases according to Definition 1.

(PV1) We have $P = \{v\}$ and $P' = \{v'\}$. Hence, the claim holds.

(PV2) Let $t \equiv a(t_1, \ldots, t_n)$ and $p \in P$. Then, we have $p = iq$ for some i and $q \in PV(t_i, \sigma, F, v)\}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$, we have $q' \prec q$ for some $q' \in PV(t_i, \sigma, F, v')$ by induction. Thus, we have $iq' \prec iq = p$ and $iq' \in P'$.

(PV3) Let $t \equiv \lambda x_1 \cdots x_n.t'$, we have $P = PV(t', \sigma', F, v)$ and $P' = PV(t', \sigma', F, v')$, where $\sigma' = \sigma|_{\overline{\{x_1, \ldots, x_n\}}}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, the claim follows from induction.

(PV4) Let $t \equiv G(t_1, \ldots, t_n)$ and $G\sigma \equiv \lambda y_1 \cdots y_n.t'$. Let $p \in P$, $Q = PV(t_i, \sigma, F, v)$, and $Q' = PV(t_i, \sigma, F, v')$. Then, $p \in PV(t', \sigma', y_i, q)$ for some $q \in Q$ and i, where $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$. Since $t\sigma >_{\rhd\beta} t_i\sigma$, we have $q' \prec q$ for some $q' \in Q'$ by induction. Since $t\sigma >_{\rhd\beta} t'\sigma'$, it follows from $q' \prec q$ by induction that $p' \prec p$ for some $p' \in PV(t', \sigma', y_i, q') \subseteq P$.

(PV5) Let $t \equiv F(t_1, \ldots, t_n)$ and $F\sigma \equiv \lambda y_1 \cdots y_n.t'$. We only check the subcase in which $p \in PT(t', \sigma', v)$, because the other subcase is similar to (PV4). $t'|_{v'}$ is a redex because $(F\sigma)|_{v'} \equiv (\lambda y_1 \cdots y_n.t')_{\_}v' \equiv t'|_{v'}$. It follows from $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$ by induction that $p' \prec p$ for some $p' \in PT(t', \sigma', v')$.

(PV6) It is obvious from $P = P' = \emptyset$.

Next, we consider (b). Let $P = PT(t, \sigma, v)$ and $P' = PT(t, \sigma, v')$. We have four cases according to Definition 2.

(PT1) $v' = \varepsilon$. Based on the assumption that $t|_{v'}$ is a redex, we have $t \equiv f(t_1, \ldots, t_n)$.Let $p \in P = PT(f(t_1, \ldots, t_n), \sigma, v)$. Then, from the definition of PT, we have $p \succ \varepsilon$. Thus, the claim follows from $P' = \{\varepsilon\}$.

(PT2) Let $v' = iw'$ and $t \equiv f(t_1, \ldots, t_n)$. We have $v = iw$ and $w' \prec w$ for some w. Let $p \in P$. Then, we have $p = iq$ for some $q \in PT(t_i, \sigma, w)$. Since

$\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$, and $t_i|_{w'}$ is a redex, we have $q' \prec q$ for some $q' \in PT(t_i, \sigma, w')$by induction. Thus, we have $iq' \prec iq = p$ and $iq' \in P'$.

(PT3) Let $t \equiv \lambda x_1 \cdots x_n.t'$. We have $P = PT(t', \sigma', v)$ and $P' = PT(t', \sigma', v')$, where $\sigma' = \sigma|_{\overline{\{x_1,\ldots,x_n\}}}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$ and $t|_{v'} \equiv t'|_v$, the claim follows from induction.

(PT4) Let $v' = iw'$, $t \equiv G(t_1, \ldots, t_n)$, and $G\sigma \equiv \lambda y_1 \cdots y_n.t'$. We have $v = iw$ and $w' \prec w$ for some w. Let $p \in P$. Then, we have $p \in PV(t', \sigma', y_i, q)$ for some $q \in PT(t_i, \sigma, w)$, where $\sigma' = \{y_1 \mapsto t_1\sigma\downarrow, \ldots, y_n \mapsto t_n\sigma\downarrow\}$. Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t_i, \sigma\rangle$, and $t_i|_{w'}$ is a redex, we have $q' \prec q$ for some $q' \in PT(t_i, \sigma, w')$ by induction. $(y_i\sigma')|_{q'} \equiv (t_i\sigma\downarrow)|_{q'}$ is a redex because $q' \in PT(t_i, \sigma, w')$ and $t_i|_{w'}$ is a redex from Lemma 16 (a). Since $\langle t, \sigma\rangle >_{\rhd\beta} \langle t', \sigma'\rangle$, it follows from $q' \prec q$ and induction that $p' \prec p$ for some $p' \in PV(t', \sigma', y_i, q')$. □

Lemma 21 Let $l \blacktriangleright r$ be a rewrite rule, let σ be a substitution, and let A be a development such that $A : l\sigma\downarrow \circ\!\!\!\rightarrow r\sigma\downarrow$. Let p and p' be redexes in $Pos(l\sigma\downarrow)$ such that $\varepsilon \prec p' \prec p$. Then, $\forall q \in p\backslash A, \exists q' \in p'\backslash A, q' \prec q$.

Proof. There exists $p_1 \in Pos_{FV}(l)$ such that $p' = p_1p_2$ and $p = p_1p_2p_2'$ for some p_2 and p_2' from orthogonality. From the definition of descendants, we have

$p'\backslash A = PV(r, \sigma, top(l|_{p_1}), p_2)$, and

$p\backslash A = PV(r, \sigma, top(l|_{p_1}), p_2p_2')$.

Here, $(top(l|_{p_1})\sigma)|_{p_2p'}$ is a redex. By Lemma 20, this claim holds. □

Lemma 22 Let $A : t \circ\!\!\!\rightarrow^{\{\varepsilon\}} t'$ be a development, Let B and D be sets of redex positions of t such that $D_{\nabla B}$, and let $\varepsilon \notin B$. Then, $(D\backslash A)_{\nabla(B\backslash A)}$.

Proof. For all $p \in D$, there exists $p' \in B$ such that $\varepsilon \prec p' \prec p$ from $D_{\nabla B}$. Therefore, this lemma holds from Lemma 21. □

Lemma 7. Let B be a set of redex positions of a term t, and let D and D' be sets of redex positions such that we can write them by $D_{\nabla B}$ and $D'_{\Delta B}$, respectively. Let $A_1 : t \circ\!\!\!\rightarrow^{D} t_1$ and $A_2 : t_1 \circ\!\!\!\rightarrow^{D'} t_2$ be developments. Then, there exist developments$A_3 : t \circ\!\!\!\rightarrow^{D'} t_3$ and $A_4 : t_3 \circ\!\!\!\rightarrow^{D''} t_2$ such that $D'' = D\backslash A_3$ can be written by $(D\backslash A_3)_{\nabla(B\backslash A_3)}$ for some t_3.

Proof. If $\varepsilon \in B$, the claim holds because D' is $\emptyset$. Consider the case in which $\varepsilon \notin B$. There exists a development A_3 corresponding to A_2. There exists development $A_4 : t_3 \circ\!\!\!\rightarrow^{D\backslash A_3} t_2$ from Lemma 3. We show that $(D\backslash A_3)_{\nabla(B\backslash A_3)}$ by induction on the definition of the development of A_3 (**Fig. 4**). The cases of (L) and (A) hold from induction. In the case of (R'), A_3 is divided into $A'; A''$, where $A' : t \circ\!\!\!\rightarrow^{D'-\{\varepsilon\}} t''$ and $A'' : t'' \circ\!\!\!\rightarrow^{\{\varepsilon\}} t_3$. Then, $(D\backslash A')_{\nabla(B\backslash A')}$ by induction. $((D\backslash A')\backslash A'')_{\nabla((B\backslash A')\backslash A'')}$ by Lemma 22. Therefore, from the definition of developments, $(D\backslash A'; A'')_{\nabla(B\backslash A';A'')}$. □

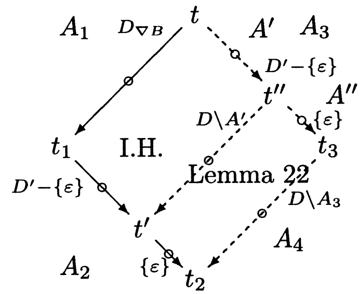

Fig. 4 Proof of Lemma 7.

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