Numerical Approach of Eigenvalue Problem of Helmholtz's Differential Equation by Utilizing Green Function

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1. *Introduction*

The eigenvalue problems expressed by Helmholtz's differential equation ($\Delta\varphi+\lambda\varphi=0$, $\varphi=0$ on boundary) can be solved for the case that the shape of boundary is easy to treat analytically. The Monte Carlo method utilizing discrete random walk process is effective for the case when the boundary is of arbitrary shape(1)-(6), but its solution has statistical fluctuations.

In this paper, to improve the weak point mentioned above, we employ the following procedures, that is, first we calculate the normal derivatives of Green function by using mass division method, and then obtain the Green function. Finally we calculate the eigenvalues of a matrix of Green function from integral equation equivalent to Helmholtz's differential equation. The matrix is symmetric, because of the reciprocity of the Green function.

This method is useful to calculate the higher order of eigenvalues more precisely than Monte Carlo method utilizing discrete random walk process.

2. *Relation between random walk process and eigenvalue problems*

Let M be a simply connected domain with a boundary Γ of an arbitrary shape and Ω be the interior of M as Fig. 1. Moreover let D be diffusion constant in M, if M is physically uniform.

At time $t=0$, a unit heat source is set at a pt. $P(x_0, y_0)$ in Ω, and the temperature on Γ is zero at all times. At time t, a function $u(x, y, t)$ representing the temperature at the pt. $Q(x, y)$ in Ω satisfies the following equation.

$$\frac{\partial u}{\partial t}=D\Delta u, \tag{1}$$

$$u(x, y, t)=\delta(x-x_0)\delta(y-y_0)\delta(t), \tag{2}$$

$$(u)_\Gamma=0. \tag{3}$$

In another expression, it satisfies(7),

$$u(x, y, t)=\sum_{j=1}^{\infty} e^{-\lambda_j Dt}\varphi_j(x_0, y_0)\varphi_j(x, y), \tag{4}$$

where λ_j and φ_j are the j-th eigenvalue and the j-th normalized (M is unit area) eigen-

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function respectively.

In order to show the relation between the eigenvalue problem and the discrete random walk process, we put M on a regular square net whose cell length is h. Let Γ^* be the net boundary and Ω^* be the interior of Γ^*, which are composed of links of the net.

Let $U(\mathrm{Q/P}, n\tau)$ be the function representing a probability that a random walk point (r. w. p.) starting from a pt. P on Ω^* at $t=0$ arrives at pt. Q(P$\neq$Q) on Ω^* at $t=n\tau$ (τ is constant). This function $U(\mathrm{Q/P}, n\tau)$ satisfies the diffusion equation approximately and is related to the function $u(x, y, t)$ as follows(7);

$$\lim_{\tau, h\to 0} \frac{U(\mathrm{Q/P}, n\tau)}{h^2} = u(x, y, t). \tag{5}$$

Let P($n\tau$) be a probability that a r. w. p. starting from a pt. P(x_0, y_0) on Ω^* does not arrive at any point on Γ^* till t=nτ. $P(n\tau)$ is given as follows(7);

$$P(n\tau) = \sum_{j=1}^{\infty} e^{-\lambda_j D n\tau} \varphi_j(x_0, y_0) \iint \varphi_j(x, y)\,dxdy. \tag{6}$$

The value of $\varphi_j(x_0, y_0)\iint\varphi_j(x, y)dxdy$ is constant when j is fixed, so the eigenvalue λ_j can be calculated from the above equation. This method having statistical fluctuations, it is less effective.

Now we introduce a Green function $G(x, y\,;\,\xi, \eta)$ such that

$$\Delta_{xy} G(x, y\,;\,\xi, \eta) = -\delta(x-\xi)\delta(y-\eta), \tag{7}$$

$$(G)_\Gamma = 0, \tag{8}$$

where $x_0=\xi$, $y_0=\eta$.

The relation between the eigenfunction φ_j and the Green function G is given(8),

$$G(x, y\,;\,\xi, \eta) = \sum_{j=1}^{\infty} c_j \varphi_j(x, y), \tag{9}$$

$$c_j = \iint \varphi_j(x, y) G(x, y\,;\,\xi, \eta)\,dxdy. \tag{10}$$

Now we introduce the following equation by using φ_j, λ_j and G,

$$\iint (G\Delta\varphi - \varphi\Delta G)\,dxdy = \iint (-\lambda\varphi G + \varphi\delta)\,dxdy = -\lambda \iint \varphi G\,dxdy + \varphi(\xi, \eta).$$

Since the left hand side of the above equation vanishes by the Green's theorem, we have

$$\varphi(\xi, \eta) = \lambda \iint \varphi(x, y) G(x, y\,;\,\xi, \eta)\,dxdy. \tag{11}$$

We represent the pt. (x, y) with k and the pt. (ξ, η) with l and for simplicity we write the functions as follows;

$$\varphi(x, y) = \phi_k, \quad \varphi(\xi, \eta) = \phi_l, \quad G(x, y\,;\,\xi, \eta) = G_{k,l}.$$

We represent Eq. (11) in difference type with $dx=dy=h$,

$$\phi_l = \lambda \sum_k \phi_k G_{kl} h^2 = \lambda \sum_k G_{lk} \phi_k h^2,$$

where $G_{kl}=G_{lk}$ from the reciprocity of the Green function.

If we express in matrix form, we have $[\boldsymbol{I}-\boldsymbol{G}h^2]\Phi=0$. Since $\Phi\neq 0$, we have

$$|\boldsymbol{I}-\lambda \boldsymbol{G}h^2|=0, \tag{12}$$

we may get the eigenvalues from the above equation without knowing the values of function Φ.

3. *Method for calculating Green function*

The relation between the random walk process and Green function was already made clear[1),(7)]. In this paper we take a physical view of random walk process.

Whenever a r. w. p. arrives at any point of Γ^*, it is absorbed there. This process is like the heat diffusion process that heat flows out of a boundary Γ. Let dE be a calorie which flows out of Γ through unit length ds on Γ during a unit time dt. We have[(10)]

$$dE=-dsD\frac{\partial u}{\partial n}dt, \tag{13}$$

where $\partial/\partial n$ denotes the normal derivative.

Let E be the total calories. From Eqs. (4) and (13), we have

$$E=-dsD\frac{\partial}{\partial n}\sum_{j=1}^{\infty}\varphi_j(x_0, y_0)\varphi_j(x, y)\int_0^{\infty}e^{-D\lambda_j t}dt$$
$$=-ds\frac{\partial}{\partial n}\sum_{j=1}^{\infty}\frac{\varphi_j(x_0, y_0)\varphi_j(x, y)}{\lambda_j}. \tag{14}$$

On the other hand, in random walk process, let $\omega(\mathrm{P}\to\mathrm{S})$ be a probability that a r. w. p. starting from a pt. $\mathrm{P}(x_0, y_0)$ on Ω^* arrives at pt. $\mathrm{S}(x_{\Gamma^*}, y_{\Gamma^*})$ on Γ^* without distinction of time. Using the probability U predescribed, $\omega(\mathrm{P}\to\mathrm{S})$ was represented as follows;

$$\omega(\mathrm{P}\to\mathrm{S})=1/4\sum_{n=0}^{\infty}U(\mathrm{Q}_h/\mathrm{P}, n\tau), \tag{15}$$

where Q_h is the point inner by h from pt. Q on Γ^*, and its co-ordinate is (x_h, y_h). From Eqs. (15), (4), (10) and (5) we have,

$$\omega(\mathrm{P}\to\mathrm{S})=1/(4\tau)\sum_{j=1}^{\infty}\varphi_j(x_0, y_0)\varphi_j(x_h, y_h)h^2,$$
$$\sum_{n=0}^{\infty}e^{-D\lambda_j n\tau}\tau\to\sum_{j=1}^{\infty}\frac{\varphi_j(x_0, y_0)\varphi_j(x_h, y_h)}{\lambda_j} \qquad (\tau, h\to 0), \tag{16}$$

where $\quad \sum_{n=0}^{\infty}e^{-D\lambda_j n\tau}\tau\to\int_0^{\infty}e^{-D\lambda_j t}dt \qquad (\tau, h\to 0).$

And we represent Eq. (14) in difference type,

$$E=-h\sum_{j=0}^{\infty}\frac{\varphi_j(x_0, y_0)}{\lambda_j}\frac{[\varphi_j(x_{\Gamma^*}, y_{\Gamma^*})-\varphi_j(x_h, y_h)]}{h}$$
$$=\sum_{j=0}^{\infty}\frac{\varphi_i(x_0, y_0)\varphi_j(x_h, y_h)}{\lambda_j}. \tag{17}$$

Since the identity of Eqs. (16) and (17), we have $\omega(\mathrm{P}\to\mathrm{S})=E$.

From Eqs. (9), (10) and (11), we have

$$G(x, y\,;\,\xi, \eta)=\sum_{j=1}^{\infty}\frac{\varphi_i(x, y)\varphi_j(\xi, \eta)}{\lambda_j}. \tag{18}$$

Here, we substitute Eq. (18) into Eq. (14) and set $\xi=x_0$, $\eta=y_0$, we have

$$\omega(\mathrm{P}\to\mathrm{S}) = -ds\frac{\partial G}{\partial n} = G(\mathrm{Q}_h, \mathrm{P}), \tag{19}$$

where we choose $\partial/\partial n$ so that ω is positive.

Now, we have following difference equations from Eqs. (17) and (8),

$$G(\mathrm{Q}, \mathrm{P}) = 1/4\sum_{i=1}^{4} G(\mathrm{Q}_i, \mathrm{P}) \qquad (\mathrm{P}\neq\mathrm{Q}) \quad \mathrm{P}, \mathrm{Q}\in\Omega^*, \tag{20}$$

$$G(\mathrm{P}, \mathrm{P}) = 1/4[\sum_{i=1}^{4} G(\mathrm{P}_i, \mathrm{P}) + 1] \qquad \mathrm{P}\in\Omega^*, \tag{21}$$

$$G(\mathrm{Q}, \mathrm{P}) = 0 \qquad \mathrm{Q}\in\Gamma^* \tag{22}$$

where Q_i and P_i are four neighbouring points of Q and P respectively. We may calculate the values of G by using Eqs. (20), (21) and (22).

There are several methods of numerical calculations to evaluate the function $\omega(\mathrm{P}\to\mathrm{S})$, such as (i) Monte Carlo method(7) (11), (ii) Mass division method(7) (Explosive method(6)). The method (i) is stochastic, on the contrary the method (ii) is deterministic. According to the purpose of this paper, we adopt the latter.

A unit mass being initially set at pt. P is divided into the four neighbouring points with equal weight (1/4). Next, each of the mass is divided into the four neighbouring points respectively. This process is continued one by one. If a piece of mass arrives at any point of Γ^*, it is absorbed there. It is known that $\omega(\mathrm{P}\to\mathrm{S})$ is the sum of mass which has been absorbed at pt. S. This process being continued infinitely, we stop dividing the mass at a proper amount of division. Accordingly the undivided masses remain on Ω^*.

Let $M(\mathrm{P}\to\mathrm{S}, \mathrm{m})$ be the mass that arrives at pt. S on Ω^* at the m-th division. From Eq. (14), we have

$$M(\mathrm{P}\to\mathrm{S}, m) = D\sum_{j=1}^{\infty}\varphi_j(\mathrm{P})\varphi_j(\mathrm{Q}_h)e^{-D\lambda_j m\tau}\tau = \sum_{j=1}^{\infty}k_j e^{-D\lambda_j m\tau}, \tag{23}$$

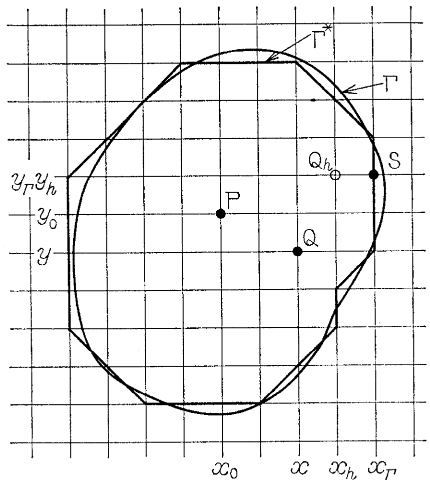

Fig. 1. Continuous boundary Γ of a simply connected domain M and discrete boundary Γ^*.

where k_j is a constant. Since the eigenvalues have the relation $\lambda_1 \leq \lambda_2 \leq \ldots$, $M(\mathrm{P} \rightarrow \mathrm{S}, \mathrm{m})$ decreases in proportion to exp $(-D\lambda_1 m\tau)$ in large m. So we can estimate the total mass that will be absorved, and we can get the approximate value of $\omega(\mathrm{P} \rightarrow \mathrm{S})$.

4. *Examples of calculation*

We choose a simple example shown in Fig. 2 for evaluating the applicability of this method. First, in Table 1 the probability $\omega(\mathrm{P} \rightarrow \mathrm{S})$ is shown. In this case the values of ω are exactly calculated by this method. We have confirmed the correspondence of the computed values and the exact values. In Table 2 the values of Green function G are shown. In Table 3. the eigenvalues are shown. By the usual Monte Carlo method, we got nothing but λ_1.

Secondly, in Table 3 a part of eigenvalues for the case that a side of squares is divided

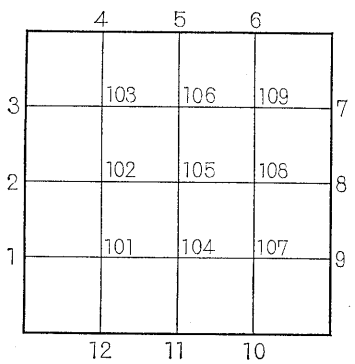

Fig. 2. An example of square boundary.

Table 1. $\omega(\mathrm{p} \rightarrow \mathrm{S})$ of square boundary shown in Fig. 1.

P \ S	1	2	3	4	5	6	7	8	9	10	11	12	
101	67	22	7	7	6	3	3	6	7	7	22	67	
102	22	74	22	22	14	6	6	10	6	6	14	22	
103	7	22	67	67	22	7	7	6	3	3	6	7	
104	22	14	6	6	10	6	6	14	22	22	74	22	
$\omega=$105	14	28	14	14	28	14	14	28	14	14	28	14	$\times\frac{1}{224}$
106	6	14	22	22	74	22	22	14	6	6	10	6	
107	7	6	3	3	6	7	7	22	67	67	22	7	
108	6	10	6	6	14	22	22	74	22	22	14	6	
109	3	6	7	7	22	67	67	22	7	7	6	3	

Table 2. $G(\mathrm{P} \rightarrow \mathrm{Q})$ of square boundary shown in Fig 1.

P \ Q	101	102	103	104	105	106	107	108	109	
101	67	22	7	22	14	6	7	6	3	
102	22	74	22	14	28	14	6	10	6	
103	7	22	67	6	14	22	3	6	7	
104	22	14	6	74	28	10	22	14	6	
$G=$105	14	28	14	28	84	28	14	28	14	$\times\frac{1}{224}$
106	6	14	22	10	28	74	6	14	22	
107	7	6	3	22	14	6	67	22	7	
108	6	10	6	14	28	14	22	74	22	
109	3	6	7	6	14	22	7	22	67	

Table 3. Eigenvalues of square by this method.

	Theo. value	Expe. value		
		division number		
		4	5	6
λ_1	19.74	17.87	19.10	19.29
λ_2	49.35	41.37	44.10	45.65
λ_3	49.35	41.37	44.10	45.65
λ_4	78.96	64.00	69.10	72.00
λ_5	98.70	64.00	75.00	81.65
λ_6	98.70	64.00	75.00	81.65
λ_7	128.3	86.63	100.0	108.0
λ_8	128.3	86.63	100.0	108.0
λ_9	167.8	110.1	100.0	117.6
λ_{10}	167.8		100.0	117.6

Monte Carlo estimate λ_1 is 22.2 (devision number is 4).

Table 4. A part of 69 eigenvalues of circle dividing the radius into 10 equal parts by this method.

	Theo. value	Expe. value		Theo. value	Expe. value
λ_1	18.17	18.02	λ_9	154.6	137.9
λ_2	46.12	44.74	λ_{10}	154.6	137.9
λ_3	46.12	44.74	λ_{11}	180.9	152.9
λ_4	82.86	76.81	λ_{12}	180.9	160.9
λ_5	82.86	79.27	λ_{13}	222.6	179.2
λ_6	96.73	89.29	λ_{14}	222.6	194.5
λ_7	127.9	115.7	λ_{15}	235.3	197.5
λ_8	127.9	115.7			

Monte Carlo estimate λ_1 is 18.5 (the starting point P is the center).

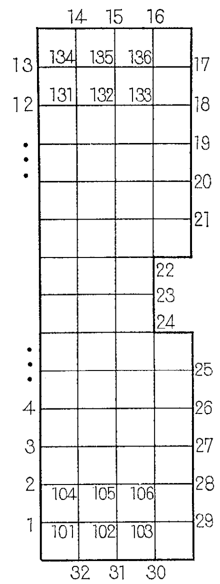

Fig. 3. An exmple of arbitrary shaped boundary.

Table 5. A part of 36 eigenvalues of figure shown in Fig. 4 by this method.

	Expe. value		Expe. value
λ_1	41.92	λ_6	117.6
λ_3	44.07	λ_7	122.2
λ_3	64.96	λ_8	125.5
λ_4	77.55	λ_9	141.7
λ_5	95.94	λ_{10}	154.9

In this case the rank of matrix is 36.

into five and six equal parts are shown. Further, in Table 4 a part of eigenvalues of a circle to be divided the radius into ten equal parts are shown. Last for the case of an arbitrary shape shown in Fig. 3, a part of 36 eigenvalues are shown in Table 5.

5. *Conclusion*

By this method we may improve a little the weak point of Monte Carlo method utilizing discrete random walk process, for we may calculate the values of Green function using the relation of Eqs. (19), (20), (21) and (22). Further we can calculate the eigenvalues of arbitrary shape more precisely, but the undivided masses remain on Ω^* and accordingly error exists.

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