On the Approximation of a Function of two Variables by the Product

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1. *The origin and the formulation of the problem*

In the solution of the boundary value problem of partial differential equations, some authors use a kind of "alternate direction approximation" ([1]). In the case of two variables, first assume that the solution $u(x, y)$ is of the form of the product $f(x)\cdot g(y)$. Solve an ordinary differential equation of $f(x)$ for a suitable $g_0(y)$ and $y=y_0$. Then solve the equation of $g(y)$ for the above solution $f_1(x)$ and suitable $x=x_0$. Repeating the process, the sequences $f_n(x)$, and $g_n(y)$ are often rapidly convergent and the method gives fairly nice approximation of the solution. In the present paper, the author would like to discuss the rationalization of the method from a stand point of the approximation of $u(x, y)$ by the product $f(x)\cdot g(y)$.

Suppose that there is given a square integrable function $u(x, y)$ (not necessarily continuous) in the domain $D=\{-a\leqq x\leqq a,\ -b\leqq y\leqq b\}$. The problem is to determine the square integrable functions $f(x)$ and $g(y)$ in $-a\leqq x\leqq a$ and in $-b\leqq y\leqq b$ respectively, which minimizes the square norm

$$J(f, g)=\int_{-b}^{b}\int_{-a}^{a}[u(x, y)-f(x)\cdot g(y)]^2 dxdy \tag{1}$$

If $a=b$ and $u(x, y)=u(y, x)$, we may assume that $g(x)=cf(x)$ (c : constant) which is called the *symmetric problem*. There is no loss of generality if we assume $c=+1$ or -1 in the symmetric case.

2. *The characterization of the least square approximation*

Let us denote by H_X, H_Y and H_Q the Hilbert spaces of all square integrable functions on the interval $-a\leqq x\leqq a$, $-b\leqq y\leqq b$ and the rectangle D respectively. Define the integral transforms:

$$\left.\begin{aligned} f(x)\to\tilde{f}(y)&=U_1 f=\int_{-a}^{a}u(x, y)f(x)dx,\\ g(y)\to\tilde{g}(x)&=U_2 g=\int_{-b}^{b}u(x, y)g(y)dy \end{aligned}\right\} \tag{2}$$

and their iterations:

$$V_X=U_2\circ U_1 : H_X\to H_X,\quad V_Y=U_1\circ U_2 : H_Y\to H_Y.$$

They all are compact (complete continuous) operators ([2], [3]). In the symmetric case,

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the operator $U_1 = U_2$ has the same property.

By an elementary discussion on the functional analysis, it is easy to prove the existence of the least square approximations, and more precisely:

Theorem 1. The least square approximation $f(x), g(y)$ are the eigenfunctions with respect to the (common positive) largest eigenvalue λ of the operators V_X, V_Y such that $\|f\|^2{}_X \cdot \|g\|^2{}_Y = \lambda$.

Corollary 1. A necessary and sufficient condition for the uniqueness (up to constant factor) of the least square approximation is the simplicity of the eigenvalue λ.

Corollary 2. A necessary and sufficient condition for the exactness of the approximation by product is that V_X, V_Y has only one positive (simple) eigenvalue other than 0.

Theorem 2. In the symmetric case ($c = \pm 1$), the least square approximation $f(x)$ is the eigenfunction of $U_1 (= U_2)$ with respect to the largest (in absolute value) eigenvalue λ with $\|f\|^2 = |\lambda|$. In this case, c has the same signature as λ. The residue $J(f, cf)$ is $\sum \lambda_i^2$ where the summation is for all eigenvalues λ_i except the largest one λ.

Therefore, the approximation will be fairly nice if V_X, V_Y has an eigenvalue λ extremely large comparing with the other eigenvalues. The estimations of the eigenvalues are possible in the various manner ([4], [5]). The above theorems may indicate how accurately we can approach the exact solution by the alternate direction approximation.

3. *Expamples*

Example 1. In the symmetric case, we put $a=1$.

$$u(x, y) = 1 - \max(|x|, |y|).$$

It is easy to see that the integral operator U_1 has the eigenvalues

$$\lambda = \lambda_n = 8/\pi^2(2n-1)^2, \quad n = 0, 1, 2, \dots$$

and the eigenfunctions are

$$f_n(x) = \frac{2\sqrt{2}}{\pi(2n-1)} \cos\left(\frac{\pi}{2}(2n-1)x\right).$$

The best approximation is

$$f_1(x) \cdot f_1(y) = \frac{8}{\pi^2} \cos\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}y\right)$$

and the ratio of the residue to λ_1 is $(\pi^4/96 - 1)^{1/2} = 0.121$. The approximation of $u(x, y)$ seems fairly good.

Example 2. Let us consider the Poisson equation

$$\Delta u = -1, \text{ the boundary value} = 0, \quad a = b = 1.$$

We may take an approximative solution (exact solution for the differential equation and the boundary value is less than 1/160; an analogue of an example in the book of Collatz [6]):

$$\frac{47}{160} - \frac{1}{4}(x^2 + y^2) - \frac{1}{20}(x^4 + y^4 - 6x^2y^2).$$

The eigenvalues are the solution of an algebraic equation

$$\mu^3/7 - (401/8)\mu^2 + 267\mu + 512 = 0, \quad \mu = 1050\lambda,$$

which gives $\mu = 345.4,\ 6.94,\ -1.50$. The first one is extremely larger than the others and so one may expect that the approximation is fairly nice. In fact, the numerical approximation of the largest eigenvalue is 0. 369 which is close to the above approximation 345. 4/1050=0. 329. The solution by alternate direction approximation coincides with the exact solution up to the relative error by 2%, and by 1% at the central part of the domain.

Example 3. As an asymmetric case, let us consider the above Poisson equation in the rectangle $a=0.5$, $b=0.8$ with the discrete approximation by the lattice with 0. 1. The eigenvalues of the matrix approximating the operators V_X, V_Y are 0. 55, 6.85×10^{-5}, 2.40×10^{-6} and all others are 0. The solutin by the alternate direction approximation coincides with the exact solution up to the relative error by 1%.

References

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