About the Outlines of Three Dimensional Objects in Computer Graphic Processing

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1. *Introduction*

Recently the "hidden-line" problem shown by I. E. Sutherland [1] becomes more important in computer graphic processing. A few thesises [2], [3] told us that this problem has some relation to the "outline", "intersection" and "shadow" problems. In our previous paper [5], we gave the basic idea about the solution of the hidden-line problem and also got a conclusion that it is very difficult to get the outlines of arbitrary surfaces.

This paper will give a simple intuitive concept, the difinitions and the nature of the outlines of various surfaces. We will give an approximate computation of outlines by using the nature of the neighborhoods of strict outlines. Also we will give an approximate computation of intersections between surfaces. Our computing method given in this paper is not point-by-point computing, but line-by-line computing.

2. *Basic Definitions*

It is likely for us to have a thought that the concept of the outlines of three dimensional objects is trivial. But we need a strict definition of outlines with mathematical representation, because we want to have the formulation and the computing method of outlines for computers. Otherwise it is very difficult to represent completely every concept we can know intuitively. So we will stand on the following points.

1. The mathematical representation of surfaces is given easily.
2. The given concept is little different from the intuitive concept of natural outlines.
3. Outlines can be computed only from the nature of surfaces and a given visual point.

Now we let Σ a set of points whose subsets are surfaces $S_1, S_2, \cdots, S_n$. $\overrightarrow{PX}$ is a line of sight between P and $X \in \Sigma$. Using $\Sigma^*(X) = \overrightarrow{PX} \cap \{\Sigma - \{X\} - \{P\}\}$, we have the following definitions.

Definition 1 visibility

If $\Sigma^*(X) \neq \Phi$, then the point X is invisible for Σ. If $\Sigma^*(X) = \Phi$, then the point

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X is visible for Σ.

Definition 2 suspected outlines

The point X on any surface S is an element of the suspected outline of the surface S, if X satisfies at least one of the following three conditions.

Condition 1 There is no tangent plane with the first order derivative form of S.

Condition 2 There is a tangent plane with the first order derivative form of S, and it contains the line of sight through the point X.

Condition 3 There is an intersection of surfaces at the point X.

Definition 3 Outlines

The set of visible points in the suspected outlines is called real outlines.

Now we can patition our problem into two problem. One is the investigation of the conditions of suspected outlines. The other is the formulation of the computing method of visibility. The latter is discussed in the paper [5]. So in this paper, we discuss about the former.

3. *The Representation of Surfaces and a Line of Sight in 3-D Space*

The representation of a surface S_i $(i=1, 2, \cdots, n)$ is given in two ways. One is direct variable representation as follows; the nature of a surface is given in the form of

$$Q(x, y, z)=0, \tag{1}$$

and its boundary condition is given in the form of

$$q_i(x, y, z)\leq 0 \qquad i=1, 2, \cdots, r. \tag{2}$$

The other is parametric representation. The nature is given in the form of

$$R=F(u, t), \tag{3}$$

using 3-D vectors $R=(x, y, z)^t$, $F(u, t)=(f_1(u, t),\ f_2(u, t),\ f_3(u, t))^t$, and its boundary condition is generally given in the form of

$$q_i(u, t)\leq 0 \qquad i=1, 2, \cdots, m, \tag{4}$$

or for formalization and simplicity

$$0\leq u, t\leq 1. \tag{5}$$

A line of sight is represented in following three ways. When a visual point P which has finite coordinates is given, it is represented as follows;

$$R=(P-X)s_1+X \qquad 0<s_1<1, \tag{6}$$

or

$$R=(X-P)s_2+P \qquad 0<s_2<1. \tag{7}$$

Each representation is different each other for the same line. Between s_1 and s_2, there is a relation

$$s_1+s_2=1. \tag{8}$$

When the visual point exists at infinity and we have its direction cosine vector Λ, it is represented as follows;

$$R=\Lambda s+X \qquad 0<s. \tag{9}$$

The formers show a group of lines of sight in an opened-up view from one point and the latter shows a group of parrallel lines of sight.

4. *Investigation of Conditions of Suspected Outlines*

4.1 Definitions and the Nature of Tangent Planes

Now we concider the representation of tangent planes of the surfaces represented with the eq. (1). If there exist continuous derivative coefficients of the first order in the neighborhood of $X=(X_1, X_2, X_3)$ on the surface Q, then Taylor expansion shows that

$$Q(X_1+h_1, X_2+h_2, X_3+h_3)=\sum_{m=0}^{n-1}\frac{1}{m!}\left(h_1\frac{\partial}{\partial h_1}+h_2\frac{\partial}{\partial h_2}+h_3\frac{\partial}{\partial h_3}\right)^m Q(X_1, X_2, X_3)$$
$$+\frac{1}{n!}\left(h_1\frac{\partial}{\partial h_1}+h_2\frac{\partial}{\partial h_2}+h_3\frac{\partial}{\partial h_3}\right) Q(X_1+\theta h_1,\ X_2+\theta h_2,\ X_3+\theta h_3)$$
$$0<\theta<1. \tag{10}$$

We neglect a extremely small amount of higher order than 2. The approximate surface resembled with a small amount of the first order in the neighborhood of the point X is called a tengent plane. And its representation is given as follows;

$$\sum_{i=1}^{3}\frac{\partial Q(X_1, X_2, X_3)}{\partial X_i}h_i=0. \tag{11}$$

This is a form of first order derivatives of a surface Q. If

$$\frac{\partial Q(X_1, X_2, X_3)}{\partial X_i}=0 \qquad i=1, 2, 3,$$

the eq. (12) losses its meaning and the point X is called a singular point. At a singular point there is no tangent plane which has a form of the first order derivatives. But if we consider the second order derivatives, there are often one or two tangent planes as follows;

$$\sum_{i,j=1}^{3}\frac{\partial^2 Q(X_1, X_2, X_3)}{\partial X_i \partial X_j}h_i h_j=0. \tag{12}$$

And if we consider the third order derivatives, so on$\cdots$. In this paper, we call only the tangent plane of the first order derivatives as the tangent plane.

Now we consider about the tangent plane of the surface with parametric representation of the eq. (3). If the whole of tangent vectors at the point X on the surface S is the two dimensional vector space (a plane), we call this space as the tangent plane at the point on the surface S. This definition requests that each coordinate component is first order partial differentialable for parameters u and t, and that some function determinant for two coordinate components is unequal to zero. For example, for the coordinate x and y,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} \end{vmatrix} \neq 0. \tag{13}$$

We will represent such a tangent plane using vector notation as follows;

$$R=\frac{\partial F(u, t)}{\partial u}\eta+\frac{\partial F(u, t)}{\partial t}\xi+F(u, t). \tag{14}$$

We can rewrite the eq. (14) into the following direct variable representation;

$$\begin{vmatrix} f_1(u, t)-x & f_2(u, t)-y & f_3(u, t)-z \\ \frac{\partial f_1(u, t)}{\partial u} & \frac{\partial f_2(u, t)}{\partial u} & \frac{\partial f_3(u, t)}{\partial u} \\ \frac{\partial f_1(u, t)}{\partial t} & \frac{\partial f_2(u, t)}{\partial t} & \frac{\partial f_3(u, t)}{\partial t} \end{vmatrix} =0. \tag{15}$$

If any function determinant is equal to zero, we know from the eq. (15) that the point X is a singular point, that the whole tangent vectors degenerates into a point or one dimensional space and that there is no tangent plane. Thus this two dimensional vector space consisted of the whole of tangent vectors can be called the tangent plane with the first order derivatives.

4.2 Investigation of the Condition 1

The following four cases are considered under the condition 1; for a given surface,

(a) there exists no surface which is adjacent to it and owns its boundary curves jointly.

(b) there exists a surface which is adjacent to it, own its boundary curves jointly and connects it unsmoothly.

(c) for notation variable——x, y and z——, or for parameters——u and t——, there exists no first order patial differential coefficients.

(d) it has singular points.

Usually we have our operation of computing outlines after synthesis and connection of surfaces. So we can previously know whether a given surface satisfies the conditions (a) and (b). In our system, we usually consider several fixed kinds of the representation of surfaces. So we can previously know the form of first order partial differential coefficients. And because the case (d) is considered as a special case of the condition 2, we do not discuss about it in detail. Finally we omit the discussion of the condition 1 in detail.

Example 1 outlines of a triangular

A triangular is represented with two parameters w and t as follows;

$$R=Awt+Bt+C \qquad 0\leq w,\ t\leq 1. \tag{16}$$

The three lines $R(w=1,\ 0\leq t\leq 1)$, $R(w=0,\ 0\leq t\leq 1)$ and $R(t=0,\ 0\leq w\leq 1)$ are the suspected outlines in the meaning of the condition 1.

Example 2 degeneration of a tangent plane

When a surface is given in the form of

$$\begin{cases} x=\left(u-\dfrac{1}{2}\right)^2\left(t-\dfrac{1}{3}\right) \\ y=\left(u-\dfrac{1}{2}\right)^2 \\ z=u+t \qquad\qquad 0\leq u,\ t\leq 1, \end{cases} \tag{17}$$

the line $R(u=1/2, (0\leq t\leq 1)$ is the set of singular points and considered as a suspected outline in the meaning of the condition 1.

4.3 Investigation of the condition 2——direct variable represention——

When a surface is given in the form of the eq. (1), the equation of the tangent plane passing through the point (X_1, X_2, X_3) on the surface is derived from the eq. (11) as follows;

$$\frac{\partial Q(X_1, X_2, X_3)}{\partial X_1}(x-X_1)+\frac{\partial Q(X_1, X_2, X_3)}{\partial X_2}(y-X_2)+\frac{\partial Q(X_1, X_2, X_3)}{\partial X_3}(z-X_3)=0. \tag{18}$$

The point is an element of suspected outlines if the eq. (18) contains the line of sight. When we use the eq. (9) as the line of sight, the equation of the suspected outlines is given as follows;

$$\begin{cases} \Lambda_1\dfrac{\partial Q}{\partial x}+\Lambda_2\dfrac{\partial Q}{\partial y}+\Lambda_3\dfrac{\partial Q}{\partial z}=0 \\ Q(x, y, z)=0, \end{cases} \tag{19}$$

where it has the boundary condition of the eq. (2). When we use the eq. (6) as the lines of sight, the equation of suspected outlines is given as follows;

$$\begin{cases} (P_1-x)\dfrac{\partial Q}{\partial x}+(P_2-y)\dfrac{\partial Q}{\partial y}+(P_3-z)\dfrac{\partial Q}{\partial z}=0 \\ Q(x, y, z)=0, \end{cases} \tag{20}$$

where it has the boundary condition of the eq. (2).

Example 3 suspected outlines of a quadric surface

A quadric surface

$$Q(x, y, z)=Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz +J=0 \tag{21}$$

has a linear equation of x, y and z corresponding to the former of the eq. (19), and a quadric equation of x, y and z corresponding to the former of the eq. (20)

4.4 Investigation of the Condition 2——parametric representation——

When a surface is given in the form of the eq. (3), the equation of the tangent plane passing through the point $(X_1=f_1(u, t),\ X_2=f_2(u, t),\ X_3=f_3(u, t))$ on the surface from the eq. (14) as follows;

$$\begin{cases} x=f_{1,u}\ \eta+f_{1,t}\ \xi+f_1 \\ y=f_{2,u}\ \eta+f_{2,t}\ \xi+f_2 \\ z=f_{3,u}\ \eta+f_{3,t}\ \xi+f_3 \quad , \end{cases} \tag{22}$$

where

$$f_{i,u}=\frac{\partial f_i(u,t)}{\partial u},\quad f_{i,t}=\frac{\partial f_i(u,t)}{\partial t}\quad (i=1,2,3). \tag{23}$$

When we use the eq. (9) as a line of sight, we get

$$J\begin{pmatrix} s \\ \eta \\ \xi \end{pmatrix}+X=0, \tag{24}$$

where

$$J\begin{pmatrix} -A_1 & f_{1,u} & f_{1,u} \\ -A_2 & f_{2,u} & f_{2,u} \\ -A_3 & f_{3,u} & f_{3,u} \end{pmatrix}. \tag{25}$$

When we use the eq. (6) as a line of sight, we get the eq. (24), where

$$J=\begin{pmatrix} -P_1+f_1 & f_{1,u} & f_{1,t} \\ -P_2+f_2 & f_{2,u} & f_{2,t} \\ -P_3+f_3 & f_{3,u} & f_{3,t} \end{pmatrix}. \tag{26}$$

Now we show that the condition of suspected outlines is given from the nature of the following $|J|$.

Theorem 1

If for a surface S with some parametric representation there exists the pair (u, t) satisfying

$$|J|=0 \qquad 0\leq u,\ t\leq 1, \tag{27}$$

the set of the points on S corresponding to the pair is a set of suspected outlines.

⟨proof⟩ When we exchange each row of J and make J^* as follows;

$$J^*=\left(\begin{array}{c|cc} J_{1\,1} & & \\ J_{1\,2} & J_s^* & \\ \hline J_{1\,3} & J_{2\,3} & J_{3\,3} \end{array}\right), \tag{28}$$

$|J_s^*|$ is one of function determinants. If for each J_s^* the equation

$$|J_s^*|=0 \tag{29}$$

follows, the eq. (27) is satisfied. And then there is no tangent plane and exists the set of singular points. So we get the set of suspected outlines in the meaning of the condition 1. If for some J^* there exists J_s^* not satisfying the eq. (29) and the eq. (27) follows, there exists a tangent plane but we can not decide the solution of the eq. (24) uniquely. So all real number is admitted to the variable s. This fact shows that the line of sight is imbedded into the tangent plane. Thus we get a set of suspected outlines in the meaning of the condition 2. Q. E. D.

Example 4 a suspected outline of some quadric surface

A quadric surface is represented as follows;

$$R=F(u,t)=Au^2+Bt^2+Cut+Du+Et+F. \tag{30}$$

Then corresponding to the eq. (25) and the eq. (27), we get quadric polynominals of u and t. Corresponding to the eq. (26) and the eq. (27), we get cubic polynominals of u and t.

4.5 Investigation of the Condition 3——Intersections——

There is a case where a surface intersects in itself and another case where surfaces intersect each other. If the surfaces are represented with the eq. (1) or the eq. (3), their intersection is the solution of the simultaneous equations whose each equation shows the nature of each surface. But it is too difficult to compute their intersection concretely. When a surface intersects in itself, if we use the direct variable representation, the intersection is the set of singular points. And if we use the parametric representation, we often get some tangent plane. It is very difficult to compute those self-intersections. But we can compute approximate self-intersections by the way of the section 5-3; in the manner of computing usual approximate intersections.

Example 5

A surface with the following parametric representation

$$\begin{cases} x=2\sin(2\pi t) \\ y=4\cos^3(2\pi t)-\cos(2\pi t) \\ z=u \qquad\qquad 0\leq u,\ t\leq 1 \end{cases} \tag{31}$$

has self-intersections; lines ($x=\pm 1$, $y=0$, $0\leq z\leq 1$). We rewrite the eq. (31) and obtain the following direct variable representation;

$$\begin{cases} Q(x, y, z)=16y^2+(x^2-1)^2(x^2-4)=0 \\ -2\leq x\leq 2,\ \ -3\leq y\leq 3,\ \ 0\leq z\leq 1. \end{cases} \tag{32}$$

In this representation, the lines ($x=\pm 1$, $y=0$, $0\leq z\leq 1$) are the set of singular points.

5. *Approximate suspected outlines*

5.1 Approximation of surfaces with triangulars

One of the generally-known parametric representations of triangulars is given as follows;

$$R=Awt+Bt+C \qquad 0\leq w,\ t\leq 1 \tag{33}$$

where

$$A=X_1-X_2,\ \ B=X_2-X_3,\ \ C=X_3\ \ (X_i\text{; vertixes}).$$

Using these triangulars, we can easily approximate the surface with the parametric representation of the eq. (3).

5.2 Approximate suspected outline

We will compute approximate suspected outlines using the nature of the neighborhoods of strict suspected outlines. Now we consider the point X on the suspected outlines in the meaning of the condition 2, a tangent plane dS passing through $X=(X_1, X_2, X_3)$, and a visual point $P=(P_1, P_2, P_3)$. If dS has the following;

$dS(x, y, z)=0,$

then the next equation follows;

$$ds(X_1, X_2, X_3)=dS(P_1, P_2, P_3)=0. \tag{34}$$

In addition, this plane dS partitions the whole space into two parts. So for any two points Y_1 and Y_2 existing on the opposite side each other on the surface S and existing in the arbitrary neighborhoods of X, at least one of the following equations follows;

$$\begin{cases} dS(Y_{1\,1}, \ Y_{1\,2}, \ Y_{1\,3})\ dS(Y_{2\,1}, \ Y_{2\,2}, \ Y_{2\,3})>0 \\ dS(Y_{1\,1}, \ Y_{1\,2}, \ Y_{1\,3})\ dS(Y_{2\,1}, \ Y_{2\,2}, \ Y_{2\,3})=0 \\ dS(Y_{1\,1}, \ Y_{1\,2}, \ Y_{1\,3})\ dS(Y_{2\,1}, \ Y_{2\,2}, \ Y_{2\,3})<0. \end{cases}$$

The first means the satisfactry condition 2. The second means that in the neighborhood of X, the surface S is equal to the plane dS. We consider this case as a special case that there exist suspected outlines. The third means no special object we want to describe.

Using these facts, we obtain approximate computation of suspected outlines. We consider a very small line $\overline{X_1X_2}$ on the surface S and the points X_3, X_4 on the surface S in the neighborhood of $\overline{X_1X_2}$. Then triangulars $\Delta X_1X_2X_3$, $\Delta X_1X_2X_4$ consist of approximate surfaces. We will decide whether $\overline{X_1X_2}$ is a suspected outline or not.

The equation of the plane passing through the line $\overline{X_1X_2}$ and containing a line of sight of the eq. (9) is represented as follows;

$$R=\Lambda s+(X_1-X_2)\eta+X_2 \qquad -\infty<s, \ \eta<+\infty. \tag{35}$$

If we use the eq. (6), the equation of the plane is represented as follows;

$$R=(P-X_1)s+(X_1-X_2)\eta+X_2 \qquad -\infty<s, \ \eta<+\infty. \tag{36}$$

These two equations are of the same type. So we use the eq. (35) in the following discussion.

We rewrite the eq. (35) and obtain the equation

$$Q(x, y, z)=Ax+By+Cz+D=0, \tag{37}$$

where

$$\begin{cases} A=\Lambda_2(X_{1\,3}-X_{2\,3})-\Lambda_3(X_{1\,2}-X_{2\,2}) \\ B=\Lambda_3(X_{1\,1}-X_{2\,1})-\Lambda_1(X_{1\,3}-X_{2\,3}) \\ C=\Lambda_1(X_{1\,2}-X_{2\,2})-\Lambda_2(X_{1\,1}-X_{2\,1}) \\ D=\Lambda_1(X_{2\,2}X_{1\,3}-X_{2\,3}X_{1\,2})+\Lambda_2(X_{1\,1}X_{2\,3}-X_{1\,3}X_{2\,1}) \\ \quad +\Lambda_3(X_{1\,2}X_{2\,1}-X_{1\,2}X_{2\,2}). \end{cases} \tag{38}$$

Thus we get the following criterion.

Theorem 2

When two triangulars $\Delta X_1X_2X_3$, $\Delta X_1X_2X_4$ which have the line $\overline{X_1X_2}$ in commen, if $Q(X_{3\,1}, X_{3\,2}, X_{3\,3})\ Q(X_{4\,1}, X_{4\,2}, X_{4\,3})>0$, the line $\overline{X_1X_2}$ is an approximate suspected outline.

⟨proof⟩ Trivial. Q. E. D.

This criterion is a line-by-line, but not point-by-point criterion.

5.3 Approximate Intersections

It is sufficient if we can develop the method to obtain the intersect-lines of arbitrary triangulars, because arbitrary surfaces can be partitioned into many triangulars approximately.

Now these two arbitrary triangulars R_1, R_2 are represented as follows;

$$R_1=A_1wt+B_1t+C_1 \qquad 0\leq w,\ \ t\leq 1 \tag{39}$$

$$R_2=A_2w't'+B_2t'+C_2 \qquad 0\leq w',\ \ t'\leq 1. \tag{40}$$

We suppose that these do not exist on one plane simultaneously. We can easily obtain an algorithm how we investigate whether they exist on one plane simultaneously or not. So we use this supposition without discussion in detail. We introduce auxiliary variables $s_i(i=1, 2)$. We can get $\Lambda_i(i=1, 2)$ usually such that each determinant of

$$Ji=\begin{pmatrix} \Lambda_{i\,1} & A_{i\,1} & B_{i\,1} \\ \Lambda_{i\,2} & A_{i\,2} & B_{i\,2} \\ \Lambda_{i\,3} & A_{i\,3} & B_{i\,3} \end{pmatrix} \qquad (i=1, 2)$$

is not equal to zero, where

$$\Lambda_1s_1+A_1wt+B_1t=A_2w't'+B_2t'+C_2-C_1 \tag{41}$$

$$\Lambda_2s_2+A_2w't'+B_2t'=A_1wt+B_1t+C_1-C_2. \tag{42}$$

Then we obtain the unique solution $s_i(i=1, 2)$ of the eq. (41) and the eq. (42) as follows;

$$s_1=-\frac{1}{|J_1|}\{D_1(A_2)w't'+D_1(B_2)t'+D_1(C_2-C_1)\} \tag{43}$$

$$s_2=-\frac{1}{|J_2|}\{D_2(A_1)wt+D_2(B_1)t+D_2(C_1-C_2)\}, \tag{44}$$

where

$$D_i(V)=\begin{vmatrix} V_1 & A_{i\,1} & B_{i\,1} \\ V_2 & A_{i\,2} & B_{i\,2} \\ V_3 & A_{i\,3} & B_{1\,3} \end{vmatrix} \qquad (i=1, 2). \tag{45}$$

The common set of the set whose points correspond to (w, t) on R_1 and of the set whose points correspond to (w', t') on R_2 appears to be the intersection of them if and only if

$$D_1(A_2)w't'+D_1(B_2)t'+D_1(C_2-C_1)=0 \tag{46}$$

$$D_2(A_1)wt+D_2(B_1)t+D_2(C_1-C_2)=0, \tag{47}$$

where

$$D_1(A_2)\neq 0,\ \ D_2(A_1)\neq 0. \tag{48}$$

If the eq. (48) is not satisfied, we can exchange the order of the vertixes and make the eq. (48) satisfied. For example, if $D_1(A_2)=0$, we exchange the coefficients as follows;

$$A_2'=A_2+B_2,\ \ B_2'=-B_2,\ \ C_2'=B_2+C_2, \tag{49}$$

then $D_1(A_2')\neq 0$. If $D_2(A_1)=0$, so on $\cdots$.

Each intersection on each triangular is a linear line and is represented as follows;

$$R_{10}(t')=\left(B_2-\frac{K_{12}}{K_{11}}A_2\right)t'+C_2-\frac{K_{13}}{K_{11}}A_2 \tag{50}$$

$$R_{20}(t)=\left(B_1-\frac{K_{22}}{K_{21}}A_1\right)t+C_1-\frac{K_{23}}{K_{21}}A_1, \tag{51}$$

where

$$K_{11}=D_1(A_2),\ \ K_{12}=D_1(B_2),\ \ K_{13}=D_1(C_2-C_1),$$
$$K_{21}=D_2(A_1),\ \ K_{22}=D_2(B_1),\ \ K_{23}=D_2(C_1-C_2).$$

In addition, we have the ristriction between w, t, w', t' as follows;

$$0\leq w,\ \ t,\ \ w',\ \ t'\leq 1.$$

So we obtain the following inequalities;

$$\begin{cases} K_{11}(K_{11}+K_{12})t'^2+K_{11}K_{13}t'\geq 0 \\ -K_{11}K_{12}t'^2-K_{11}K_{13}t'\geq 0 \\ 0\leq t'\leq 1 \end{cases} \tag{52}$$

$$\begin{cases} K_{21}(K_{21}+K_{22})t^2+K_{21}K_{23}t\geq 0 \\ -K_{21}K_{22}t^2-K_{21}K_{23}t\geq 0 \\ 0\leq t\leq 1. \end{cases} \tag{53}$$

We can easily obtain the solution α, β, γ and δ of the above inequalities in the form of

$$0\leq\alpha\leq t'\leq\beta\leq 1 \tag{54}$$

$$0\leq\gamma\leq t\leq\delta\leq 1. \tag{55}$$

We do not discuss the algorithm of computing the inequalities (52) and (53) in detail.

We can get a seeking line if we know the points on the line of the eq. (51) corresponding to the terminal points (there $t'=\alpha,\ \beta$) represented with the eq. (50) and the eq. (54). So we modify the eq. (51) and we obtain

$$R_{20}(t)-X=Yt, \tag{56}$$

where

$$X=C_1-\frac{K_{23}}{K_{21}}A_1,\quad Y=B_1-\frac{K_{22}}{K_{21}}A_1.$$

Now we substitute α and β into the eq. (56) and obtain the corresponding t_1 and t_2 on the line of the eq. (51) as follows;

$$\begin{aligned} R_{10}(\alpha)-X=Z_1=Yt_1 \\ R_{10}(\beta)-X=Z_2=Yt_2. \end{aligned} \tag{57}$$

And then we obtain each t_i as follows;

$$t_i=\frac{Z_{i1}Y_1+Z_{i2}Y_2+Z_{i3}Y_3}{Y_1^2+Y_2^2+Y_3^2}\qquad (i=1,2) \tag{58}$$

so that the function φ_i

$$\varphi_i=(t_iY_1-Z_{i1})^2+(t_iY_2-Z_{i2})^2+(t_iY_3-Z_{i3})^2 \tag{59}$$

should be minimal.

Then we can easily obtain the common area of the area (γ, δ) obtained from the eq. (55) and of the area (t_1, t_2) obtained from the above.

Example 6 approximate intersection

Vertixes of two triangulars are given as follows;

$$X_{11}=(0, 0, 2), \quad X_{12}=(3, 0, -1), \quad X_{13}=(-3, 0, -1),$$
$$X_{21}=(-2, 1, 0), \quad X_{22}=(2, 1, 0), \quad X_{23}=(0, -1, 0).$$

We obtain finally

$$R_{20}\left(\frac{1}{2}\right)=\begin{pmatrix}-1\\0\\0\end{pmatrix}, \qquad R_{20}\left(\frac{5}{6}\right)=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

So the line between (1, 0, 0) and (−1, 0, 0) appears to be the intersection.

6. *Conclusion*

This thesis gave the concrete algorithm obtaining approximate suspected outlines and intersections. Then we know that we can treat the algorithm if we base on triangular approximation of arbitrary surfaces. So we can suggest that the data form in graphic information processing system is desired to be easily transformed to or partitioned to many triangulars.

As a principle, we can estimate the solution of the shadow problem basing on our discussion, but do not discuss about it in detail.

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Reference

[1] Sutherland, I. E., Computer Graphics——Ten Unsolved Problems, *Datamation, Vol. 12, No.* 5, 22–27 (May 1966).

[2] Freeman, H. et al., An Algorithm for the Solution of the Two-Dimensional "Hidden-Line" Problem, *IEEE Trans. EC-16,* 6, Dec. (1967), 784–790.

[3] Comba, P. G., A Procedure for Detecting Intersections of Three-Dimensional Objects, *J. of ACM, 15,* 3 (July 1968), 354–366.

[4] Appel, A., Some Technique for Shading Machine Rendering of Solids, *Proc. SJCC* (1968), 37–45.

[5] Kamiuchi, H., About the Three-Dimensional "Hidden-Line" Problem, *Joho-Shori, 11,* 3, (May 1970), 144–154 (in Japanese).

[6] Coons, S., Surfaces for Computer Aided Designing of Space Form, *Project MAC TR-41,* (June 1967).

[7] Hoshaka, M. and M. Endo, On Generation, Storage and Processing of Curve-Connected Patterns, *Joho-Shori, 6,* 3, (May 1965), 129–139 (in Japanese).

[8] Emerging Concepts in Computer Graphics 1967, *University of Illinois Conference,* W. A. Benjamin Inc. (1968).