Analysis of Accuracy Decreasing in Polynomial Remainder Sequence with Floating-point Number Coefficients

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Let $(P_1, P_2, P_3, \ldots)$ be the univariate polynomial remainder sequence with floating-point number coefficients. Let λ roots of P_1 be close to μ roots of P_2, and let $\deg(P_k)=\min\{\lambda, \mu\}$. Then, the accuracy of the coefficients of P_{k+i}, $i>0$, decreases significantly. The accuracy decreasing in P_{k+1} was investigated in a previous paper. This paper almost clarifies the phenomenon of accuracy decreasing in P_{k+i}, $i=1, 2, \ldots$, under the restriction that degrees of initial polynomials are not large. It is shown that if the close roots are concentrated at one point then the accuracy decreases at each calculation of P_{k+i}, $i>0$. If the close roots are distributed around r points, $r>1$, which are mutually well-distant then the accuracy decreases each time the degree of remainder decreases by r. Furthermore, the amount of decrease of accuracy is clarified, with emphasis on the case $P_2(x) \propto dP_1(x)/dx$.

1. Introduction

Let F and G be polynomials in single variable x.

$$F(x)=f_l x^l+\cdots+f_0, \quad f_l\neq 0,$$
$$G(x)=g_m x^m+\cdots+g_0, \quad g_m\neq 0, \quad l\geqq m. \qquad (1.1)$$

The l and f_l are called *degree* and *leading coefficient*, respectively, of F and abbreviated to deg (F) and lc (F): deg $(F)=l$, lc $(F)=f_l$. By quo (F, G) and rem (F, G), we mean the *quotient* and the *remainder*, respectively, of F and G. Given F and G, we calculate the *polynomial remainder sequence* (PRS in short)

$$(P_1=F, \quad P_2=G, \quad P_3, \cdots)$$

by the iterative formula (c_i is a number specified in **2.**)

$$c_i P_{i+1}=\text{rem}\,(P_{i-1}, P_i), \quad i=2, 3, \ldots. \qquad (1.2)$$

The PRS has been used for many years to separate real roots of univariate real polynomial (in this case, PRS must be calculated as a Strum sequence [1]), and so on. Furthermore, one of the authors and Noda showed recently that PRS can be used to separate close as well as multiple roots nicely [2]. In these applications, PRS is often computed efficiently by treating its coefficients as fixed-precision floating-point numbers. In this paper, we call PRS with floating-point number coefficients *approximate PRS*.

It is known that the accuracy of the coefficients of approximate PRS decreases significantly in some cases, and such a case happens when the initial polynomials P_1 and P_2 have mutually close roots. Therefore, we must treat approximate PRS carefully, otherwise we may be lead to a wrong answer. On the other hand, the phenomenon of accuracy decreasing gives us good information about the initial polynomials, which may be utilized variously. In fact, in [2], we showed that approximate PRS allows us to calculate not only the number of but also the mutual distances of close roots of a given polynomial. Similarly, [3] discusses the usefulness of approximate Strum sequence. In order to get such information, analysis of accuracy decreasing is indispensable.

The phenomenon of accuracy decreasing has been analyzed considerably in [2] from the viewpoint of approximate GCD (*greatest common divisor*), but the analysis is quite incomplete. This paper almost clarifies the phenomenon, under the restriction that degrees of initial polynomials are not large. We will see that the phenomenon of accuracy decreasing is not simple but changeable according to the distribution of close roots. Furthermore, the amount of decrease of accuracy is related closely with the distance of mutually close roots.

In **2.** we define necessary notions and discuss the treatment of polynomials with floating-point number coefficients. In **3.** ~ **6.** we analyze the accuracy decreasing in four typical cases: 1) the close roots are concentrated at one point, 2) the close roots of different mutual distances are concentrated at one point, 3) the close roots constitute r distant clusters, with the same number of roots in each cluster, and 4) the close roots constitute r distant clusters, with different number of roots in each cluster. Two examples are given for each case.

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2. Treatment of Approximate Polynomials

The definitions and discussion in this paper follow to [2] basically, but they are slightly more general and rigorous than those in [2].

Definition 1 [maximum magnitude coefficient]. The absolute value of the maximum magnitude coefficient of $F(x)$ is written as mmc(F):

$$\text{mmc}(F)=\max\{|f_l|, \ldots, |f_0|\}. // \qquad (2.1)$$

The mmc (F) is nothing but the infinite "norm" of F, i.e., mmc $(F)=\|F\|_\infty$.

Definition 2 [numbers of similar magnitude]. Let f and g be numbers (may be complex), with $g\neq 0$. By $f=O(g)$, we mean that $1/c\leqq |f/g|\leqq c$, where c is a positive number not much different from 1. (Usually, "O" denotes Landau's notation, and we are using "O" in some different meaning.)//

Let F and G, given in (1.1), be univariate polynomials with floating-point number coefficients. If F and G contain coefficients of very different magnitudes, then successive application of algebraic operations to F and G may lead to numerical instability. A practical way of handling such polynomials is to "regularize" them, as was done in [2].

Definition 3 [regularity]. The $F(x)$ in (1.1) is called *regular* if

$$|f_l|=O(1) \quad \text{and}$$
$$\max\{|f_{l-1}|, \ldots, |f_0|\}=\text{either } O(1) \text{ or } 0. \qquad (2.2)$$

The set $\{F(x), G(x)\}$, with F and G given in (1.1), is called *regular* if

$$|f_l|=O(1), \quad |g_m|=O(1), \quad \text{and}$$
$$\max\{|f_{l-1}|, \ldots, |f_0|, |g_{m-1}|, \ldots, |g_0|\} =\text{either } O(1) \text{ or } 0. // \qquad (2.3)$$

Note. Any polynomial $F(x)$ can be made regular by the transformation $F(x)\to\xi F(\eta x)$, with ξ and η numbers. Similarly, any polynomial set $\{F(x), G(x)\}$ can be made regular by the transformation $F(x)\to\xi_F F(\eta x)$ and $G(x)\to\xi_G G(\eta x)$.

In the analysis of polynomial arithmetic, estimation of the magnitude of coefficients is quite messy. If we consider the upper bound of coefficients rigorously, the estimated upper bound is often unrealistically large. For example, even the simple polynomial $(x+1)^{2n}$ gives a rather large coefficient ${}_{2n}C_n$. In this paper, we adopt the following very simple and practical policy to avoid the complexity of discussing the magnitude of coefficients rigorously.

Policy [treatment of the magnitude of coefficients]. Let F, G and H be univariate polynomials satisfying $F=GH$. Then, we regard that

$$\text{mmc}(F)=O(\text{mmc}(G)\times\text{mmc}(H)). // \qquad (2.4)$$

Remark 1. This equality is surely valid if deg (F) is small or medium, but it may be invalid for polynomials of large degrees. Thus, *our discussion in this paper is valid only for polynomials of low or medium degrees.* With this policy, analysis of accuracy decreasing becomes very simple and clear, as we will see below.

The following theorem is well-known in algebra.

Theorem. Let the roots of F, given in (1.1), be $\alpha_1, \ldots, \alpha_l$, then

$$\max\{|\alpha_1|, \ldots, |\alpha_l|\} \leqq 1+\max\{|f_{l-1}/f_l|, \ldots, |f_0/f_l|\}. //$$

With this theorem, we see that the roots of regular polynomial or regular set of polynomials are located in a circle of radius $O(1)$. Therefore, we can define the close roots as follows.

Definition 4 [close roots]. Let α_i and α_j be roots of regular polynomial $F(x)$ or regular set of polynomials $\{F(x), G(x)\}$. If $0\leqq|\alpha_i-\alpha_j|\ll 1$ then α_i and α_j are (mutually) close roots.//

Following [2], we calculate the approximate PRS as follows.

Algorithm for approximate PRS. Let $\{P_1, P_2\}$ be regular set of univariate polynomials, with deg $(P_1)\geqq$ deg (P_2). We calculate the remainder sequence $(P_1, P_2, P_3, \ldots)$ by the following iteration formula.

$$\begin{cases} Q_i\leftarrow \text{quo}(P_{i-1}, P_i), \quad i=2, 3, \ldots, \\ P_{i+1}\leftarrow \text{rem}(P_{i-1}, P_i)/\max\{1, \text{mmc}(Q_i)\}. // \end{cases} \qquad (2.5)$$

Remark 2. If mmc $(P_{i-1})\gg$ mmc (P_i) or $|\text{lc}(P_i)|\ll$ mmc (P_i) then mmc (Q_i) may become much larger than 1, and mmc(rem (P_{i-1}, P_i)) may become much larger than mmc (P_i). With the above formula, however, we have mmc $(P_{i+1})\leqq O(\text{mmc}(P_i))$.

The formula (2.5) has very nice properties as follows, see [2] for the proof.

Property 1. Suppose the accuracy of the coefficients of P_{i+1} decreases by M' bits by the cancellation of almost equal numbers in the calculation of rem (P_{i-1}, P_i), then we have

$$\text{mmc}(P_{i+1})/\text{mmc}(P_i)=O(2^{-M'}). \qquad (2.6)$$

Property 2. Let $\{P_1, P_2\}$ be regular and the coefficients of P_1 and P_2 contain errors less than or equal to ε, with ε a small positive number. By error (f), with f a floating-point number, we mean the error of f. Then, so long as mmc $(P_i)\gg\varepsilon$, we have

$$\text{error(coefficients of } P_i)\leqq O(\varepsilon). // \qquad (2.7)$$

Property 3. There exist polynomials A_i and B_i satisfying

$$A_iP_1+B_iP_2=P_i+\Delta P_i, \quad \mathrm{mmc}\,(\Delta P_i)=O(\varepsilon),$$
$$\deg(A_i)<\deg(P_2)-\deg(P_i),$$
$$\deg(B_i)<\deg(P_1)-\deg(P_i),$$
$$\mathrm{mmc}\,(A_i)\leqq O(1) \text{ and } \mathrm{mmc}\,(B_i)\leqq O(1). \tag{2.8}$$

Usually, we have $\mathrm{mmc}\,(A_i)=O(1)$ and $\mathrm{mmc}\,(B_i)=O(1)$.//

In the calculation of approximate PRS, the accuracy decreasing due to the cancellation of almost equal numbers is crucially important and, in this paper, we analyze only this kind of accuracy decreasing. Then, according to Property 1, we have only to investigate the magnitude of coefficients of P_i, $i=3, 4, \ldots$.

Example 1 shows an approximate PRS calculated by (2.5) with double-precision floating-point arithmetic. We see a strong reduction of mmc (P_i) at $i=6$ and $i=7$, and the PRS suggests strongly that there exists a definite relationship between the magnitude reduction and the distance of mutually close roots.

Example 1.

$P_1=(X-0.500)*(X-0.502)$
$\quad *(X+1)*(X-2)*(X-1.5)$
$P_2=(X-0.501)*(X-0.503)$
$\quad *(X-1)*(X+2)*(X+1.5)$
$P_3=-4.998*X**4+5.013997*X**3$
$\quad +4.7414925*X**2-\cdots$
$P_4=6.97880794E-1*X**3-7.01037162E-1*X**2$
$\quad +1.78930204E-1*X-\cdots$
$P_5=8.40067492E-1*X**2-8.41442765E-1*X$
$\quad +2.10704053E-1$
$P_6=1.87196957E-3*X-9.38795693E-4$
$P_7=-1.39801471E-9$

3. Close Roots Concentrated at One Point

Let δ be a small positive number, $0<\delta\ll 1$, representing the average distance of mutually close roots. We first note a simple fact which we will use in the following discussions without referring to.

Remark 3. Let $C(x)$ be as follows.

$$C(x)=x^\nu+\delta c_{\nu-1}x^{\nu-1}+\cdots+\delta^\nu c_0,$$

where $c_i=O(1)$ or 0, $i=0, 1, \ldots$. Let $A(x)$ be a polynomial whose coefficients are numbers of $O(1)$ or 0. If $A(x)=x^\kappa\tilde{A}(x)$, with $\kappa\geqq 0$ and $\tilde{A}(0)=O(1)$, then

$$A(x)\cdot C(x)=\sum_{i=\nu+\kappa}^{l} a_i'x^i+\delta a'_{\nu+\kappa-1}x^{\nu+\kappa-1}+\cdots+\delta^\nu a_\kappa' x^\kappa,$$

where $a_i'=O(1)$ or 0, $i=\kappa, \kappa+1, \ldots$. Similarly, given $B(x)$ of the form

$$B(x)=\sum_{i=\nu+\kappa}^{l} b_ix^i+\delta b_{\nu+\kappa-1}x^{\nu+\kappa-1}+\cdots+\delta^{\nu+\kappa}b_0x^0,$$

where $b_i=O(1)$ or 0, $i=0, 1, \ldots, l$, we have

$$\mathrm{rem}\,(B(x), C(x))=\delta^{\kappa+1}d_{\nu-1}x^{\nu-1}+\cdots+\delta^{\kappa+\nu}d_0x^0,$$

where $d_i=O(1)$ or 0, $i=0, 1, \ldots, \nu-1$.//

Let F and G be the following univariate polynomials.

$$F(x)=\tilde{F}(x)\cdot(x-w+\delta_1)\cdots(x-w+\delta_\lambda),$$
$$|\delta_i|=O(\delta), \quad i=1, \ldots, \lambda,$$
$$G(x)=\tilde{G}(x)\cdot(x-w+\delta_1')\cdots(x-w+\delta_\mu'),$$
$$|\delta_i-\delta_{i'}'|=O(\delta), \quad i'=1, \ldots, \mu. \tag{3.1}$$

Let the expanded form of the above close root factors be

$$(X+\delta_1)\cdots(X+\delta_\rho)=X^\rho+c'_{\rho-1}X^{\rho-1}+\cdots+c_0',$$
$$c'_{\rho-1}=\sum_{i=1}^{\rho}\delta_i, \quad c'_{\rho-2}=\sum_{i=1}^{\rho-1}\sum_{j=i+1}^{\rho}\delta_i\delta_j, \text{ etc.},$$

where $X=x-w$ and $\rho=\lambda$ or $\rho=\mu$. Hence, we see $c'_{\rho-i}=O(\delta^i)$ or 0. Let $\nu=\min\{\lambda, \mu\}$, and assume that $c'_{\rho-1}=\cdots=c'_{\rho-\kappa+1}=0$ for both F and G while $c'_{\rho-\kappa}\neq 0$ for either F or G ($\kappa=1$ usually). Let C be defined as

$$C(x)=X^\nu+\delta^\kappa c_{\nu-\kappa}X^{\nu-\kappa}+\cdots+\delta^\nu c_0, \tag{3.2}$$

where $c_i=O(\delta^0)$ or 0, $i=0, 1, \ldots$. Dividing F and G by C, respectively, we can decompose F and G as

$$F(x)=\hat{F}(x)\cdot C(x)+\Delta F(x),$$
$$G(x)=\hat{G}(x)\cdot C(x)+\Delta G(x). \tag{3.3}$$

Noting Remark 3, we find that ΔF and ΔG are given as

$$\Delta F=\delta^\kappa\times\sum_{i=\lambda-\kappa}^{l} f_iX^i+\delta^{\kappa+1}f_{\lambda-\kappa-1}X^{\lambda-\kappa-1}+\cdots+\delta^\lambda f_0,$$
$$\Delta G=\delta^\kappa\times\sum_{i=\mu-\kappa}^{m} g_iX^i+\delta^{\kappa+1}g_{\mu-\kappa-1}X^{\mu-\kappa-1}+\cdots+\delta^\mu g_0, \tag{3.4}$$

where, f_i and g_i, $i=0, 1, \ldots$, are numbers of $O(\delta^0)$ or 0. (We have $f_i=g_i=0$ if $i\geqq\nu$, but we leave the terms f_iX^i and g_iX^i, $i\geqq\nu$, in (3.4) for the corollaries to Theorems 1 and 2 given later.)

Lemma 1. Let F and G satisfy mmc $(F)=O(1)$, mmc $(G)=O(1)$, and equalities in (3.3) with (3.2) and (3.4), where $\nu=\min\{\lambda, \mu\}$. Furthermore, we assume that no root of $\hat{F}$ is close to any root of $\hat{G}$ ($\hat{F}$ or $\hat{G}$ itself may contain mutually close or multiple roots), and that if $\lambda\leqq\mu$ then $\hat{F}(w)=O(\delta^0)$ else if $\mu\leqq\lambda$ then $\hat{G}(w)=O(\delta^0)$. Let $H=\mathrm{rem}\,(F, G)/\max\{1, \mathrm{mmc}\,(\mathrm{quo}\,(F, G))\}$. Unless $\deg(\hat{G})=0$ and $\lambda\geqq\mu$ (this case will be considered in Lemma 2), we can express H as

$$H(x)=\hat{H}(x)\cdot C(x)+\Delta H(x), \quad \text{mmc}(\hat{H})=O(1) \text{ or } 0, \tag{3.5}$$

$$\Delta H=\delta^{\kappa}\times \sum_{i=\nu'-\kappa}^{m-1} h_i X^i+\delta^{k+1}h_{\nu'-\kappa-1}X^{\nu'-\kappa-1} + \cdots +\delta^{\nu'}h_0. \tag{3.6}$$

Here, $\nu \leqq \nu' \leqq \max\{\lambda, \mu\}$, $|h_i| \leqq O(\delta^0)$, $i=0, 1, \ldots,$ and $\nu'=\nu$ unless special relation holds among coefficients of $\hat{F}$ and $\hat{G}$.

Proof. Put $Q=\text{quo}(F, G)\approx \text{quo}(\hat{F}, \hat{G})$, then

$$\begin{aligned}\text{rem}(F, G)&=(\hat{F}C+\Delta F)-Q\times(\hat{G}C+\Delta G)\\&=(\hat{F}-Q\hat{G})\cdot C+(\Delta F-Q\cdot\Delta G).\end{aligned}$$

Eliminating the terms, of degrees $\geqq \deg(G)$, of $\Delta F-Q\cdot\Delta G$ by C as

$$\Delta F-Q\cdot\Delta G=\Delta Q\cdot C+\Delta R, \quad \deg(\Delta R)<\deg(G), \tag{3.7}$$

we find that $\hat{H}$ and ΔH are given by

$$\begin{aligned}&\hat{H}=(\hat{F}-Q\hat{G}+\Delta Q)/\max\{1, \text{mmc}(Q)\},\\&\Delta H=\Delta R/\max\{1, \text{mmc}(Q)\}.\end{aligned} \tag{3.8}$$

The $\Delta F-Q\cdot\Delta G$ is of the same form as the r.h.s. of (3.6). Put

$$Q=q_{l-m}(x-w)^{l-m}+\cdots+q_0,$$

then the case $\nu'>\nu$ occurs only when $\lambda>\mu$ and $|q_0|\leqq O(\delta)$. If $\lambda>\mu$ then $\nu=\mu$ and $\hat{G}(w)=O(\delta^0)$. Since $Q\approx \text{quo}(\hat{F}, \hat{G})$, conditions $\deg(\hat{G})>0$ and $\hat{G}(w)=O(\delta^0)$ mean that $q_0=O(\delta^0)$ unless some relation holds among coefficients of $\hat{F}$ and $\hat{G}$.

Next, we consider ΔR in (3.7). Since $\text{lc}(C)=\text{mmc}(C)=1$, division by C does not increase the magnitude of coefficients of ΔR. On the other hand, if $|\text{lc}(G)|<O(1)$ then magnitude of coefficients of $Q\cdot\Delta G$ may become larger than that of ΔG by the amount of mmc (Q). This magnitude increase is, however, exactly cancelled by the denominator in (3.8). This proves the lemma.//

Note. The h_i in (3.6) is usually of $O(\delta^0)$ or 0. The case $0<|h_i|<O(\delta^0)$ occurs only when mmc $(Q)>O(1)$ and $\Delta F-Q\cdot\Delta G$ is dominated by ΔF. Hence, the case is rare to occur if F and G are two successive elements of PRS.

Lemma 1′. Let $\lambda=\mu+1$ and ΔF and ΔG in (3.3) be given by

$$\begin{aligned}\Delta F&=\delta^{\kappa}\times\sum_{i=\lambda-\kappa}^{l} f_i X^i+\delta^{\kappa+1}f_{\lambda-\kappa-1}X^{\lambda-\kappa-1}+\cdots+\delta^{\lambda}f_0,\\ \Delta G&=\delta^{\kappa}\times\sum_{i=\lambda-\kappa}^{m} g_i X^i+\delta^{\kappa+1}g_{\lambda-\kappa-1}X^{\lambda-\kappa-1}+\cdots+\delta^{\lambda}g_0,\end{aligned} \tag{3.4′}$$

then ΔH in (3.5) is given by

$$\Delta H=\delta^{\kappa}\times\sum_{i=\lambda-\kappa}^{m-1} h_i X^i+\delta^{\kappa+1}h_{\lambda-\kappa-1}X^{\lambda-\kappa-1}+\cdots+\delta^{\lambda}h_0. \tag{3.6′}$$

Proof. Similar to the proof of Lemma 1.//

Lemma 2. Let $\lambda\geqq\mu$ and F and G be

$$\begin{aligned}&F(x)=\hat{F}(x)\cdot[(x-w)^{\lambda}+\delta f_{\lambda-1}(x-w)^{\lambda-1}+\cdots+\delta^{\lambda}f_0],\\&G(x)=(x-w)^{\mu}+\delta g_{\mu-1}(x-w)^{\mu-1}+\cdots+\delta^{\mu}g_0,\end{aligned} \tag{3.9}$$

where mmc $(\hat{F})=O(\delta^0)$, $\hat{F}(w)=O(\delta^0)$, and f_i and g_i, $i=0, 1, \ldots,$ are numbers of $O(\delta^0)$ or 0. Let $H=\text{rem}(F, G)/\max\{1, \text{mmc}(\text{quo}(F, G))\}$, then

$$H(x)=\delta^{\lambda-\mu+1}h_{\mu-1}(x-w)^{\mu-1}+\cdots+\delta^{\lambda}h_0, \tag{3.10}$$

where $h_i=O(\delta^0)$ or 0, $i=0, 1, \ldots.$

Proof: This lemma is obvious if we note that

$$\text{quo}(F, G)=\sum_{i=\lambda-\mu}^{l-\mu} q_i(x-w)^i+\delta q_{\lambda-\mu-1}(x-w)^{\lambda-\mu-1}+\cdots+\delta^{\lambda-\mu}q_0, \tag{3.11}$$

where $q_i=O(\delta^0)$ or 0, $i=0, 1, \ldots.$//

Theorem 1. Let F and G satisfy (3.1), where $\{F, G\}$ is regular, $\deg(F)\geqq\deg(G)$, and $\tilde{F}$ and $\tilde{G}$ have no other mutually close (as well as multiple) root. Let $(P_1=F, P_2=G, P_3, \ldots)$ be an approximate PRS generated by formula (2.5). Let k and k' be integers such that $k<k'$ and $\deg(P_k)=\min\{\lambda, \mu\}$. Then,

$$\text{mmc}(P_i)\leqq O(\delta^0), \quad i=3, \ldots, k-1, \tag{3.12}$$

$$\text{mmc}(P_{k+1})/\text{mmc}(P_k)\leqq O(\delta^{\kappa}), \quad \kappa \text{ is a positive integer}, \tag{3.13}$$

$$\text{mmc}(P_{k'+1})/\text{mmc}(P_{k'})\leqq O(\delta^{d}), \tag{3.14}$$

where

$$\begin{aligned}d=&\min\{\deg(P_{k'-1}), \deg(P_k)-\kappa\}\\&-\min\{\deg(P_{k'+1}), \deg(P_k)-\kappa\}\end{aligned}$$

(if $\kappa=1$ then $d=\deg(P_{k'-1})-\deg(P_{k'+1})$).

Here, if $\deg(\tilde{G})=0$ and $\lambda>\mu$ then $\kappa\geqq\lambda-\mu+1$ else $\kappa\geqq 1$, where equalities hold unless special relation holds among close roots (hence, $\kappa=\lambda-\mu+1$ or $\kappa=1$ usually). Furthermore, in (3.12 ~ 14), inequalities hold only rarely.

Proof. The case of $\deg(\tilde{G})=0$ and $\lambda>\mu$ is obvious from Lemma 2, hence we omit this case. We first note that the PRS contains P_k such that $\deg(P_k)=\min\{\lambda, \mu\}$, because $\hat{F}$ and $\hat{G}$ are relatively prime and GCD$(F, G)\to P_k$ as $\delta\to 0$. By assumption, F and G can be decomposed as (3.3) with (3.2) and (3.4), where either $\hat{F}(w)=O(\delta^0)$ or $\hat{G}(w)=O(\delta^0)$. For integer i, $3\leqq i\leqq k$, put

$$P_i=\tilde{P}_i\times[C(x) \text{ in } (3.2)]+\Delta P_i, \quad \text{mmc}(\Delta P_i\leqq O(\delta),$$

then either $\hat{P}_{i-1}(w)=O(\delta^0)$ or $\hat{P}_i(w)=O(\delta^0)$, because if not so then P_{i-1} and P_i have more than ν approximately common roots. Therefore, we can apply Lemma 1 successively, finding $P_k \propto [C(x) \text{ in } (3.2)]$, mmc $(P_k)=O(\delta^0)$.

This proves (3.12). Then, successive application of Lemma 2 to $P_{k'-1}/\mathrm{mmc}(P_{k'-1})$ and $P_{k'}/\mathrm{mmc}(P_{k'})$ gives (3.13) and (3.14). Note that if $\kappa>1$ and $P_{k-1}=\hat{P}_{k-1}C+\Delta F$ and $P_k=\mathrm{const}\times C+\Delta G$, with ΔF and ΔG given by (3.4), then

$$P_{k+1}=\delta^\kappa f_{\nu-1}X^{\nu-1}+\cdots+\delta^\kappa f_{\nu-\kappa}X^{\nu-\kappa}+\delta^{\kappa+1}f_{\nu-\kappa-1}X^{\nu-\kappa-1}+\cdots.$$

This makes d in (3.14) complicated as in the theorem.//

Corollary. If $P_2(x)\propto dP_1(x)/dx$ then

$$\mathrm{mmc}\,(P_{k+1})/\mathrm{mmc}\,(P_k)=O(\delta^\kappa),\quad \kappa\geqq 2. \tag{3.15}$$

Proof. We choose w to be the center of λ close roots of $P_1=F$. Put

$$P_1(x)=\tilde{P}(x)\cdot\tilde{C}(x),\quad \tilde{C}(x)=(x-w+\delta_1)\cdots(x-w+\delta_\lambda),$$

then $\delta_1+\cdots+\delta_\lambda=0$. Put $X=x-w$ and expand $\tilde{C}$ as

$$\tilde{C}(x)=X^\lambda+\delta^2c_{\lambda-2}X^{\lambda-2}+\cdots+\delta^\lambda c_0.$$

We decompose $\hat{C}$ as $\tilde{C}(x)=(x-w)C(x)+\Delta C(x)$, where

$$C(x)=[d\tilde{C}(x)/dx]/\lambda,$$
$$\Delta C(x)=[2\delta^2c_{\lambda-2}X^{\lambda-2}+\cdots+(\lambda-1)\delta^{\lambda-1}c_1X+\lambda\delta^\lambda c_0]/\lambda. \tag{3.16}$$

Using these expressions, we can decompose P_1 and $P_1'\equiv dP_1/dx$ as

$$P_1(x)=(x-w)\tilde{P}(x)\cdot C(x)+\tilde{P}(x)\cdot\Delta C(x),$$
$$P_1'(x)=[(x-w)d\tilde{P}/dx+\lambda\tilde{P}(x)]\cdot C(x)+[d\tilde{P}/dx]\cdot\Delta C(x). \tag{3.17}$$

Hence, $\kappa\geqq 2$ for decompositions in (3.3) with (3.2) and (3.4').//

Theorem 1 tells us that usually we have $O(\delta)$-decrease at $P_k\to P_{k+1}$ and $O(\delta^2)$-decrease at $P_{k+i}\to P_{k+i+1}$, $i=1, 2, \ldots$. This can be observed in Example 1. On the other hand, if $P_2(x)\propto dP_1(x)/dx$ then we have $O(\delta^2)$-decrease usually in the step $P_k\to P_{k+1}$ also, as can be seen in Example 2 below.

Example 2.

$P_2=(dP_1/dx)/\deg(P_1)$, where

$P_1=(X+1)*(X-2)*(X-0.50)*(X-0.49)*(X-0.52)*(X-0.53)$

$P_3=-7.50188889E-1*X**4+1.53538122*X**3-1.17763848*X**2+\cdots$

$P_4=-1.12461684*X**3+1.72067626*X**2-8.77270065E-1*X+\cdots$

$P_5=5.62488182E-4*X**2-5.73738847E-4*X+1.4621363E-4$

$P_6=5.06392428E-8*X-2.58261579E-8$

$P_7=-8.10292702E-12$

4. Close Roots of Different Mutual Distances

Restriction A. Below, we neglect the case $\kappa>1$, with κ defined in (3.2 ~ 4'), except for the cases $G\propto dF/dx$ and $\deg(\tilde{G})=0$. The neglected case is not important practically while it makes the theorem ugly.

Let $\delta_1, \ldots, \delta_r$ be small positive numbers satisfying

$$1\gg\delta_1\gg\cdots\gg\delta_r>0. \tag{4.1}$$

Putting $X=x-w$ as before, we consider the following product.

$$(X+\delta_{11})\cdots(X+\delta_{1\nu})\times\cdots\times(X+\delta_{r1})\cdots(X+\delta_{r\nu}),$$
$$|\delta_{ij}|=O(\delta_i),\quad i=1,\ldots,r,\quad j=1,\ldots,\nu. \tag{4.2}$$

Let the expanded form of the above product be

$$X^{r\nu}+c'_{r\nu-1}X^{r\nu-1}+\cdots+c'_0X^0,$$

then $c'_{r\nu-1}=O(\delta_1)$, unless accidental cancellation occurs, because

$$c'_{r\nu-1}=c'_{(r-1)\nu+(\nu-1)}=\sum_{i=1}^{r}\sum_{j=1}^{\nu}\delta_{ij}\cong\sum_{j=1}^{\nu}\delta_{1j}.$$

Similarly, we find

$$c'_{(r-i)\nu+(\nu-j)}=O(\delta_1^\nu\cdots\delta_{i-1}^\nu\delta_i^j)\quad\text{or}\quad 0,$$
$$1\leqq i\leqq r,\quad 0<j\leqq\nu.$$

Therefore, putting $c'_{r\nu-1}=\delta_1c_{r\nu-1}$, etc., we find

$$(X+\delta_{11})\cdots(X+\delta_{1\nu})\times\cdots\times(X+\delta_{r1})\cdots(X+\delta_{r\nu})$$
$$=X^{r\nu}+\delta_1c_{rn-1}X^{r\nu-1}+\cdots+\delta_1^\nu c_{r\nu-\nu}X^{r\nu-\nu}+\cdots+\delta_1^\nu\cdots\delta_{r-1}^\nu\delta_rc_{\nu-1}X^{\nu-1}+\cdots+\delta_1^\nu\cdots\delta_{r-1}^\nu\delta_r^\nu c_0X^0, \tag{4.3}$$

where $c_j=O(\delta^0)$ or 0, $j=0, 1, \ldots, r\nu-1$.

Now, we analyze the following case.

$$F(x)=\tilde{F}(x)\times(X+\delta_{11})\cdots(X+\delta_{1\lambda})\times\cdots\times(X+\delta_{r1})\cdots(X+\delta_{r\lambda}),$$
$$G(x)=\tilde{G}(x)\times(X+\delta'_{11})\cdots(X+\delta'_{1\mu})\times\cdots\times(X+\delta'_{r1})\cdots(X+\delta'_{r\mu}), \tag{4.4}$$

where $X=x-w$ as above and

$$|\delta_{ij}|=O(\delta_i),\quad i=1,\ldots,r,\quad j=1,\ldots,\lambda,$$
$$|\delta_{ij}-\delta'_{i'j'}|=O(\delta_i),\quad i'=1,\ldots,r,\quad j'=1,\ldots,\mu. \tag{4.5}$$

(The actual case may not be so simple as above, but the analysis is almost the same.) Theorem 1 is generalized as follows.

Theorem 2. Let F and G satisfy (4.4) with (4.5), as well as conditions given in Theorem 1 and Restriction A. Let $(P_1=F, P_2=G, P_3, \ldots)$ be an approximate PRS generated by formula (2.5). Let $\nu=\min\{\lambda, \mu\}$ and n be an integer such that $0<n<r$. Let k, k' and k'' be integers such that $k<k'<k''$ and

$$\deg(P_k) = r\nu,$$
$$(r-n-1)\nu < \deg(P_{k''}) < \deg(P_{k'}) = (r-n)\nu. \quad (4.6)$$

Let $d_j = \deg(P_{j-1}) - \deg(P_j)$, $j = 2, 3, \ldots,$ then,

$$\text{mmc}(P_i) \leqq O(\delta^0), \; i = 3, \ldots, k-1, \quad (4.7)$$

$$\text{mmc}(P_{k+1})/\text{mmc}(P_k) \leqq O(\delta_1^\kappa), \; \kappa \text{ is a positive integer}, \quad (4.8)$$

$$\text{mmc}(P_{k'+1})/\text{mmc}(P_{k'}) \leqq O(\delta_n^{d_{k'}} \delta_{n+1}^{d_{k'+1}}), \quad (4.9)$$

$$\text{mmc}(P_{k''+1})/\text{mmc}(P_{k''}) \leqq O(\delta_{n+1}^{d_{k''}+d_{k''+1}}). \quad (4.10)$$

Here, if $\deg(\tilde{G}) = 0$ and $\lambda > \mu$ then $\kappa \geqq \lambda - \mu + 1$, else if $G \propto dF/dx$ then $\kappa \geqq 2$, else $\kappa \geqq 1$, and equalities hold for κ usually. Furthermore, in (4.7 ~ 10), inequalities hold only rarely.

Proof. We first note that, for any integer n, $0 \leqq n \leqq r$, the PRS contains $P_{k'}$ satisfying $\deg(P_{k'}) = (r-n)\nu$, because $P_{k'} \to \text{GCD}(F, G)$ if we set $\delta_{n+1} \to \cdots \to \delta_r \to 0$. The $P_{k'-1}$ and $P_{k'}$ are of the forms

$$P_{k'-1}/\text{mmc}(P_{k'-1}) = X^{(r-n)\nu+1} + \delta_n f_{(r-n)\nu} X^{(r-n)\nu} + \delta_n \delta_{n+1} f_{(r-n)\nu-1} X^{(r-n)\nu-1} + \cdots,$$

$$P_{k'}/\text{mmc}(P_{k'}) = X^{(r-n)\nu} + \delta_{n+1} g_{(r-n)\nu-1} X^{(r-n)\nu-1} + \cdots.$$

Therefore, the proof goes similarly to that of Theorem 1.//

We can observe the magnitude reduction predicted by Theorem 2 in Examples 3 and 4. Note that $\delta_1 \approx 10^{-2}$ and $\delta_2 \approx 10^{-3}$ and $O(\delta_1\delta_2)$ magnitude reduction in the step $P_6 \to P_7$ in Example 3 and step $P_5 \to P_6$ in Example 4. The reduction in $P_6 \to P_7$ in Example 4 is $O(\delta_2^2)$, while that in $P_7 \to P_8$ in Example 3 is not $O(\delta_2^2)$ because of the lack of accuracy; P_8 is fully erroneous.

Example 3.

$P_1 = (X-1)*(X+2)*(X-0.500)*(X-0.501)$
$\quad *(X-0.490)*(X-0.510)$

$P_2 = (X+1)*(X-1.5)*(X-0.499)*(X-0.502)$
$\quad *(X-0.485)*(X-0.509)$

$P_3 = 1.494*X**5 - 3.489457*X**4$
$\quad + 3.25253548*X**3 - \cdots$

$P_4 = -1.55940005*X**4 + 3.10932274*X**3$
$\quad - 2.32464915*X**2 + \cdots$

$P_5 = 7.40940408E-3*X**3 - 1.11680386E-2*X**2$
$\quad + 5.61101433E-3*X - \cdots$

$P_6 = 4.24593305E-7*X**2 - 4.24820146E-7*X$
$\quad + 1.06263786E-7$

$P_7 = -1.91637941E-12*X + 9.90828638E-13$

$P_8 = 5.47657254E-16$

Example 4.

$P_2 = (dP_1/dx)/\deg(P_1)$, where

$P_1 = (X+1)*(X-2)*(X-0.500)*(X-0.501)$
$\quad *(X-0.51)*(X-0.52)$

$P_3 = -7.50056806E-1*X**4 + 1.5272393*X**3$
$\quad - 1.16593856*X**2 + \cdots$

$P_4 = -1.12485476*X**3 + 1.71343808*X**2$
$\quad - 8.69926399E-1*X + \cdots$

$P_5 = 1.46663127E-4*X**2 - 1.48330917E-4*X$
$\quad + 3.75005483E-5$

$P_6 = 3.8167845E-9*X - 1.91141596E-9$

$P_7 = -9.29903772E-15$

5. Clusters of the Same Number of Close Roots

Let w_i, $i = 1, \ldots, r$, be numbers such that $|w_i| \leqq O(1)$ and

$$|w_i - w_j| \gg \delta \text{ for any } i \neq j. \quad (5.1)$$

Putting $X_i = x - w_i$, $i = 1, \ldots, r$, we consider the product

$$(X_1 + \delta_{11}) \cdots (X_1 + \delta_{1\nu}) \times \cdots \times (X_r + \delta_{r1}) \cdots (X_r + \delta_{r\nu}),$$
$$|\delta_{ij}| = O(\delta), \quad i = 1, \ldots, r, \quad j = 1, \ldots, \nu. \quad (5.2)$$

We can expand this product uniquely as follows (the expansion is made by successive division by $(X_1 \cdots X_r)$, which guarantees the uniqueness).

$$(X_1 \cdots X_r)^\nu + C'_{\nu-1}(x) \cdot (x_1 \cdots X_r)^{\nu-1} + \cdots + C'_0(x) \cdot (X_1 \cdots X_r)^0, \quad \deg(C'_k) < r,$$
$$k = 0, 1, \ldots, \nu-1. \quad (5.3)$$

Lemma 3. Unless accidental cancellation occurs, we have

$$\text{mmc}(C'_k) = O(\delta^{\nu-k}) \text{ or } 0, \quad k = 0, 1, \ldots, \nu-1. \quad (5.4)$$

Proof. We first note that, for each i, $1 \leqq i \leqq r$, we have

$$(X_i + \delta_{i1}) \cdots (X_i + \delta_{i\nu}) = X_i^\nu + \delta c_{i,\nu-1} X_i^{\nu-1} + \cdots + \delta^\nu c_{i,0},$$
$$c_{i,j} = O(\delta^0) \text{ or } 0, \quad j = 0, 1, \ldots, \nu-1.$$

Expansion of the product in (5.2) gives the term

$$\left(\prod_{i=1}^{r} \delta^{\nu-k_i} c_{i,k_i}\right) \cdot X_1^{k_1} \cdots X_r^{k_r}. \quad (5.5)$$

Let $k = \min\{k_1, \ldots, k_r\}$, then this term does not contribute to $C'_{k-1}, \ldots, C'_0$ in (5.3), because it is proportional to $(X_1 \cdots X_r)^k$. On the other hand, it gives nonzero contribution to C'_k. This can be seen as follows. Suppose, for example,

$$k_1 - 2 = k_2 - 1 = \cdots = k_{r-1} - 1 = k_r, \text{ hence } k_r = k.$$

Since $X_1 = X_r + (w_r - w_1)$, we can transform the term in (5.5) as follows.

$$X_1^{k+2} X_2^{k+1} \cdots X_{r-1}^{k+1} X_r^k$$
$$= (X_1 \cdots X_r)^{k+1} + (w_r - w_1) \cdot X_1 \cdots X_{r-1} \cdot (X_1 \cdots X_r)^k.$$

Continuing this kind of transformation, we see that (5.5) contributes to C_k'. This means that the largest contribution to C_k' is given by the terms

$$(X_1\cdots X_r/X_i)^\nu\cdot\delta^{\nu-k}c_{i,k}X_i^k,\quad i=1,\ldots,r,$$

whose coefficients are of $O(\delta^{\nu-k})$. This proves the lemma.//

Now, we analyze the case that the close roots constitute r distant clusters, with the same number of roots in each cluster. For simplicity, we consider the case

$$\begin{aligned}F(x)&=\tilde{F}(x)\times(X_1+\delta_{11})\cdots(X_1+\delta_{1\lambda})\times\cdots\\&\quad\times(X_r+\delta_{r1})\cdots(X_r+\delta_{r\lambda}),\\G(x)&=\tilde{G}(x)\times(X_1+\delta'_{11})\cdots(X_1+\delta'_{1\mu})\times\cdots\\&\quad\times(X_r+\delta'_{r1})\cdots(X_r+\delta'_{r\mu}),\end{aligned}\tag{5.6}$$

where $X_i=x-w_i$, $i=1,\ldots,r$, as above and

$$\begin{aligned}&|\delta_{ij}|=O(\delta),\quad i=1,\ldots,r,\quad j=1,\ldots,\lambda,\\&|\delta_{ij}-\delta'_{i'j'}|=O(\delta),\quad i'=1,\ldots,r,\quad j'=1,\ldots,\mu.\end{aligned}\tag{5.7}$$

Below, we put $\bar{X}=X_1\cdots X_r$. Let $\nu=\min\{\lambda,\mu\}$ and $C(x)$ be defined as

$$\begin{aligned}&C(x)=\bar{X}^\nu+\delta C_{\nu-1}(x)\cdot\bar{X}^{\nu-1}+\cdots+\delta^\nu C_0(x),\\&\deg(C_i)<r,\quad \mathrm{mmc}(C_i)=O(\delta^0)\text{ or }0,\quad i=0,1,\ldots.\end{aligned}\tag{5.8}$$

Generalizing Remark 3 given in **3.** we see that, by the division by $C(x)$, $F(x)$ and $G(x)$ in (5.6) can be decomposed as

$$\begin{aligned}F(x)&=\hat{F}(x)\cdot C(x)+\Delta F(x),\\G(x)&=\hat{G}(x)\cdot C(x)+\Delta G(x).\end{aligned}\tag{5.9}$$

Here, ΔF and ΔG are the following polynomials.

$$\begin{aligned}\Delta F&=\delta\times\sum_{i=\lambda-1}^{l}F_i\bar{X}^i+\delta^2F_{\lambda-2}\bar{X}^{\lambda-2}+\cdots+\delta^\lambda F_0,\\\Delta G&=\delta\times\sum_{i=\mu-1}^{m}G_i\bar{X}^i+\delta^2G_{\mu-2}\bar{X}^{\mu-2}+\cdots+\delta^\mu G_0,\end{aligned}\tag{5.10}$$

where F_i and G_i are polynomials such that

$$\begin{aligned}&\deg(F_i)<r,\quad \mathrm{mmc}(F_i)=O(\delta^0)\text{ or }0,\\&\qquad\qquad i=0,1,\ldots,l,\\&\deg(G_i)<r,\quad \mathrm{mmc}(G_i)=O(\delta^0)\text{ or }0,\\&\qquad\qquad i=0,1,\ldots,m.\end{aligned}\tag{5.11}$$

Lemmas 1 and 1′ are now generalized as follows.

Lemma 4. Let F and G satisfy (5.9) with (5.8) and (5.10), as well as other conditions given in Lemma 1. Let $H=\mathrm{rem}(F,G)/\max\{1,\mathrm{mmc}(\mathrm{quo}(F,G))\}$ and $\nu=\min\{\lambda,\mu\}$, then H can be expressed as

$$H(x)=\hat{H}(x)\cdot C(x)+\Delta H(x),\quad \mathrm{mmc}(\hat{H})=O(\delta^0),\tag{5.12}$$

$$\Delta H=\delta\times\sum_{i=\nu'-1}^{m}H_i\bar{X}^i+\delta^2H_{\nu'-2}\bar{X}^{\nu'-2}+\cdots+\delta^{\nu'}H_0,\tag{5.13}$$

$\deg(H_i)<r$, $\mathrm{mmc}(H_i)=O(\delta^0)$ or 0, $i=0,1,\ldots,m$, where $\nu\leqq\nu'\leqq\max\{\lambda,\mu\}$ and $\nu'=\nu$ usually.//

Lemma 4′. Let $\lambda=\mu+1$ and ΔF and ΔG in (5.9) be given by

$$\begin{aligned}\Delta F&=\delta^\kappa\times\sum_{i=\lambda-\kappa}^{l}F_i\bar{X}^i+\delta^{\kappa+1}F_{\lambda-\kappa-1}\bar{X}^{\lambda-\kappa-1}\\&\quad+\cdots+\delta^\lambda F_0,\\\Delta G&=\delta^\kappa\times\sum_{i=\lambda-\kappa}^{m}G_i\bar{X}^i+\delta^{\kappa+1}G_{\lambda-\kappa-1}\bar{X}^{\lambda-\kappa-1}\\&\quad+\cdots+\delta^\lambda G_0,\end{aligned}\tag{5.10′}$$

then ΔH in (5.12) is given by

$$\begin{aligned}\Delta H&=\delta^\kappa\times\sum_{i=\lambda-\kappa}^{m}H_i\bar{X}^i+\delta^{\kappa+1}H_{\lambda-\kappa-1}\bar{X}^{\lambda-\kappa-1}\\&\quad+\cdots+\delta^\lambda H_0.//\end{aligned}\tag{5.13′}$$

Similary, Lemma 2 can be generalized as follows.

Lemma 5. Let F and G be the following polynomials.

$$\begin{aligned}&F(x)=F_l(x)\bar{X}^l+\delta F_{l-1}(x)\bar{X}^{l-1}+\cdots+\delta^lF_0(x),\\&G(x)=G_m(x)\bar{X}^m+\delta G_{m-1}(x)\bar{X}^{m-1}+\cdots+\delta^mG_0(x),\end{aligned}\tag{5.14}$$

where $\bar{X}=X_1\cdots X_r$, $\deg(F)\geqq\deg(G)$, and F_i and G_i, $i=0,1,\ldots,$ satisfy (5.11), hence $l\geqq m$. Let $(P_1=F, P_2=G,\ldots,P_{k-1}, P_k\propto H,\ldots)$ be an approximate PRS such that $nr\leqq\deg(H)<(n+1)r$ for an integer n, $0\leqq n<m$. Then, polynomials A and B satisfying

$$A(x)F(x)-B(x)G(x)=H(x),\tag{5.15}$$

$$\begin{cases}\deg(A)=\deg(G)-\deg(P_{k-1}),\quad \mathrm{mmc}(A)=O(1),\\\deg(B)=\deg(F)-\deg(P_{k-1}),\quad \mathrm{mmc}(B)=O(1),\end{cases}\tag{5.16}$$

are of the following forms.

$$\begin{aligned}A&=A_{m-n-1}\bar{X}^{m-n-1}+\delta A_{m-n-2}\bar{X}^{m-n-2}\\&\quad+\cdots+\delta^{m-n-1}A_0,\\B&=B_{l-n-1}\bar{X}^{l-n-1}+\delta B_{l-n-2}\bar{X}^{l-n-2}\\&\quad+\cdots+\delta^{l-n-1}B_0,\end{aligned}\tag{5.17}$$

where $\mathrm{mmc}(A_i)=O(\delta^0)$ or 0, $\mathrm{mmc}(B_i)=O(\delta^0)$ or 0, $i=0,1,\ldots.$ Furthermore,

$$\begin{aligned}&H=\delta^dH_n\bar{X}^n+\delta^{d+1}H_{n-1}\bar{X}^{n-1}+\cdots+\delta^{d+n}H_0,\\&d=l+m-2n-1,\end{aligned}\tag{5.18}$$

where $\mathrm{mmc}(H_i)=O(\delta^0)$ or 0, $i=0,1,\ldots.$

Proof. We first note that, so long as the degree of H is fixed, A, B, and H satisfying (5.15) and (5.16) are determined uniquely up to the normalization factor, see [4].

Secondly, we note that (5.15) is valid for any value of δ, hence we can regard δ as a variable. We consider (5.15) by neglecting terms of order δ^i, $i=1, 2, \ldots$, successively. By the neglect of $O(\delta)$ terms, (5.15) becomes

$$AF_l - BG_m = 0. \quad (5.19)$$

The solution of (5.19) satisfying (5.16) is

$$A = U\cdot(G_m/D)\bar{X}^{m-n-1}, \quad B = U\cdot(F_l/D)\bar{X}^{l-n-1},$$

where $D=\mathrm{GCD}(F_l, G_m)$ and U is an arbitrary polynomial satisfying (5.16). Next, assume for integer i, $1\leqq i<l-n-1$, that

$$A' = A'_{m-n-1}\bar{X}^{m-n-1} + \delta A'_{m-n-2}\bar{X}^{m-n-2} + \cdots + \delta^{i-1}A'_{m-n-i}\bar{X}^{m-n-i},$$
$$B' = B'_{l-n-1}\bar{X}^{l-n-1} + \delta B'_{l-n-2}\bar{X}^{l-n-2} + \cdots + \delta^{i-1}B'_{l-n-i}\bar{X}^{l-n-i} \quad (5.20)$$

satisfy the equation

$$A'F - B'G = \tilde{H}^{(i)}, \ \mathrm{mmc}\,(\tilde{H}^{(i)}) \leqq O(\delta^i), \quad (5.21)$$

where $\mathrm{mmc}\,(A'_j)=O(\delta^0)$ or 0, $\mathrm{mmc}\,(B'_j)=O(\delta^0)$ or 0, $j=0, 1, \ldots$, and $A'_j=B'_j=0$ for $j<0$. With this assumption, we determine A and B satisfying

$$AF - BG = \tilde{H}^{(i+1)}, \quad \mathrm{mmc}(\tilde{H}^{(i+1)}) \leqq O(\delta^{i+1}). \quad (5.22)$$

Putting $A=A'+\Delta A$ and $B=B'+\Delta B$, neglecting terms of order δ^{i+1}, and using the induction assumption, we obtain

$$\Delta A\cdot F - \Delta B\cdot G = \delta^i H^{(i)}\bar{X}^{l+m-n-i-1},$$
$$H^{(i)} = A'_{m-n-1}F_{l-i} - B'_{l-n-1}G_{m-i} + \cdots. \quad (5.23)$$

Expressing A, B, and H in terms of unknown coefficients, we can rewrite (5.15) with (5.16) to a linear system on the unknown coefficients. Similarly, (5.22) with (5.16) gives a linear system which is a subsystem of that given by (5.15). Hence, (5.23) has solutions (in fact, infinitely many solutions unless $i=d$ $(=l+m-2n-1)$) and we find that

$$\Delta A = A'_{m-n-i-1}\bar{X}^{m-n-i-1}, \quad \mathrm{mmc}\,(A'_{m-n-i-1}) = O(\delta^0),$$
$$\Delta B = B'_{l-n-i-1}\bar{X}^{l-n-i-1}, \quad \mathrm{mmc}\,(B'_{l-n-i-1}) = O(\delta^0).$$

Thus, we see that the solution of (5.15) is expressed as (5.17). Eq. (5.18) is a direct consequence of the representation (5.17).//

Theorem 3. Let F and G satisfy (5.6) with (5.7), as well as conditions given in Theorem 1 and Restriction A given in **4.** Let $(P_1=F, P_2=G, P_3, \ldots)$ be an approximate PRS generated by formula (2.5). let $\nu=\min\{\lambda, \mu\}$ and k and k' be integers such that $k<k'$ and

$$\deg(P_k)=\nu r, \quad \nu_1 r \leqq \deg(P_{k+1}) < \nu r,$$
$$\nu' r \leqq \deg(P_{k'}) < (\nu'+1)r, \quad (5.24)$$

for some integers ν_1 and ν', $\nu'\leqq\nu_1<\nu$. Then, we have

$$\mathrm{mmc}\,(P_i) \leqq O(\delta^0), \quad i=3, \ldots, k-1, \quad (5.25)$$
$$\mathrm{mmc}\,(P_{k+1})/\mathrm{mmc}\,(P_k) \leqq O(\delta^\kappa), \quad (5.26)$$
$$\mathrm{mmc}\,(P_{k'})/\mathrm{mmc}\,(P_k) \leqq O(\delta^{\kappa+d}),$$
$$d=\nu+\nu_1-2\nu'-1. \quad (5.27)$$

Here, if $\deg(\tilde{G})=0$ and $\lambda>\mu$ then $\kappa\geqq\lambda-\mu+1$ else $\kappa\geqq 1$, and equalities hold for κ usually. Furthermore, in (5.25 ~ 27), inequalities hold only rarely.

Proof. Noting Property 3 in **2.** we see that Lemma 5 is applicable to P_k and P_{k+1}. Hence, the proof goes similarly to that of Theorem 1.//

Corollary. If $P_2(x)\propto dP_1(x)/dx$ then

$$\mathrm{mmc}\,(P_{k+1})/\mathrm{mmc}\,(P_k) = O(\delta^\kappa), \quad \kappa\geqq 2. \quad (5.28)$$

Proof. We give a proof which can be easily generalized to the case considered in the next section, although a proof similar to that in **3.** is possible.

We choose w_i, $i=1, \ldots, r$, to be the center of λ roots which are close to $x=w_i$, hence $\delta_1+\cdots+\delta_\lambda=0$. Putting

$$P_1(x)=\tilde{P}(x)\cdot\tilde{C}(x), \quad \tilde{C}(x)=\prod_{i=1}^{r}(X_i+\delta_{i1})\cdots(X_i+\delta_{i\lambda}),$$

and expanding $\tilde{C}$ as before, we have

$$\tilde{C}(x)=\bar{X}^\lambda+\delta^2 C_{\lambda-2}\cdot\bar{X}^{\lambda-2}+\cdots+\delta^\lambda C_0.$$

We decompose $\tilde{C}$ as $\tilde{C}(x)=\bar{X}\cdot C(x)+\Delta C(x)$, where

$$C(x)=\bar{X}^{\lambda-1}, \quad \Delta C(x)=\delta^2 C_{\lambda-2}\cdot\bar{X}^{\lambda-2}+\cdots+\delta^\lambda C_0. \quad (5.29)$$

Similarly, we decompose $d\tilde{C}/dx$ as

$$d\tilde{C}/dx=\lambda C(x)\cdot d\bar{X}/dx+\Delta C'(x),$$
$$\Delta C'(x)=\delta^2(dC_{\lambda-2}/dx)\bar{X}^{\lambda-2} + (\lambda-2)\delta^2 C_{\lambda-2}\bar{X}^{\lambda-3}(d\bar{X}/dx)+\cdots.$$

Using these, we can decompose P_1 and $P_1'\equiv dP_1/dx$ as

$$P_1(x)=\tilde{P}(x)\bar{X}\cdot C(x)+\tilde{P}(x)\cdot\Delta C(x),$$
$$P_1'(x)=[(d\tilde{P}/dx)\bar{X}+\lambda\tilde{P}(x)(d\bar{X}/dx)]\cdot C(x) + (d\tilde{P}/dx)\cdot\Delta C(x)+\tilde{P}(x)\cdot\Delta C'(x). \quad (5.30)$$

Hence, $\kappa\geqq 2$ for decompositions in (5.9) with $C=\bar{X}^{\lambda-1}$ and (5.10′).//

Theorem 3 tells that, usually, $\mathrm{mmc}\,(P_i)$ decreases by $O(\delta)$ at $P_k\to P_{k+1}$ and by $O(\delta^2)$ at $P_{k+nr}\to P_{k+nr+1}$, $n=1, 2, \ldots$. Example 5 shows the case that $\delta\approx 10^{-2}$, $r=3$, and $k=3$. We see that the prediction of Theorem 3 is well verified. Example 6 shows the case that $P_2\propto dP_1/dx$, with $\delta\approx 10^{-3}$ and $k=4$, and the result is again consistent with the theory.

Example 5.

$$P_1=(X+1)*(X+0.01)*(X+0.02)$$
$$*(X+0.49)*(X+0.508)$$
$$*(X-1.50)*(X-1.52)$$
$$P_2=(X-1)*(X-0.01)*(X-+0.00)$$

$*(X+0.50)*(X+0.515)$
$*(X-1.49)*(X-1.51)$

$P_3=2.003*X**6-4.01475*X**5-1.0710285*X**4$
$+\cdots$

$P_4=2.63715546E-2*X**5-3.02182613E-2*X**4$
$-3.12331779E-2*X**3+\cdots$

$P_5=-8.8104723E-3*X**4-4.23378864E-3*X**3$
$+1.96697023E-2*X**2+\cdots$

$P_6=9.91195816E-3*X**3-9.86421814E-3*X**2$
$-7.57658018E-3*X-\cdots$

$P_7=-3.14053031E-6*X**2-2.38925971E-6*X$
$-2.76898269E-7$

$P_8=8.64468426E-7*X+2.70013245E-7$

$P_9=4.48641788E-8$

Example 6.

$P_2=(dP_1/dx)/\deg(P_1)$, where

$P_1=(X-1)*(X-0.300)*(X-0.302)*(X-0.299)$
$*(X+0.510)*(X+0.512)*(X+0.509)$

$P_3=-2.9033898E-1*X**5$
$+2.54147949E-2*X**4+\cdots$

$P_4=-6.63230111E-2*X**4$
$-2.78542633E-2*X**3+\cdots$

$P_5=1.21561063E-6*X**3+6.36471234E-8*X**2$
$-2.26089335E-7*X+\cdots$

$P_6=1.16335442E-7*X**2+2.44298257E-8*X$
$-1.77855571E-8$

$P_7=-4.38922505E-13*X+6.87152132E-14$

$P_8=-4.19157134E-14$

6. Clusters of Different Numbers of Close Roots

Let w_i, $i=1,\ldots,r$, be numbers such that $|w_i|\leqq O(1)$ and

$$|w_i-w_j|\gg\delta \quad \text{for any } i\neq j. \qquad (6.1)$$

Putting $X_i=x-w_i$, $i=1,\ldots,r$, we consider the product

$$(X_1+\delta_{11})\cdots(X_1+\delta_{1\nu_1})\times\cdots\times(X_r+\delta_{r1})\cdots(X_r+\delta_{r\nu_r}),$$
$$|\delta_{ij}|=O(\delta),\quad i=1,\ldots,r,\quad j=1,\ldots,\nu_i. \qquad (6.2)$$

We define functions $e(n)$ and $\sigma(n)$, with integer n, as

$$e(n)=\max\{0,n\},\quad \sigma(n)=\sum_{i=1}^{r} e(\nu_i-n). \qquad (6.3)$$

By the successive division by $(X_1^{e(\nu_1-i)}\cdots X_r^{e(\nu_r-i)})$, which is of degree $\sigma(i)$, $i=0,1,\ldots$, we can expand the product in (6.2) uniquely as follows.

$$(X_1^{\nu_1}\cdots X_r^{\nu_r})+\sum_{i=1}^{\nu}\delta^i C_{\nu-i}(x)\cdot(X_1^{e(\nu_1-i)}\cdots X_r^{e(\nu_r-i)}), \qquad (6.4)$$

$$\deg(C_{\nu-i})<\sigma(i-1)-\sigma(i),\quad i=1,2,\ldots,\nu, \qquad (6.5)$$

where $\nu=\max\{\nu_1,\ldots,\nu_r\}$.

Similarly to Lemma 3, we obtain the following lemma.

Lemma 6. Unless accidental cancellation occurs, we have

$$\mathrm{mmc}(C_{\nu-i})=O(\delta^0) \text{ or } 0,\quad i=1,2,\ldots,\nu.// \qquad (6.6)$$

Now, we analyze the case that the close roots constitute r distant clusters, with different number of roots in each cluster. For simplicity, we consider the case

$$F(x)=\tilde{F}(x)\times(X_1+\delta_{11})\cdots(X_1+\delta_{1\lambda_1})\times\cdots$$
$$\times(X_r+\delta_{r1})\cdots(X_r+\delta_{r\lambda_r}),$$
$$G(x)=\tilde{G}(x)\times(X_1+\delta'_{11})\cdots(X_1+\delta'_{1\mu_1})\times\cdots$$
$$\times(X_r+\delta'_{r1})\cdots(X_r+\delta'_{r\mu_r}), \qquad (6.7)$$

where $X_i=x-w_i$, $i=1,\ldots,r$, as above and

$$|\delta_{ij}|=O(\delta),\quad i=1,\ldots,r,\quad j=1,\ldots,\lambda_i,$$
$$|\delta_{ij}-\delta'_{i'j'}|=O(\delta),\quad i'=1,\ldots,r,\quad j'=1,\ldots,\mu_i. \qquad (6.8)$$

The λ_i and ν_i are numbers of close roots of F and G, respectively, located around $x=w_i$. We put

$$\nu_i=\min\{\lambda_i,\mu_i\},\quad i=1,\ldots,r,$$
$$\nu=\max\{\nu_1,\ldots,\nu_r\}. \qquad (6.9)$$

The analysis in **5.** is directly applicable to the above case, and we obtain the following theorem.

Theorem 4. Let F and G satisfy (6.7) with (6.8), as well as other conditions given in Theorem 1 and Restriction A given in **4.** Let $(P_1=F,\ P_2=G,\ P_3,\ldots)$ be approximate PRS generated by (2.5). Let ν be defined as (6.9). Let k and k' be integers such that $k<k'$ and

$$\deg(P_k)=\sigma(0)=\sum_{i=1}^{r}\nu_i,$$
$$\sigma(\nu-\hat{\nu})\leqq\deg(P_{k+1})<\sigma(0),$$
$$\sigma(\nu-\nu')\leqq\deg(P_{k'})<\sigma(\nu-\nu'-1) \qquad (6.10)$$

for some integers $\hat{\nu}$ and ν' such that $\nu'\leqq\hat{\nu}<\nu$, where $\hat{\nu}$ is chosen as large as possible hence $\hat{\nu}=\nu-1$ usually. Then, we have

$$\mathrm{mmc}(P_i)\leqq O(\delta^0),\quad i=3,\ldots,k-1, \qquad (6.11)$$
$$\mathrm{mmc}(P_{k+1})/\mathrm{mmc}(P_k)\leqq O(\delta^{\kappa}), \qquad (6.12)$$
$$\mathrm{mmc}(P_{k'})/\mathrm{mmc}(P_k)\leqq O(\delta^{\kappa+d}),$$
$$d=\nu+\hat{\nu}-2\nu'-1. \qquad (6.13)$$

Here, if $\deg(\tilde{G})=0$ and $\lambda>\mu$ then $\kappa\geqq\lambda-\mu+1$ else $\kappa\geqq1$, and equalities hold for κ usually. Furthermore, in $(6.11\sim13)$, inequalities hold only rarely.//

Note. If $\nu_1>\nu_2\geqq\nu_3\geqq\cdots$ then some $P_{k'}$ satisfies deg

$(P_{k'}) < v_1 - v_2$ and every root of $P_{k'}$ is close to $x = w_1$. This case was already considered in **3.**

Corollary. If $P_2(x) \propto dP_1(x)/dx$ then

$$\text{mmc}\,(P_{k+1})/\text{mmc}\,(P_k) = O(\delta^{\kappa}), \quad \kappa \geqq 2. // \tag{6.14}$$

Example 7.

$P_1 = (X-1)*(X+2.0)*(X+0.510)*(X+0.512)$
$\quad *(X-0.500)*(X-0.502)*(X-0.504)$

$P_2 = (X+1)*(X-1.5)*(X+0.511)*(X+0.509)$
$\quad *(X-0.499)*(X-0.501)*(X-0.503)$

$P_3 = 1.499*X**6 - 1.227542*X**5$
$\quad -5.3697623E-1*X**4 + \cdots$

$P_4 = -1.55461576*X**5 + 7.48009375E-1*X**4$
$\quad +8.10339649E-1*X**3 - \cdots$

$P_5 = 7.69315715E-4*X**4 + 2.89658028E-3*X**3$
$\quad -1.81360615E-3*X**2 - \cdots$

$P_6 = -3.22924748E-3*X*3 + 1.59041136E-3*X**2$
$\quad +8.41318405E-4*X - \cdots$

$P_7 = 2.87715338E-10*X**2 + 1.25063642E-10*X$
$\quad -1.35080302E-10$

$P_8 = -1.76078175E-10*X + 8.83034966E-11$

$P_9 = 5.232599E-16$

Example 8.

$P_2 = (dP_1/dx)/\deg(P_1)$, where

$P_1 = (X-1)*(X+0.300)*(X+0.301)*(X+0.302)$
$\quad *(X-0.497)*(X-0.499)*(X-0.500)*(X-0.502)$

$P_3 = -2.15148859E-1*X**6$
$\quad +3.47949658E-1*X**5 - \cdots$

$P_4 = -2.95982284E-2*X**5$
$\quad +2.65355441E-2*X**4 + \cdots$

$P_5 = 5.31977343E-7*X**4 - 4.28249859E-7*X**3$
$\quad +1.24986698E-8*X**2 + \cdots$

$P_6 = 8.51917176E-8*X**3 - 5.94635217E-8*X**2$
$\quad -4.36163135E-9*X + \cdots$

$P_7 = -5.20783763E-13*X**2 + 1.4973954E-13*X$
$\quad +5.51406285E-14$

$P_8 = -3.29858454E-14*X + 1.64762159E-14$

$P_9 = 1.42599702E-19$

7. Concluding Remark

We have not analyzed the general case that the close roots constitute r distant clusters, where each cluster contains different number of roots of different mutual distances. The analysis in **4.** and **6.** shows that describing such a general case is complicated, but we can easily imagine the result from the analysis in **4.** and **6.** Thus, we may say that we have almost clarified the phenomenon of accuracy decreasing in approximate PRS's of low and medium degrees. However, analysis of high degree PRS's is postponed as a future research theme.

The accuracy decreasing in high degree PRS will not be expressed so clearly as in this paper. The reason is as follows. If many roots are distributed within a circle of radius $O(1)$ then the average distance of neighboring roots is small, hence we cannot clearly distinguish mutually close roots from distant roots.

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(Received March 27, 1989; revised October 11, 1989)