Research Contribution

Phase-Lag Analysis of Diagonally Implicit Runge-Kutta Methods

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This paper concerns an approximation property of diagonally implicit Runge-Kutta methods when they are applied to a system of ordinary differential equations with periodic solutions. In order to characterize the property, phase errors are studied for a certain class of rational approximations to exp (z) and several conditions for reducing the phase error are derived. A-acceptability is also considered for the rational approximations in the same class and higher order A-acceptable rational approximations with reduced phase errors are obtained.

1. Introduction

We discuss an approximation property of implicit Runge-Kutta methods when they are applied to a system of ordinary differential equations (ODEs) of the form

$$\frac{du}{dt}=f(t, u), \quad u(t_0)=u_0 \tag{1.1}$$

with periodic solutions. More specifically, we analyze the phase errors introduced by the methods when the linear test equation

$$\frac{du}{dt}=i\omega u, \quad \omega\in \mathbf{R}. \tag{1.2}$$

is integrated. A similar analysis of numerical methods for second order ODEs is well known and called *phase-lag analysis* (cf. [2], [4], [5], [6], [7], [12], [13], [14], [16]).

The phase-lag analysis of an implicit Runge-Kutta method is, as far as the test equation (1.2) is concerned, equivalent to an analysis of its stability function, a rational approximation to the exponential function. In Section 2, the phase error in the numerical solution of the test equation (1.2) is represented by the stability function and expanded as

$$\Phi(y)=\sum_{j=0}^{\infty} C_{P,j}y^j, \tag{1.3}$$

where $y=\omega h$ and $h>0$ is the step-size. Then, a new order of the method is defined as an integer q satisfying $C_{P,j}=0$ for $j=0, 1, \ldots, q$ and $C_{P,q+1}\neq 0$. This q is called a phase order and plays a character significant for the accuracy with respect to the phase component.

In Section 3, we give a relation between the phase order q and the usual order p defined for the stability function. It is shown that q is equal to p if p is even and q is an even integer greater than or equal to $p+1$ if p is odd. That is, q is determined only by p if p is even but it is not so if p is odd. Another condition is hence derived to estimate q when p is odd.

In Section 4, we study the phase order of diagonally implicit Runge-Kutta (DIRK) methods [1], which are characterized by the fact that the stability function has the form

$$R(z)=\frac{P_m(z)}{(1-z/\lambda)^m}, \quad \lambda\in \mathbf{R}. \tag{1.4}$$

Here m is the stage-number and $P_m(z)$ is a polynomial of degree at most m. For a rational approximation of the form (1.4), the attainable order of approximation is equal to $m+1$ [10], and, in this case, the phase order is determined as $q=m+1$ if m is odd and $q=m+2$ if m is even. However, the highest order approximation is not necessarily the best with respect to the phase order. In fact, Van der Houwen and Sommeijer [15] show that there are rational approximations with higher phase order: (i) $m=2$, $p=1$, $q=6$, (ii) $m=3$, $p=3$, $q=6$ and (iii) $m=4$, $p=3$, $q=8$. Furthermore, they have constructed A-stable DIRK methods in the cases (ii) and (iii).

We give a generalization of their results: If $m(>1)$ is odd, then there is a rational approximation of the form (1.4) with $p=m$ and $q=m+3$. If m is even, then there is a rational approximation of the form (1.4) with $p=m-1$ and $q=m+4$.

In Section 5, we investigate the A-acceptability of such rational approximations with reduced phase errors. As a result, we obtain new A-acceptable rational approximations in the cases: $m=5$, $p=5$, $q=8$ and $m=6$, $p=5$, $q=10$.

2. Preliminaries

For the system of ODEs (1.1), an m-stage implicit

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Runge-Kutta method is written as

$$u_n=u_{n-1}+h\sum_{j=1}^{m} b_j f(t_{n-1}+c_jh,\ U_j)$$

$$U_j=u_{n-1}+h\sum_{k=1}^{m} a_{j,k}f(t_{n-1}+c_kh,\ U_k),\qquad j=1,2,\ldots,m, \tag{2.1}$$

where $a_{j,k}$, b_j and c_j are real numbers. When $a_{j,k}=0$ for $j<k$, the method is said to be semi-implicit. Furthermore, if $a_{j,j}$ is constant for $j=1,2,\ldots,m$, the semi-implicit method is said to be diagonally implicit [1, 3].

Let A and A^*, $m\times m$ matrices, be defined by

$$A=(a_{j,k})\quad (1\leqq j,\ k\leqq m)$$

and

$$A^*=(a_{j,k}-b_k)\quad (1\leqq j,\ k\leqq m),$$

respectively. When the method (2.1) is applied to the linear equation

$$\frac{du}{dt}=\zeta u,\quad \zeta\in \boldsymbol{C}, \tag{2.2}$$

the numerical solution is given by

$$u_n=R_m(z)u_{n-1},\quad R_m(z)=\frac{\det(I-zA^*)}{\det(I-zA)},\quad z=h\zeta \tag{2.3}$$

(cf., e.g., [11]). Here $R_m(z)$ is the stability function, which is a rational approximation to the exponential function and plays an important role in the stability analysis of Runge-Kutta methods [3]. In particular, the stability function of a diagonally implicit method has the form (1.4).

In order to characterize a property of Runge-Kutta methods for a system of ODEs with periodic solutions, we give several definitions for a rational approximation to the exponential function.

A rational approximation $R(z)$ to $\exp(z)$ is said to be of order p if

$$C_j=0\quad\text{for}\quad j=0,1,\ldots,p\quad\text{and}\quad C_{p+1}\neq 0 \tag{2.4}$$

for the coefficients of the Taylor expansion of $\exp(z)-R(z)$,

$$\exp(z)-R(z)=\sum_{j=0}^{\infty} C_j z^j. \tag{2.5}$$

When the rational approximation is the stability function of a Runge-Kutta method, the order is called the linear order of the Runge-Kutta method.

Definition. For a rational approximation $R(z)$ to $\exp(z)$, the function

$$\Phi(y)=y-\arg(R(iy)),\quad y\in \boldsymbol{R}, \tag{2.6}$$

is called a phase error function.

Since $\Phi(y)$ is real-analytic in a neighborhood of the origin, it is expanded as (1.3). Based on the expansion, another order is defined as follows.

Definition. A rational approximation $R(z)$ to exp (z) is said to be of phase order q if

$$C_{P,j}=0\quad\text{for}\quad j=0,1,\ldots,q\quad\text{and}\quad C_{P,q+1}\neq 0 \tag{2.7}$$

in (1.3).

The phase order of the Runge-Kutta method can be also defined as the phase order q of its stability function.

3. Fundamental Property of Phase Order

Let $\Psi(y)=\tan(y)-(\mathrm{Im}(R(iy))/\mathrm{Re}(R(iy)))$. Then, $\Psi(y)$ can be expanded as

$$\Psi(y)=\sum_{j=0}^{\infty} C_{T,j}y^j. \tag{3.1}$$

Lemma 1. The phase order of the rational approximation $R(z)$ to exp (z) is equal to q if and only if

$$C_{T,j}=0\quad\text{for}\quad j=0,1,\ldots,q\quad\text{and}\quad C_{T,q+1}\neq 0. \tag{3.2}$$

Proof. Using the addition theorem, we have

$$\tan(\Phi(y))=\Psi(y)\Big/\left\{1+\tan(y)\frac{\mathrm{Im}(R(iy))}{\mathrm{Re}(R(iy))}\right\}. \tag{3.3}$$

Since $\tan(z)=z+O(z^3)$, it follows from (3.3) that (3.2) is equivalent to (2.4).

Q.E.D.

The following lemma gives a fundamental relation between the linear order and the phase order.

Lemma 2. Let p be the order of the rational approximation $R(z)$. Then, the phase order q is equal to p if p is even and q is an even integer greater than or equal to $p+1$ if p is odd.

Proof. $\Psi(y)$ is rewritten as

$$\Psi(y)=\frac{\mathrm{Re}(R(iy))\sin(y)-\mathrm{Im}(R(iy))\cos(y)}{\mathrm{Re}(R(iy))\cos(y)}. \tag{3.4}$$

Since (2.5) together with the Taylor expansions of sin (y) and cos (y) yields

$$\mathrm{Re}(R(iy))\sin(y)-\mathrm{Im}(R(iy))\cos(y)=\sum_{k=0}^{\infty}(-1)^k\left\{\sum_{j=0}^{k}\frac{C_{2j+1}}{(2k-2j)!}-\frac{C_{2j}}{(2k-2j+1)!}\right\}y^{2k+1} \tag{3.5}$$

and

$$\mathrm{Re}(R(iy))\cos(y)=\sum_{k=0}^{\infty}(-1)^k\left\{\sum_{j=0}^{k}\times\left(\frac{1}{(2j)!}-C_j\right)\frac{1}{(2k-j)!}\right\}y^{2k}, \tag{3.6}$$

$\Psi(y)$ is expanded in the form

$$\Psi(y)=\sum_{k=0}^{\infty} C_{T,2k+1}v^{2k+1}. \tag{3.7}$$

Hence, the phase order q is always even. Furthermore, if (2.7) is satisfied, it follows from (3.5) and (3.6) that

$$\Psi(y)=(-1)^l C_{p+1}y^{p+1}+O(y^{p+3}) \tag{3.8}$$

when p is even, and

$$\Psi(y)=(-1)^l(C_{p+2}-C_{p+1})y^{p+2}+O(y^{p+1}) \qquad (3.9)$$

when p is odd, where l is the integral part of $(p+1)/2$. The result thus follows from Lemma 1.

Q.E.D.

Lemma 2 shows that q is determined only by p if p is even, but it is not so if p is odd. Thus, other conditions are required to determine q when p is odd. On a general method to estimate q, refer to [9]. Here, we describe simple characterization of the phase order q when the order p is odd.

From (3.5) and (3.6) the precise estimate of $\Psi(y)$ is expressed as

$$\Psi(y)=C_{T,p+2}y^{p+2}+C_{T,p+4}y^{p+4}+O(y^{p+6}),$$
$$C_{T,p+2}=(-1)^k(C_{p+2}-C_{p+1}),$$
$$C_{T,p+4}=(-1)^{k+1}(C_{p+4}-C_{p+3}+C_{p+2}/2 -C_{p+1}/6)+(-1)^k(C_{p+2}-C_{p+1})(1+C_2), \qquad (3.10)$$

when p is odd. Therefore, by the same argument as in the proof of Lemma 2, we conclude that

$$\text{if } C_{p+2}-C_{p+1}=0, \text{ then } q\geqq p+3 \qquad (3.11)$$

and

$$\text{if } C_{p+2}-C_{p+1}=0 \qquad (3.12)$$

and

$$C_{p+4}-C_{p+3}+C_{p+2}/2-C_{p+1}/6=0, \quad \text{then } q\geqq p+5.$$

4. Phase order of DIRK Methods

In this section, we study the phase order of rational approximations of the form (1.4) in order to characterize that of DIRK methods.

Let $L_m(\lambda)$ be the Laguerre polynomial of degree m,

$$L_m(\lambda)=\sum_{j=0}^{m}(-\lambda)^j\frac{m!}{(m-j)!(j!)^2}, \qquad (4.1)$$

and, for a positive integer k, let $L_m^{(k)}(\lambda)$ denote the kth derivative of $L_m(\lambda)$. For $k=0, -1, -2, \ldots,$ define $L_m^{(k)}(\lambda)$ inductively by

$$L_m^{(0)}(\lambda)=L_m(\lambda),\ L_m^{(k)}(\lambda)=\int_0^{\lambda}L_m^{(k+1)}(\mu)\,\mathrm{d}\mu,$$
$$k=-1, -2, \ldots. \qquad (4.2)$$

With this notation, the following equality holds.

Lemma 3. ([3], p. 246)

$$(1-z/\lambda)^m\exp(z)=(-1)^m\sum_{j=0}^{\infty}L_m^{(m-j)}(\lambda)(z/\lambda)^j. \qquad (4.3)$$

Using this lemma, we obtain an expression for the rational approximation of the form (1.4) with order $p\geqq m$

$$R(z)=\frac{P_m(z)}{(1-z/\lambda)^m},$$

$$P_m(z)=(-1)^m\sum_{j=0}^{m}L_m^{(m-j)}(\lambda)(z/\lambda)^j,\ \lambda\in \boldsymbol{R}, \qquad (4.4)$$

with the error term

$$\exp(z)-R(z) =C_{m+1}z^{m+1}+C_{m+2}z^{m+2}+C_{m+3}z^{m+3}+O(z^{m+4}),$$
$$C_{m+1}=(-1)^mL_m^{(-1)}(\lambda)/\lambda^{m+1},$$
$$C_{m+2}=(-1)^m\{L_m^{(-2)}(\lambda)+mL_m^{(-1)}(\lambda)\}/\lambda^{m+2},$$
$$C_{m+3}=(-1)^m\left\{L_m^{(-3)}(\lambda)+mL_m^{(-2)}(\lambda) +\frac{m+1}{2}L_m^{(-1)}(\lambda)\right\}/\lambda^{m+3}. \qquad (4.5)$$

Furthermore, from standard identities involving Laguerre polynomials (cf., e.g., [8]),

$$L_m^{(-1)}(\lambda)=-\frac{\lambda}{m+1}L'_{m+1}(\lambda), \qquad (4.6)$$

and thereby the factor C_{m+1} is rewritten as

$$C_{m+1}=(-1)^{m+1}\frac{1}{m+1}L'_{m+1}(\lambda)/\lambda^m. \qquad (4.7)$$

From this representation of C_{m+1} it follows that the rational approximation (4.4) is of order $m+1$ if and only if λ is a root of $L'_{m+1}(\lambda)$.

For the rational approximations of order $m+1$, we obtain the following characterization of the phase order.

Theorem 1. Suppose that the order of a rational approximation of the form (1.4) is equal to $m+1$. Then, the phase order q is equal to $m+1$ if m is odd and equal to $m+2$ if m is even.

Proof. If m is odd then $p=m+1$ is even, and q is thus equal to $m+1$ by Lemma 2. Let's consider the case m is even.

Using (4.6),

$$L_m^{(-2)}(\lambda)=\frac{\lambda}{m+2}L'_{m+2}(\lambda)-\frac{\lambda}{m+1}L'_{m+1}(\lambda) \qquad (4.8)$$

and

$$L_m^{(-3)}(\lambda)=-\frac{\lambda}{m+3}L'_{m+3}(\lambda)+\frac{2\lambda}{m+2}L'_{m+2}(\lambda) -\frac{\lambda}{m+1}L'_{m+1}(\lambda), \qquad (4.9)$$

we obtain

$$C_{m+3}-C_{m+2}=\frac{\lambda-m-1}{m+3}L_{m+1}(\lambda)/\lambda^{m+2}. \qquad (4.10)$$

Here we have also used

$$L'_{m+2}(\lambda)=L'_{m+1}(\lambda)-L_{m+1}(\lambda) \qquad (4.11)$$

and so on.

Since all the roots of $L_{m+1}(\lambda)$ are simple, $C_{m+3}-C_{m+2}\neq 0$ whenever $L'_{m+1}(\lambda)=0$, and q is thus equal to $m+2$.

Q.E.D.

When m is an odd integer greater than 1, there are rational approximations of the form (1.4) which are of order m but exceed the highest order approximation with respect to the phase order.

Theorem 2. Let $m(>1)$ be odd. Then, there is a rational approximation of the form (1.4) with $p=m$ and $q\geqq m+3$, where p is the order and q is the phase order.

Proof. Let

$$f_m(\lambda)=\frac{1}{m+2}L'_{m+2}(\lambda)-\frac{m+1-\lambda}{m+1}L'_{m+1}(\lambda). \qquad (4.12)$$

Then, (4.6) and (4.8) imply that $C_{m+2}-C_{m+1}=-f_m(\lambda)/\lambda^{m+1}$. Thereby, if $f_m(\lambda)$ has a real root λ_0, (4.4) for $\lambda=\lambda_0$ gives a rational approximation with $p=m$ and $q\geqq m+3$. Hence, it suffices to show that $f_m(\lambda)$ has a real root.

Using (4.11), we have

$$f_m(\lambda)=-\frac{1}{m+2}L_{m+1}(\lambda) \qquad (4.13)$$

when $L'_{m+1}(\lambda)=0$. Since $L_{m+1}(\lambda)$ has a positive extremal value if $m\geqq 3$, (4.13) implies that $f_m(\lambda)<0$ for some λ. On the other hand, since $L'_m(0)=-m$ for any m, $f_m(0)=m>0$. Thus, the polynomial $f_m(\lambda)$ has a real root if $m\geqq 3$.

Q.E.D.

When m is even, the phase order q of the rational approximation of order m is equal to m by Lemma 2, and thus it does not exceed that of the approximation of order $m+1$. However, when the order is lower, there exist approximations with higher phase order.

When m is even, a rational approximation of the form (1.4) with $p\geqq m-1$ is written as

$$R(z)=\frac{P_m(z)}{(1-z/\lambda)^m},$$
$$P_m(z)=\sum_{j=0}^{m-1}L_m^{(m-j)}(\lambda)(z/\lambda)^j+\mu(z/\lambda)^m,$$
$$\lambda,\ \mu\in \boldsymbol{R}, \qquad (4.14)$$

where the error term is given by

$$\exp(z)-R(z)=C_m z^m+C_{m+1}z^{m+1}+C_{m+2}z^{m+2}+C_{m+3}z^{m+3}+O(z^{m+4}),$$
$$C_m=(L_m(\lambda)-\mu)/\lambda^m,$$
$$C_{m+1}=\{L_m^{(-1)}(\lambda)+m(L_m(\lambda)-\mu)\}/\lambda^{m+1},$$
$$C_{m+2}=\left\{L_m^{(-2)}(\lambda)+mL_m^{(-1)}(\lambda)+\frac{m(m+1)}{2}\times(L_m(\lambda)-\mu)\right\}\Big/\lambda^{m+2},$$
$$C_{m+3}=\left\{L_m^{(-3)}(\lambda)+mL_m^{(-2)}(\lambda)+\frac{m(m+1)}{2}L_m^{(-1)}(\lambda)+\frac{m(m+1)(m+2)}{6}(L_m(\lambda)-\mu)\right\}\Big/\lambda^{m+3}. \qquad (4.15)$$

Theorem 3. Let m be even. Then, there is a rational approximation of the form (1.4) with $p=m-1$ and $q\geqq m+4$, where p is the order and q is the phase order.

Proof. We show that the rational approximation (4.14) for some μ, λ satisfies

$$C_{m+1}-C_m=0 \quad \text{and}$$
$$C_{m+3}-C_{m+2}+C_{m+1}/2-C_m/6=0. \qquad (4.16)$$

Since $C_{m+1}-C_m$ is written as

$$C_{m+1}-C_m=\{L_m^{(-1)}(\lambda)+(m-\lambda)(L_m(\lambda)-\mu)\}/\lambda^{m+1}, \qquad (4.17)$$

the parameters μ and λ must satisfy

$$\mu=L_m(\lambda)+L_m^{(-1)}(\lambda)/(m-\lambda) \qquad (4.18)$$

for $C_{m+1}-C_m$ to vanish. Furthermore, this condition together with (4.6), (4.8) and (4.9) yields

$$(\lambda-m)\lambda^{m+2}(C_{m+3}-C_{m+2}+C_{m+1}/2-\Psi_\mu/6)$$
$$=\frac{m-\lambda}{m+3}L'_{m+3}(\lambda)-\frac{(m+2-\lambda)(m-\lambda)}{m+2}L'_{m+2}(\lambda)+\frac{m(m+1)(m+2)/3-(m+1)^2\lambda+(m+1)\lambda^2-\lambda^3/3}{m+1}\times L'_{m+1}(\lambda) \qquad (4.19)$$

Hence, it suffices to show that the polynomoial given by the right side of (4.19) has a real root.

Let $g_m(\lambda)$ denote the polynomial. Since $L'_m(0)=-m$ for any m, $g_m(0)=m(1-m^2)/3<0$. On the other hand, the equality

$$\lim_{\lambda\to\infty}L'_{m+1}(\lambda)/\lambda^m=(-1)^{m+1}/m! \qquad (4.20)$$

implies that

$$\lim_{\lambda\to\infty}g_m(\lambda)/\lambda^{m+3}=\frac{m(m+2)}{3(m+3)!}>0 \qquad (4.21)$$

Therefore, $g_m(\lambda)$ has at least a real root.

Q.E.D.

5. *A*-acceptability

In this section we investigate the A-acceptability of the rational approximations appearing in the preceding section.

A rational approximation $R(z)$ to exp (z) is said to be A-acceptable if

$$|R(z)|\leqq 1 \text{ for any } z\in \boldsymbol{C} \text{ with } \operatorname{Im} z\leqq 0. \qquad (5.1)$$

Furthermore, a Runge-Kutta method is said to be A-stable if its stability function is A-acceptable [3]. When the rational approximation $R(z)$ has the form (1.4), the A-acceptability is determined as follows.

It is clear that λ must be positive for $R(z)$ to be A-acceptable. When λ is positive $R(z)$ is regular in the left half complex plane, and hence, by the maximum modulus principle, if

$$|(1-iy/\lambda)^m|^2-|P_m(iy)|^2\geqq 0 \qquad (5.2)$$

for any $y \in \boldsymbol{R}$, then the condition (5.1) is satisfied.

Writing $P_m(z)$ as

$$P_m(z)=\sum_{j=0}^{m} a_j z^j,\ a_j \in \boldsymbol{R}, \tag{5.3}$$

we have

$$|P_m(iy)|^2=\left\{\sum_{k=0}^{[m/2]} a_{2k}(-y^2)^k\right\}^2 + y^2\left\{\sum_{k=0}^{[(m-1)/2]} a_{2k+1}(-y^2)^k\right\}^2, \tag{5.4}$$

where $[x]$ represents the integral parts of x. Using this equality, together with

$$|(1-iy/\lambda)^m|^2=(1+y^2/\lambda^2)^m, \tag{5.5}$$

we express the left side of (5.2) as

$$E_m(u)=(1+u)^m-\left\{\sum_{k=0}^{[m/2]}(-u)^k(a_{2k}\lambda^{2k})\right\}^2 - u\left\{\sum_{k=0}^{[(m-1)/2]}(-u)^k(a_{2k+1}\lambda^{2k+1})\right\}^2 \tag{5.6}$$

where $u=y^2/\lambda^2$. Consequently, $R(z)$ of the form (1.4) is A-acceptable if and only if $\lambda>0$ and $E_m(u)\geqq 0$ for any $u\geqq 0$.

On the rational approximations of order $m+1$, it is known that the A-acceptability is possible only when $m=1$, 2, 3 or 5 [17]. We investigate the other rational approximations, described in Theorem 2 and Theorem 3, in the cases $m\leqq 6$.

By Theorem 2, we obtain rational approximations of the form (1.4) with $p=3$ and $q=6$ when $m=3$ and with $p=5$ and $q=8$ when $m=5$. These approximations are given by (4.4) with the real roots of the polynomial $f_m(\lambda)$ appearing in the proof of Theorem 2. Similarly, by Theorem 3, we obtain rational approximations with $p=1$ and $q=6$ when $m=2$, with $p=3$ and $q=8$ when $m=4$ and with $p=5$ and $q=10$ when $m=6$. These approximations are given by (4.14) and (4.17) with the real roots of the polynomial $g_m(\lambda)$ appearing in the proof of Theorem 3.

In the five cases above, we find the real roots of $f_m(\lambda)$ or $g_m(\lambda)$ numerically, and determine the A-acceptability of $R(z)$ corresponding to each real root by investigating $E_m(u)$ which is approximately obtained.

(i) The case $m=2$, $p=1$, $q=6$.

The polynmial

$$g_2(\lambda)=-2+4\lambda-(10/3)\lambda^2+(4/3)\lambda^3 -(4/15)\lambda^4+\lambda^5/45. \tag{5.7}$$

has the real root, $\lambda_1=3.51909015....$ We obtain approximately

$$E_2(u)=u(2.1085-0.4594u). \tag{5.8}$$

Thus, the corresponding rational approximation is not A-acceptable.

(ii) The case $m=3$, $p=3$, $q=6$.

The polynomial

$$f_3(\lambda)=3-5\lambda+(5/2)\lambda^2-\lambda^3/2+\lambda^4/30 \tag{5.9}$$

has the two real roots, $\lambda_1=1.02493188...$ and $\lambda_2=7.33493979....$ In this case, we obtain approximately

$$E_3(u)=u^2(0.1169+0.5396u) \tag{5.10}$$

for λ_1, and

$$E_3(u)=u^2(-6.6801-35.8984u) \tag{5.11}$$

for λ_2. Thus, only the rational approximation corresponding to λ_1 is A-acceptable.

(iii) The case $m=4$, $p=3$, $q=8$.

The polynomial

$$g_4(\lambda)=-20+48\lambda-42\lambda^2+(56/3)\lambda^3-(14/3)\lambda^4 +(2/3)\lambda^5-(16/315)\lambda^6+\lambda^7/630 \tag{5.12}$$

has the three real roots,

$\lambda_1=0.88516994...,\quad \lambda_2=5.34281697...$

and $\lambda_3=9.94198815....$

We obtain

$$E_4(u)=u^2(0.0553+0.24780u+0.5716u^2),$$
$$E_4(u)=u^2(1.8419-1.6480u-10.4180u^2) \tag{5.13}$$

and

$$E_4(u)=u^2(-8.0558-38.8468u-187.7797u^2),$$

for λ_1, λ_2 and λ_3, respectively. Thus, the A-acceptability of $R(z)$ is obtained for λ_1.

(iv) The case $m=5$, $p=5$, $q=8$.

The polynomial

$$f_5(\lambda)=5-13\lambda+10\lambda^2-(10/3)\lambda^3+(13/24)\lambda^4 -\lambda^5/24+\lambda^6/840 \tag{5.14}$$

has the four real roots,

$\lambda_1=0.63700032...,\quad \lambda_2=2.21458814...,$

$\lambda_3=7.90620732...$ and $\lambda_4=14.00659861....$

We obtain

$$E_5(u)=u^3(-0.0219-0.2966u+0.6936u^2),$$
$$E_5(u)=u^3(0.1783+1.4482u+0.1643u^2),$$
$$E_5(u)=u^3(-9.5705-39.3597u-46.9197u^2) \tag{5.15}$$

and

$$E_5(u)=u^3(82.7562+2217.0072u-20139.7094u^2),$$

for λ_1, λ_2, λ_3 and λ_4, respectively. Thus, the A-acceptability of $R(z)$ is obtained for λ_2.

(v) The case $m=6$, $p=5$, $q=10$.

The polynomial

$$g_6(\lambda)=-70+228\lambda-248\lambda^2+(400/3)\lambda^3 -(165/4)\lambda^4+(47/6)\lambda^5-(167/180)\lambda^6 +\lambda^7/15-\lambda^8/378+\lambda^9/22680 \tag{5.16}$$

has the five real roots,

$\lambda_1=0.56664636\ldots,\quad \lambda_2=1.98010163\ldots,$
$\lambda_3=6.9445970\ldots,\quad \lambda_4=10.22462450\ldots$

and

$$\lambda_5=16.98815977.\ldots$$

We obtain

$$E_6(u)=u^3(-0.0076-0.0670u-0.4538u^2+0.7079u^3),$$
$$E_6(u)=u^3(0.0709+0.4871u+1.9509u^2+0.2743u^3),$$
$$E_6(u)=u^3(-0.3481-12.1442u-36.4719u^2-38.9178u^3),$$
$$E_6(u)=u^3(-12.3466-51.1070u-140.2317u^2-229.3559u^3)$$

and

$$E_6(u)=u^3(76.6586+1201.2564u+22930.8473u^2-205517.7805u^3), \tag{5.17}$$

for λ_1, λ_2, λ_3, λ_4 and λ_5, respectively. Thus, the A-acceptability of $R(z)$ is obtained for λ_2.

5. Concluding Remark

In this paper, we have constructed higher order A-acceptable rational approximations with reduced phase errors, but have not discussed the corresponding A-stable DIRK methods. In practice, it is important to construct such A-stable DIRK methods and this is a future problem.

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References

1. Alexander, R. Diagonally Implicit Runge-Kutta Methods for Stiff O. D. E.'s, *SIAM J. Numer. Anal.*, **14** (1977), 1006–1021.
2. Brusa, L. and Nigro, L. A One-Step Method for Direct Integration of Structural Dynamic Equations, *Internat. J. Numer. Methods Eng.* **15** (1980), 685–699.
3. Butcher, J. C. The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods, *John Wiley and Sons*, Chichester, 1987.
4. Cash, J. R. Higher Order, P-stable Formulae for the Numerical Integration of Periodic Initial Value Problems, *Numer. Math.*, **37** (1981), 355–370.
5. Chawla, M. M. and Rao, P. S. A Numerov-Type Method with Minimal Phase-Lag for the Integration of Second Order Periodic Initial Value Problems II. Explicit Method, *J. Comput. Appl. Math.*, **15** (1986), 329–337.
6. Chawla, M. M., Rao, P. S. and Neta, B. Two-Step Fourth Order P-stable Methods with Phase-Lag of Order Six for $y''=f(t, y)$, *J. Comput. Appl. Math.*, **16** (1986), 233–236.
7. Gladwell, I and Thomas, R. M. Damping and Phase Analysis of Some Methods for Solving Second Order Ordinary Differential Equations, *Internat. J. Numer. Methods Eng.*, **19** (1983), 495–503.
8. Hildebrand, F. B. Introduction to Numerical Analysis, *McGraw-Hill*, New York, 1956.
9. Koto, T. Phase Lag Analysis of Implicit Runge-Kutta Methods (in Japanese), *Trans. IPS Japan*, **30** (1989), 1357–1363.
10. Nørsett, S. P. Restricted Padé Approximation to the Exponential Function, *SIAM J. Numer. Anal.*, **15** (1978), 1008–1029.
11. Scherer, R. A Necessary Condition for B-stability, *BIT*, **19** (1979), 111–115.
12. Thomas, R. M. Phase Properties of High Order, Almost P-stable Formulae, *BIT*, **24** (1984), 225–238.
13. Twizell, E. H. Phase-Lag Analysis for a Family of Two-Step Methods for Second Order Periodic Initial Value Problems, *J. Comput. Appl. Math.*, **15** (1986), 261–263.
14. van der Houwen, P. J. and Sommeijer, B. P. Explicit Runge-Kutt (-Nyström) Methods with Reduced Phase Errors for Computing Oscillating Solutions, *SIAM J. Numer. Anal.*, **24** (1987), 595–617.
15. van der Houwen, P. J. and Sommeijer, B. P. Phase-Lag Analysis of Implicit Runge-Kutta Methods, *SIAM J. Numer. Anal.*, **26** (1989), 214–229.
16. van der Houwen, P. J. and Sommeijer, B. P. Diagonally Implicit Runge-Kutta-Nyström Methods for Oscillatory Problems, *SIAM J. Numer. Anal.*, **26** (1989), 414–429.
17. Wanner, G., Hairer, E. and Nørsett, S. P. Order Stars and Stability Theorems, *BIT*, **18** (1978), 475–489.

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