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5-ブレイドの3－並行化不変量のための行列表現のコンピュータによる構成

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ヘッケ環H(q,15)の行列表現から、5－ブレイドで与えられる結び目の3－並行化不変量（村上不変量）を計算するために必要な表現行列をコンピュータを利用して構成した。村上不変量をコンピュータで計算するために、表現の適当な部分空間を作り、それから計算可能な表現を構成した。作られた部分空間の行列表現から、5－ブレイド表現を持つ結び目の村上不変量を、妥当な時間内に実際に計算可能であることを示した。また、寺坂－樹下結び目とコンウェー結び目の5－ブレイド表現から村上不変量を計算し、4－ブレイドの場合と同じ計算結果を得た。

Computational construction of representation matrices for 3-pararell version polynomial invariants of 5-braids

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We construct by computer necessary representation matrices to compute 3-pararell version polynomial invariants, called Murakami invariants, of knots with 5-braid forms using W-graphs of Hecke algebras H(q,15). All matrices, corresponding to irreducible representations of Hecke algebras H(q,15), had already been given in [OK] but no direct calculations of Murakami invariants to use them is well adapted for computational computations as it involves product calculations of very big matrices. Hence we construct certain subspaces of representation matrices of the irreducible representation matrices of H(q,15). Furthermore, we verified to able to compute in adequate time Murakami's invariants of knots with 5-braid forms, including Terasaka-Kinosita knot and Conway knot, using such the matrix representations.

1 Introduction

V. Jones [Jo1] discovered a polynomial invariant in one variable which is an invariant of oriented links, and also defined in [Jo2] another two-variable invariant $X_L(q,\lambda)$ of an oriented link L given by the following formula

$$X_L(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e tr(\pi(\alpha))$$

, where α is any element of the braid group B_n with $\hat{\alpha} = L$, e being the exponent sum of α and π is the representation of B_n in the Hecke algebra $H(q,n)$ sending the standard generators of B_n to those of $H(q,n)$. But it is already known that no polynomial invariants of Conway type can recognize two different mutant knots and so that no Jones invariants in one or two variable can distinguish Kinoshita-Terasaka knot KT and Conway knot KC. Under such situation, J. Murakami [M] found that 3-parallel version of 2-variable Jones polynomials distinguish certain mutant knots and later 3-parallel version of a special type of 2-variable Jones polynomial of 4 braids can distinguish KT and KC [OM].

2 3-parallel version of a special type of 2-variable Jones polynomial

Let L be a link, α be an element of the braid group B_n with $\hat{\alpha} = L$, and β be the 3-parallel version of α (see [M]). Then the following Laurent polynomial

$$X_L^{(3)}(q,\lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e tr(\pi(\beta))$$

is a polynomial invariant of L (see [M]).

Let $H(q,n)$ be a C-algebra with a unit defined by the following relations:

$$H(q,n) = \langle g_1, g_2, \cdots, g_{n-1} | \quad \begin{aligned} g_i^2 &= (q-1)g_i + q, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\ g_i g_j &= g_j g_i, \text{if} |i-j| \geq 2 \rangle \end{aligned}$$

Then $H(q,n)$ is called a Hecke algebra of type A_{n-1} and each generator g_i is called a standard generator of it. Let $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ be the standard generators of B_n and $\Psi_n^{(3)} : CB_n \to H(q,3n)$ be the algebra homomorphism defined by

$\Psi_n^{(3)}(\sigma_i) = g(3i-2,3i-1)^{-3} g(3i,3i+2) g(3i-1,3i+1) g(3i-2,3i)$

where $g(i,j) = g_i g_{i+1} \cdots g_j (1 \leq i < j \leq n-1)$.

In [OK], we gave the irreducible representations of $H(q,9)$, $H(q,12)$, and $H(q,15)$ and so can compute $X_L^{(3)}(q,\lambda)$ for every link L whose braid form has braid index of 3, 4, and 5.

It is necessary for a direct calculation of $X_L^{(3)}(q,\lambda)$ of a 5 braid of length r to compute about $10r$ many times products of matrices of size n up to 292864. But Murakami method [M] and [OM] needs only the similar products of matrices of size n up to 1449.

Let Y be a Young diagram for the set $\Lambda(n)$ of partitions of a positive integer n , and let $X = \{x_1, x_2, ..., x_s\}$ be the collection of words induced from the standard Young tableaux generated by Y (see [OK]). For each element x of X, define I-invariant $I(x)$ as the set of $i \in \{1,2,...,n-1\}$ such that the row containing i is above the one containing $i+1$ in x

([G1,2,3]). Let Y be a Young diagram associated with $\Lambda(3n)$, and $G(Y)$ be the W-graph with the vertex set $V(Y) = \{\chi_1, \chi_2, ..., \chi_s\}$ labelled by $I(G(Y)) = \{I(\chi_1), I(\chi_2), ..., I(\chi_s)\}$ corresponding to Y, where $I(\chi_i)$ is the I-invariant of χ_i. Then define a subset $V^3(Y)$ of $V(Y)$ as follows:

Each vertex χ in $V(Y)$ is included in $V^3(Y)$ if and only if $I(\chi)$ contains all numbers k in $I = \{1, 2, ..., 3n\}$ with $k \equiv 2(mod\ 3)$ and no numbers k in I with $k \equiv 1(mod\ 3)$.

Let π_Y be the representation given by the W-graph $G(Y)$, let π_Y^* be the restriction to $\pi_Y \circ \Psi^{(3)}$ on the subspace U spanned by the basis corresponding to $V^3(Y)$. Then it is well known that π_Y^* gives a representation of B_n (see [M]). Then by Theorem 1.5.1 of [M], the following $X_L^{(3)}(q,\lambda)^*$ is an invariant of L.

$$X_L^{(3)}(q,\lambda)^* = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e \textstyle\sum_Y W_Y(q,\lambda)\omega_Y^{(3)}(\alpha)$$

,where $W_Y(q,\lambda) = S(q,\lambda)/Q(q)$ and $\omega_Y^{(3)}$ is the trace of π_Y^*.

Later we will compute only the special type $X_L^{(3)}(q,q^3)^*$ called by a Murakami-invariant of L.

$Q(q)$, $S(q,\lambda)$ are as follows:

Let Y be a Young diagram $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then each node of Y has a value, called by hook length, which is the total number of nodes which exist on the right and downward direction. For an example, a Young diagram (10, 4, 1) has the following hook length.

12	10	9	8	6	5	4	3	2	1
5	3	2	1						
1									

Then we have that

$Q(q) = (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^8)(1-q^9)(1-q^{10})(1-q^{12})$ $(1-q)(1-q^2)(1-q^3)(1-q^5)(1-q)$

and

$S(q,\lambda) = (1-\lambda q)(q-\lambda q)(q^2-\lambda q)(q^3-\lambda q)(q^4-\lambda q)(q^5-\lambda q)(q^6-\lambda q)(q^7-\lambda q)(q^8-\lambda q)(q^9-\lambda q)(1-\lambda q^2)(q-\lambda q^2)(q^2-\lambda q^2)(q^3-\lambda q^2)(1-\lambda q^3)$.

$1-\lambda q$	$q-\lambda q$	$q^2-\lambda q$	$q^3-\lambda q$	$\cdots$
$1-\lambda q^2$	$q-\lambda q^2$	$q^2-\lambda q^2$	$\cdots$	
$1-\lambda q^3$	$q-\lambda q^3$	$\cdots$	$\cdots$	
$1-\lambda q^4$	$\vdots$	$\vdots$		
$\vdots$				

3 Computational results

By computational computation using the result of [OK], we got the following list which means that the first column gives a Young diagram Y, the second gives the dimension of π_Y, and the third gives the one of π_Y^* :

[[14 1], 14, 0], [[13 2], 90, 0],
[[13 1 1], 91, 0], [[12 3], 350, 0],
[[12 2 1], 715, 0], [[12 1 1 1], 364, 0],
[[11 4], 910, 0], [[11 3 1], 2835, 0],

[[11 2 2], 1925, 0], [[11 2 1 1], 2925, 0],
[[11 1 1 1 1], 1001, 0], [[10 5], 1638, 1],
[[10 4 1], 7007, 4], [[10 3 2], 9100, 5],
[[10 3 1 1], 11088, 6], [[10 2 2 1], 9450, 5],
[[10 2 1 1 1], 7722, 4], [[10 1 1 1 1 1], 2002, 1],
[[9 6], 2002, 4], [[9 5 1], 11375, 20],
[[9 4 2], 22113, 36], [[9 4 1 1], 25025, 40],
[[9 3 3], 12740, 20], [[9 3 2 1], 42042, 64],
[[9 3 1 1 1], 26950, 40], [[9 2 2 2], 13650, 20],
[[9 2 2 1 1], 24948, 36], [[9 2 1 1 1 1], 14300, 20],
[[9 1 1 1 1 1 1], 3003, 4], [[8 7], 1430, 5],
[[8 6 1], 11583, 36], [[8 5 2], 32032, 94],
[[8 5 1 1], 35100, 100], [[8 4 3], 35035, 100],
[[8 4 2 1], 91000, 250], [[8 4 1 1 1], 53625, 140],
[[8 3 3 1], 57330, 155], [[8 3 2 2], 58968, 156],
[[8 3 2 1 1], 100100, 255], [[8 3 1 1 1 1], 44550, 105],
[[8 2 2 2 1], 42042, 104], [[8 2 2 1 1 1], 43120, 100],
[[8 2 1 1 1 1 1], 19305, 40], [[8 1 1 1 1 1 1 1], 3432, 6],
[[7 7 1], 5005, 20], [[7 6 2], 25025, 100],
[[7 6 1 1], 27027, 104], [[7 5 3], 45045, 180],
[[7 5 2 1], 108108, 416], [[7 5 1 1 1], 61425, 220],
[[7 4 4], 25025, 100], [[7 4 3 1], 135135, 520],
[[7 4 2 2], 112112, 424], [[7 4 2 1 1], 184275, 660],
[[7 4 1 1 1 1], 75075, 240], [[7 3 3 2], 90090, 340],
[[7 3 3 1 1], 122850, 440], [[7 3 2 2 1], 159250, 560],
[[7 3 2 1 1 1], 150150, 480], [[7 3 1 1 1 1 1], 51975, 140],
[[7 2 2 2 2], 34398, 116], [[7 2 2 2 1 1], 70070, 220],
[[7 2 2 1 1 1 1], 51975, 140], [[7 2 1 1 1 1 1 1], 19305, 40],
[[7 1 1 1 1 1 1 1 1], 3003, 4], [[6 6 3], 21450, 100],
[[6 6 2 1], 50050, 225], [[6 6 1 1 1], 28028, 116],
[[6 5 4], 30030, 145], [[6 5 3 1], 128700, 600],
[[6 5 2 2], 100100, 460], [[6 5 2 1 1], 162162, 699],
[[6 5 1 1 1 1], 63700, 240], [[6 4 4 1], 80080, 380],
[[6 4 3 2], 175175, 820], [[6 4 3 1 1], 231660, 1020],
[[6 4 2 2 1], 243243, 1056], [[6 4 2 1 1 1], 221130, 855],
[[6 4 1 1 1 1 1], 70070, 220], [[6 3 3 3], 50050, 235],
[[6 3 3 2 1], 210210, 920], [[6 3 3 1 1 1], 156000, 610],
[[6 3 2 2 2], 112112, 474], [[6 3 2 2 1 1], 221130, 855],
[[6 3 2 1 1 1 1], 150150, 480], [[6 3 1 1 1 1 1 1], 43120, 100],
[[6 2 2 2 2 1], 63700, 240], [[6 2 2 2 1 1 1], 75075, 240],
[[6 2 2 1 1 1 1 1], 44550, 105], [[6 2 1 1 1 1 1 1 1], 14300, 20],
[[6 1 1 1 1 1 1 1 1 1], 2002, 1], [[5 5 5], 6006, 32],
[[5 5 4 1], 54054, 283], [[5 5 3 2], 96525, 500],
[[5 5 3 1 1], 126126, 612], [[5 5 2 2 1], 125125, 600],
[[5 5 2 1 1 1], 112112, 474], [[5 5 1 1 1 1 1], 34398, 116],
[[5 4 4 2], 81081, 432], [[5 4 4 1 1], 100100, 500],

[[5 4 3 3], 75075, 400], [[5 4 3 2 1], 292864, 1448],
[[5 4 3 1 1 1], 210210, 920], [[5 4 2 2 2], 125125, 600],
[[5 4 2 2 1 1], 243243, 1056], [[5 4 2 1 1 1 1], 159250, 560],
[[5 4 1 1 1 1 1 1], 42042, 104], [[5 3 3 3 1], 100100, 500],
[[5 3 3 2 2], 126126, 612], [[5 3 3 2 1 1], 231660, 1020],
[[5 3 3 1 1 1 1], 122850, 440], [[5 3 2 2 2 1], 162162, 699],
[[5 3 2 2 1 1 1], 184275, 660], [[5 3 2 1 1 1 1 1], 100100, 255],
[[5 3 1 1 1 1 1 1 1], 24948, 36], [[5 2 2 2 2 2], 28028, 116],
[[5 2 2 2 2 1 1], 61425, 220], [[5 2 2 2 1 1 1 1], 53625, 140],
[[5 2 2 1 1 1 1 1 1], 26950, 40], [[5 2 1 1 1 1 1 1 1 1], 7722, 4],
[[5 1 1 1 1 1 1 1 1 1 1], 1001, 0], [[4 4 4 3], 24024, 138],
[[4 4 4 2 1], 75075, 400], [[4 4 4 1 1 1], 50050, 235],
[[4 4 3 3 1], 81081, 432], [[4 4 3 2 2], 96525, 500],
[[4 4 3 2 1 1], 175175, 820], [[4 4 3 1 1 1 1], 90090, 340],
[[4 4 2 2 2 1], 100100, 460], [[4 4 2 2 1 1 1], 112112, 424],
[[4 4 2 1 1 1 1 1], 58968, 156], [[4 4 1 1 1 1 1 1 1], 13650, 20],
[[4 3 3 3 2], 54054, 283], [[4 3 3 3 1 1], 80080, 380],
[[4 3 3 2 2 1], 128700, 600], [[4 3 3 2 1 1 1], 135135, 520],
[[4 3 3 1 1 1 1 1], 57330, 155], [[4 3 2 2 2 2], 50050, 225],
[[4 3 2 2 2 1 1], 108108, 416], [[4 3 2 2 1 1 1 1], 91000, 250],
[[4 3 2 1 1 1 1 1 1], 42042, 64], [[4 3 1 1 1 1 1 1 1 1], 9450, 5],
[[4 2 2 2 2 2 1], 27027, 104], [[4 2 2 2 2 1 1 1], 35100, 100],
[[4 2 2 2 1 1 1 1 1], 25025, 40], [[4 2 2 1 1 1 1 1 1 1], 11088, 6],
[[4 2 1 1 1 1 1 1 1 1 1], 2925, 0], [[4 1 1 1 1 1 1 1 1 1 1 1], 364, 0],
[[3 3 3 3 3], 6006, 32], [[3 3 3 3 2 1], 30030, 145],
[[3 3 3 3 1 1 1], 25025, 100], [[3 3 3 2 2 2], 21450, 100],
[[3 3 3 2 2 1 1], 45045, 180], [[3 3 3 2 1 1 1 1], 35035, 100],
[[3 3 3 1 1 1 1 1 1], 12740, 20], [[3 3 2 2 2 2 1], 25025, 100],
[[3 3 2 2 2 1 1 1], 32032, 94], [[3 3 2 2 1 1 1 1 1], 22113, 36],
[[3 3 2 1 1 1 1 1 1 1], 9100, 5], [[3 3 1 1 1 1 1 1 1 1 1], 1925, 0],
[[3 2 2 2 2 2 2], 5005, 20], [[3 2 2 2 2 2 1 1], 11583, 36],
[[3 2 2 2 2 1 1 1 1], 11375, 20], [[3 2 2 2 1 1 1 1 1 1], 7007, 4],
[[3 2 2 1 1 1 1 1 1 1 1], 2835, 0], [[3 2 1 1 1 1 1 1 1 1 1 1], 715, 0],
[[3 1 1 1 1 1 1 1 1 1 1 1 1], 91, 0], [[2 2 2 2 2 2 2 1], 1430, 5],
[[2 2 2 2 2 2 1 1 1], 2002, 4], [[2 2 2 2 2 1 1 1 1 1], 1638, 1],
[[2 2 2 2 1 1 1 1 1 1 1], 910, 0], [[2 2 2 1 1 1 1 1 1 1 1 1], 350, 0],
[[2 2 1 1 1 1 1 1 1 1 1 1 1], 90, 0], [[2 1 1 1 1 1 1 1 1 1 1 1 1 1], 14, 0]

Let us consider the output datum in the above list:

For an example, we adopt the case of Young diagram $Y_{13} = [10\ 4\ 1]$ and its adjoint $Y_{164} = [3222111111]$. In this case, there happends a collection $X_{13} = \{\chi_1, \chi_2, \ldots, \chi_{7007}\}$ (res. $X_{164} = \{\chi_1^*, \chi_2^*, \ldots, \chi_{7007}^*\}$) of words such that only χ_{437}, χ_{1823} , χ_{1997} , and χ_{2017} (res. χ_{2927}^*, χ_{2941}^* , χ_{3065}^* , and χ_{4045}^*) are included in $V^3(Y_{13})$ (res. $V^3(Y_{164})$), where

$$\chi_{437} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 8 & 10 & 11 & 13 & 14 \\ \hline 3 & 6 & 9 & 12 \\ \cline{1-4} 15 \\ \cline{1-1} \end{array} \qquad (1)$$

,

$$\chi_{1823} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 8 & 10 & 11 & 13 & 14 \\ \hline 3 & 6 & 9 & 15 \\ \cline{1-4} 12 \\ \cline{1-1} \end{array} \tag{2}$$

,

$$\chi_{1997} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 8 & 10 & 11 & 13 & 14 \\ \hline 3 & 6 & 12 & 15 \\ \cline{1-4} 9 \\ \cline{1-1} \end{array} \tag{3}$$

, and

$$\chi_{2017} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 8 & 10 & 11 & 13 & 14 \\ \hline 3 & 9 & 12 & 15 \\ \cline{1-4} 6 \\ \cline{1-1} \end{array} \tag{4}$$

(resp. 省略).

Really χ_{437}, χ_{1823} , χ_{1997} , and χ_{2017} (res. χ^*_{2927}, χ^*_{2941} , χ^*_{3065} , and χ^*_{4045}) are included in $V^3(Y_{13})$ (resp. $V^3(Y_{164})$) , because all of them have an I-invariant $\{2,5,8,11,14\}$ (resp. $\{2,3,5,6,8,9,11,12,14\}$) which satifies with the subspace condition mentioned above and every word other than those don't satisfy with the one.

For an example,

$$\chi = \begin{array}{|c|c|c|} \hline 1 & 3 & 14 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} 6 & 11 \\ \cline{1-2} 7 \\ \cline{1-1} 9 \\ \cline{1-1} 10 \\ \cline{1-1} 12 \\ \cline{1-1} 13 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array} \tag{5}$$

has an I-invariant $\{3,5,6,8,9,11,12,14\}$, and then it don't satisfy with the condition because it's I-invarint don't contain 2.

As another computational result, we have the following matrix representation corresponding to Y_{13} (resp. Y_{164}):

$$\sigma_1 \to \begin{pmatrix} q^4 & 0 & 0 & 0 \\ 0 & q^4 & 0 & 0 \\ 0 & 0 & q^4 & 0 \\ 0 & 0 & q^3\sqrt{q} & -q^3 \end{pmatrix} \tag{6}$$

$$\sigma_2 \to \begin{pmatrix} q^4 & 0 & 0 & 0 \\ 0 & q^4 & 0 & 0 \\ 0 & q^3\sqrt{q} & -q^3 & 0 \\ 0 & 0 & 0 & q^4 \end{pmatrix} \tag{7}$$

$$\sigma_3 \to \begin{pmatrix} q^4 & 0 & 0 & 0 \\ q^3\sqrt{q} & -q^3 & q^3\sqrt{q} & 0 \\ 0 & 0 & q^4 & 0 \\ 0 & 0 & 0 & q^4 \end{pmatrix} \tag{8}$$

$$\sigma_4 \rightarrow \begin{pmatrix} -q^3 & q^3\sqrt{q} & 0 & 0 \\ 0 & q^4 & 0 & 0 \\ 0 & 0 & q^4 & 0 \\ 0 & 0 & 0 & q^4 \end{pmatrix} \tag{9}$$

(resp. 省略).

KT (resp. CK) is the closure of a 5 braid $\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^2\sigma_3^{-1}\sigma_1^{-4}\sigma_2^2\sigma_4$ (resp. $\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^2\sigma_3^{-1}\sigma_1\sigma_2^{-3}\sigma_4$). Then we tried to compute $X_{KT}^{(3)}(q,q^3)^*$ and $X_{CK}^{(3)}(q,q^3)^*$ and got the same result as given in [OM]. On a personal computer with DEC Alpha 300 Hz CPU, our program to compute such the invariant needed about 7 minutes and 68 M bytes main memory.

In conclusion, we hope to have complete lists and also our computer programs (KnotTheorybyComputer Windows 95 version) available on the ftp server in the site ftp@ics.nara-wu.ac.jp at the directory /export2/ftp/pub/ochiai so that readers in this field of knot theory, representation theory, and etc. can use them.

Reference

[G1] A. Gyoja, On the Existence of a W-graph for an Irreducible Representation of a Coxeter Group, J. Algebra, **86** (1984), 422-438.

[G2] A. Gyoja, Topological Invariants of Links and Representations of Hecke Algebras, preprint.

[G3] A. Gyoja, Topological Invariants of Links and Representations of Hecke Algebras II, preprint.

[Jo1] V.F.R.Jones, A polynomial invariants for knots via von Neumann algebras, Bull. A. M. S. ,**12** (1985), 103-111.

[Jo2] V.F.R.Jones, Hecke algebra representations of braid groups and link polynomias, Annals of Mathematics, **126** (1987), 335-388.

[M] J. Murakami, The Parallel Version of Polynomial Invariants of Links, Osaka J. **26** (1989),1-55.

[OM] M. Ochiai and J.Murakami, Subgraphs of W-graphs and 3-parallel invariants of knots, Proc. of Japan Acad. VOL. LXX, Sere. A No. 8 267-270(1994).

[OK] M. Ochiai and F. Kako, Computational Construction of W-graphs of Hecke algebras H(q,n) for n up to 15, Experimental Mathematics Vol. 4 61-67(1995).