数理モデル化と問題解決 27－4
(1999. 11. 25)

生命と知能の情報物理学 (2) 信号保存論理によるべき乗則分布の証明

稲垣 耕作
京都大学大学院情報学研究科

Information Physics of Life and Intelligence (2)
— A Proof of Power-Law Distribution by Signal Conservation Logic —

Kosaku INAGAKI
Graduate School of Informatics, Kyoto University

Abstract This paper investigates power-law distribution exhibited by cellular automata based on signal conservation logic (SCL) and proves its theoretical exactness. SCL is a model of logic for the physical world subject to the matter conservation law. Power-law distribution is caused by a kind of discrete relaxation via some logical transformations of white noise. The proposed model gives us various insights into this law including: (1) power-law distribution is generated by positive logic, which means its irrelevance to computational universality in the sense of elemental universality, (2) context-free grammar can generate power-law distribution, (3) a hypothetical model for $1/f$ spectrum, (4) relation between discrete power laws caused by perfectly inelastic collisions and the entropy problem.

1 Introduction

A problem variously called such as power law, $1/f$ noise, scaling law, Zipf's law, Pareto distribution, and pink noise is investigated. The $1/f$ noise was discovered by J. B. Johnson in 1925 [1]. Similar power-law distributions appear in word frequencies found by G. K. Zipf [2] and in many physical, biological, and social phenomena including fractals and V. F. D. Pareto's discovery concerning income distribution [3, 4, 5, 6].

We still do not know the very origin of this famous law. Since the Pareto distribution, $\alpha k^{\alpha} x^{-(\alpha+1)}$, is defined for $\alpha > 0$, the true $1/f$ distribution cannot exist in mathematical meaning. Simply speaking, it is because the integral of $1/f$ gives infinity.

This paper focuses on $1/x$-type power-law distributions that have a power exactly -1. Here x does not necessarily mean frequency and the function form is generally $1/(ax+b)$. The signal conservation logic (SCL) proposed by the author [7] is employed to prove an exact power law exhibited by cellular automata.

Since positive logic can generate $1/x$-type distribution, in general it has no connection with elemental universality, a kind of computational universality proposed by the author as a basic law for life and intelligence [7, 8, 9, 10].

For definitions of SCL, see Inagaki [7].

2 Collision Automaton

Here we shall define a very simple cellular automaton based on SCL. We easily know that the simplest SCL element is $z_1 = x_1$, which is trivial. The second simplest SCL element, B, is defined as: $z_1 = x_1 + x_2, z_2 = x_1 x_2$. We call this SCL element the *collision gate* or *collision cell.*

We construct a one-dimensional cellular automaton by repetitive use of the collision gates. Each gate B_j has a unit delay. We call this automaton the *collision automaton.* At time t, the inputs and outputs of cell B_j are expressed by $x_{j,i}^{(t)}$ and $z_{j,i}^{(t)}$. Then $x_{j,1}^{(t)} = z_{j-1,2}^{(t)}$ and $x_{j,2}^{(t)} = z_{j,2}^{(t)}$. From these values $z_{j,i}^{(t+1)}$'s at the next time step will be calculated. Namely, the neighborhood of cell B_j contains only two cells B_{j-1} and B_j.

The number of cells B_j in a one-dimensional cellular automaton is generally unbounded in left and right directions. Here we consider a collision automaton only in the region $j = 1, 2, \cdots, n$, i.e., bounded in both directions.

At time -1, initial inputs to these n cells are applied such that $x_{j,1}^{(-1)}$ and $x_{j,2}^{(-1)}$ are random numbers 0 or 1 with equal probability 1/2. The input patterns to cells are $(0,0), (0,1), (1,0)$, and $(1,1)$ with equal probability 1/4. The cells outside this region are all assumed fed by $(0,0)$.

Note that the average Hamming weight of the input vector for a cell in this region is approximately 1. This property is conserved at any time, except the signal flow at both boundaries of the region.

Input-output transformation by a cell itself is: $(0,0) \rightarrow (0,0); (0,1), (1,0) \rightarrow (1,0); (1,1) \rightarrow (1,1)$. Each cell takes on one of $(0,0), (1,0)$, and $(1,1)$ at each time step, which means the cell is a three-state machine. The cells at time 0 and afterwards have these values.

We regard state $(1,0)$, whose Hamming weight is 1, as a basic state of this cellular automaton. It is easily known that, if B_j assumes $(1,0)$ at time t and its neighborhood cell B_{j-1} assumes state $(0,0)$ or $(1,0)$, then B_j remains $(1,0)$ at time $t+1$.

In Fig. 1, binary vectors $(0,0), (1,0)$, and $(1,1)$ are represented by 0, 1 and 2, respectively, i.e., the Hamming weights of the vectors. The cells outside this region are omitted by regarding them as assuming state 0, namely, state $(0,0)$.

time 0:	1	0	1	2	1	1
time 1:	1	0	1	1	2	1
time 2:	1	0	1	1	1	2
		(a)			(b)	

time 0:	1	2	1	1	1	0	1
time 1:	1	1	2	1	1	0	1
time 2:	1	1	1	2	1	0	1
time 3:	1	1	1	1	2	0	1
time 4:	1	1	1	1	1	1	1
time 5:	1	1	1	1	1	1	1
			(c)				

Figure 1: Behaviors of collision cells.

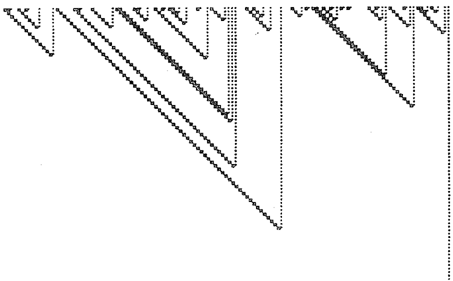

Figure 2: A Pattern exhibited by a collision automaton.

Fig. 1(a) shows the behavior of a state 0 surrounded by states 1's. It remains 0 without any interaction with 1's. Fig. 1(b) depicts the behavior of a state 2 among states 1's. It moves right step by step with time. Then Fig. 1(c) shows the interaction between states 0 and 2 surrounded by 1's. These two states collide with each other and at last vanish. Since states 0 and 2 proceed solely straight and then eventually collide, this automaton is named collision automaton.

A pattern exhibited by a collision automaton is exemplified in Fig. 2. This pattern is generated by a finite 128-cell automaton, and the signals outside this region are assumed to be 0's.

In this figure, slanting lines correspond to state 2's. Vertical dotted lines represent state 0's. The background white region means basic state 1's.

Triangles constructed by slanting and vertical lines are seen in this figure. Smaller triangles are contained in a larger one, and they never intersect with each other.

The initial assignment of this automaton happens to contain one excessive state 0, which corresponds to the rightmost vertical line. This state 0 does not collide with any state 2. However, if the automaton is unbounded to the left and the cell states are randomly assigned, some state 2 will at last collide with this state 0.

Similarly, if the number of state 2's is excessive and the defined region is finite, excessive 2's will eventually get out of the bounded region. However, if the automaton is unbounded to the right and the cell states are randomly assigned, some 0's will eventually collide with such 2's.

3 Power Law Exhibited by Collision Automata

Although the self-organized criticality caused by cellular automata [12] shows patterns very similar to the $1/x$-type power law, complete proof has not been reported. Here we can give a proof of an exact power law.

Let k be the length, or the duration time, of a state 2 that gradually moves with time and at last vanishes by collision with a state 0. The lengths of 2 and 0 that collide are equal.

Let state 2 correspond to symbol '+1', and state 0 to symbol '−1'. State 1 is omitted as a null symbol. Then we can regard the initial state assignment to a collision automaton as a random '+1' and '−1' sequence, where the symbols '+1' and '−1' occur with equal probability.

Assume that the first symbol is '+1'. If the next symbol is '−1', they collide. The collision condition is easily understood: when S, the arithmetic sum of +1's and −1's, becomes 0 for the first time, collision with the first symbol '+1' takes place.

We employ a coordinate system (j, S) such that j is the number of symbols in a sequence and S is the sum above mentioned.

When k '+1'symbols and k '−1'symbols are arranged in a sequence, the number of possible paths from $(0,0)$ to $(2k,0)$ is $N_0 = C(2k,k)$. Here $C(n,r)$ means the number of combinations of r from n objects. Similarly, the number of all paths from $(1,1)$ to $(2k,0)$ is $N_1 = C(2k-1,k)$.

We want to obtain the number N of paths from $(1,1)$ to $(2k,0)$ that do not touch or cross the j-axis except the last point. We divide such a path into two parts: a path from $(1,1)$ to $(2k-1,1)$ and that from $(2k-1,1)$ to $(2k,0)$. Since the number of latter paths is 1, N also means the number of possible paths from $(1,1)$ to $(2k-1,1)$ that do not touch or cross the j-axis.

Such numbers of paths are usually analyzed in random walk theory. We apply such theory to this problem. The reflection principle quoted in W. Feller [11] (Vol. I, Chap. III) can be utilized.

This principle regards the j-axis as a mirror. Then we can know that the number of paths from $(1,1)$ to $(2k-1,1)$ that touch or cross the j-axis is equal to the number of all paths from $(1,-1)$ to $(2k-1,1)$, which is $N_a = C(2k-2,k)$.

The number of all paths from $(1,1)$ to $(2k-1,1)$ is $N_2 = C(2k-2,k-1)$. Then N is calculated as follows:

$$\begin{aligned} N &= N_2 - N_a \\ &= N_2 - (k-1)N_2/k \\ &= N_2/k. \end{aligned} \tag{1}$$

The number N_1 of all paths from $(1,1)$ to $(2k,0)$ contains the paths that go through $(2k-1,-1)$, the number of which is again N_a. Then,

$$\begin{aligned} N &= N_1 - 2N_a \\ &= N_1 - 2(k-1)N_1/(2k-1) \\ &= N_1/(2k-1). \end{aligned} \tag{2}$$

Since $N_0 = 2N_1$, the relation between N and N_0 is:

$$N = N_0/2(2k-1). \tag{3}$$

In an unbounded sequence composed of symbols '+1' and '−1', subsequences of length $2k$ occurs substantially the same times for any k. For example, from coordinates $(0,0)$ we have sequences of any length k. Hence the probability of the collision of length k is mathematically proportional to $1/(2k-1)$.

In patterns generated by the collision automata, state 1's are randomly scattered among states 0's and 2's. Such states do not affect the power-law property, if only they are thoroughly randomly scattered. Or we may not assign such states to the collision automata. States '1' are used to make Fig. 2 easy to be observed. The following theorem holds:

Theorem 1 *The collision automaton exhibits the power-law distribution whose power is* -1.

Computer experiments of collision automata make a good accordance with this theorem.

4 Discussions

Mathematical Reality but Physical Approximation: The equation (3) is a mathematical reality. If collision cells are placed boundlessly in both directions, the power law holds as a theoretical reality. Such an unbounded cellular automaton will be classified as a so-called open system.

However, a mathematical difficulty is associated with such infinite systems. The Pareto distribution, or any legitimate distribution, cannot be defined for power -1 in spite of doubtless existence of such distribution.

If the number of cells is finite, the power law must be regarded as a mere approximation. Such a cellular automaton is definitely a closed system, and can be regarded as a physically plausible system. Such finite systems can by no means exhibit real power laws.

Some discrepancy between mathematics and physics, or between finiteness and infinity, may exist in this problem. We do not have enough mathematical tools to investigate such infinite or open systems as yet. See also the author's discussion about the mismatching between mathematics and physics [10].

Positive Logic and Nonlinearity: We should pay attention to the logic realized by a collision gate. The gate realizes OR and AND. They are both positive functions.

It means that the collision cell cannot realize NOT function. Thus collision gates are not sufficient parts to construct a universal computer.

The collision gate does not have elemental universality [8, 9, 10, 7], a kind of computational universality proposed by the author. He hypothesizes that biological life is based on the elemental universality. If so, the power law may not be a principal law of life, although deeper discussions must be done continually.

OR and AND are classified as nonlinear functions in elemental universality theory. The author thinks that linear functions (i.e., exclusive OR's, typically) never exhibit power-law distributions because of their characteristics. It means that the power law is a nonlinear property.

Relation to Context-Free Grammars: If we change the state 2 to a symbol '(' and the state 0 to a symbol ')', then the theory can be interpreted as 'grammar of parentheses,' which belongs to context-free grammars.

It means that context-free languages can exhibit power-law distributions. The following rewriting rule generates all legal sequences of parentheses: $S \to SS \mid (S) \mid ()$, where S is a nonterminal start symbol and $\mid$ means 'or'. This rule already contains the constraint that the probabilities of '(' and ')' are equal.

The symbol immediately after the symbol '(' must be '(' or ')' with equal probability. Then the occurrence probabilities of '(S)' and '()' must be equal, because '(S)' eventually produces '(('.

We can know that 'SS' will eventually produce ')(' and that '(S)' will produce '))'. The symbol immediately after ')' must be '(' or ')' with equal probability.

Thus we obtain the result that the rules 'SS', '(S)', and '()' must be used independently with probability 1/3.

We can conclude that some origins of the power law are equal to context-free grammars, which are not so powerful as Turing machines.

Noise Transformation: The collision automaton tells us a typical mechanism that causes power-law distribution:

1. The seed of the power-law distribution is noise, in particular, the white noise.
2. Noise is transformed by some logical, or equivalently logical, functions.
3. The logical functions are not combinational but sequential, i.e., utilize some memory effect.
4. The logical functions may not include negations, but must be nonlinear.

Various simple models of power-law distributions have been proposed, such as an infinite ladder model consisting of resistors and capacitors [14] (an infinite ladder circuit has square-root impedance). However, many of those models do not necessarily present us a clear image of this law. The author thinks that the collision automaton model is a comparatively good one that can give us various new insights into this law.

Discrete Relaxation with Stable Distribution: We will be able to think of the power law as a kind of discrete relaxation process [5, 10]. The fact that the number of trees in graph theory follows an exact $1/x$-type power law [13] will strengthen this hypothetical view.

Continuous relaxation usually exhibits an exponential function with a finite time constant. On the other hand, discrete relaxation often exhibits power laws.

If so, various concrete examples of power laws observed in physical and logical systems may be so important clues to investigate discrete phenomena in nature that we should not make little of them.

However, the power-law distribution belongs to a class called the stable distribution [11], in which the sum of independent random variables also follows the same distribution under the transformation of parameters.

It means that the power law is somewhat simple because such random variables have linear characteristics. The discussion about positive logic also makes us suspect the simplicity inherently associated with this law.

We know that power-law relaxation is very ubiquitous in our world. The author thinks that it is simple because it is so ubiquitous. For example, the domino automaton [7] exhibits approximately $x^{-2/3}$-type power-law distribution. The author would like to propose the next conjecture: the power law (with the power -1) is caused by discrete phenomena or some discretizing process.

A Hypothetical Model for $1/f$ Spectrum: We shall consider an important problem, mechanism of the $1/f$ spectrum.

The collision automaton is a collision model that can be expressed by a metaphor 'grow gradually or die at once' principle. We can obtain a hypothetical explanation of $1/f$ noise in electrical circuits.

We analyze the collisions of electrons against positively charged atoms in a solid sample under uniform electric field E. The electric field accelerates electrons, and the kinetic energy of an electron grows gradually. Such kinetic energy of electrons is transformed into harmonic oscillations of atoms by sudden collisions.

In Fig. 2, we regard the state 0 as a charged atom, and the state 2 as an electron. The vertical axis means *quantized* kinetic energy K of an electron. The horizontal axis means length l in the direction along E. Charged atoms are fixed at constant positions.

Let m: electron mass, q: electron charge, and v: electron velocity. An electron with initial velocity 0 at position 0 obtains velocity $v = qEt/m$ at time t and arrives at position $l = qEt^2/2m$. The kinetic energy K of an electron is $q^2E^2t^2/2m$. Then we obtain

$$K = qEl, \text{ or } \Delta K = qE\Delta l. \quad (4)$$

The relation between K and l is linear. This property conforms with the electron behavior in Fig. 2. The difference equation form in (4) tells us a fact that the moving length of an electron is constant with respect to its quantized kinetic energy. Note that the collision probability of an electron is proportional to its moving length. Thus such collision probability is constant with respect to the quantized kinetic energy of an electron.

Coulomb's force F between an electron and a charged atom (positive ion) is proportional to r^{-2}, where r is the distance between them. If the centrifugal force $K/2r < F$, i.e., $K < C/r$ for a constant C, then an electron will collide with the atom. Let $K_k = C/r_k$, where K_k means quantized energy and r_k, the collision radius.

An electron in rectilinear motion collides with an atom with probability proportional to πr^2, the cross-sectional area. Let K_{Tk} be the total sum of the kinetic energy K_k associated with the electrons within collision radius r_k. Then $K_{Tk} \propto r_k^2 C/r_k = Cr_k$, and consequently

$$K_k K_{Tk} = \text{constant}. \quad (5)$$

Since f_k is generally proportional to quantized energy, we can surmise the following $1/f$-spectrum law in the light of equations (4) and (5):

$$f_k K_{Tk} = \text{constant}. \quad (6)$$

Entropy Problem: Shannon's entropy of the collision automaton at time 0 is easily calculated as 1 bit/cell. In equilibrium, it decreases to 0 bit/cell. This model may lead us to an idea that 'perfectly inelastic collisions can decrease the entropy of a physical system.'

The second law of thermodynamics asserts that heat cannot be taken in at a certain temperature and converted into work *with no other changes* in the system or the surroundings. The above observation does not contradict this law, because the system has changed. Inelastic collisions convert part of kinetic energy into thermal energy.

Theory of entropy in statistical mechanics, however, may not be well formulated. Consider an isolated adiabatic system with only two particles, 1 and 2. The energy of the system is conservative. If ergodicity is assumed, the particles at last unite by a perfectly inelastic collision. The system changes (ergodicity is broken).

According to Boltzmann, entropy with respect to the positions and the velocities of particles decreases from H_{O1} to H_{O2}, where $H_{O2} \approx H_{O1}/2$. The degree of internal freedom of the particles changes from $n_1 + n_2$ for particles 1 and 2 to n_3 for a combined particle 3. If the entropy of the total system has increased, it must be kept in the internal freedom of the combined particle. Then, if the spacial volume of the system is different, should we think that different amount of entropy is held in internal freedom? Or can we discuss only about ensemble average?

In this system, irreversibility originates from inelastic collisions. Order emerging from disorder, e.g., seen in the creation of stars, planets, and galaxies in the big-bang universe, is a result of such collisions. The author is now interested in the relation between apparent order brought about by inelastic collisions and the entropy of such systems.

Acknowledgment The author expresses his sincere thanks to Yoshisuke Ueda, Masatoshi Shima, Keisuke Ito, and Hirokazu Fujisaka for their valuable discussions and suggestions.

References

[1] Johnson, J. B.: *Phys. Rev.*, Vol. 26, pp. 71–85 (1925).

[2] Zipf, G. K.: *Human Behavior and the Principle of Least Effort*, Addison-Wesley (1949).

[3] Hooge, F. N.: *Physica*, Vol. 83B, pp. 14–23 (1976).

[4] Mandelbrot, B. B.: *The Fractal Geometry of Nature*, W. H. Freeman (1977).

[5] Musha, T.: *Oyo Buturi*, Vol. 46, No. 12, pp. 1144-1155 (1977).

[6] Schroeder, M.: *Fractals, Chaos, Power Laws*, W. H. Freeman, (1991).

[7] Inagaki, K.: *IPSJ SIG Notes*, MPS-26-8 (1999).

[8] Inagaki, K.: *IEICE Trans. Inf. & Syst.*, Vol. E81-D, No. 8, pp. 767–772 (1998).

[9] Inagaki, K.: *IEICE Trans. A*, Vol. J81-A, No. 9, pp. 1230–1237 (1998). English translation: *Electronics and Communications in Japan, Part III*, Scripta Technica, Vol. 82, No. 11, pp. 1–8 (1999).

[10] Inagaki, K.: *IPSJ Trans.*, Vol. 40, No. SIG2(TOM1), pp. 68-75 (1999).

[11] Feller, W.: *An Introduction to Probability Theory and Its Applications*, John Wiley (1957, 66).

[12] Bak, P., Tang, C., and Wiesenfeld, K.: *Physical Review A*, Vol. 38, pp. 364–374 (1988).

[13] Inagaki, K.: *IPSJ Trans.*, Vol. 40, No. SIG2(TOM1), pp. 76–81 (1999).

[14] Motoishi, K. and Koga, T.: *IECE Trans. (A)*, Vol. J65-A, No. 3, pp. 237–244 (1982).