アルゴリズム 1－4
（1988. 5. 23）

グラフの最小線分数による直交線分描画

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平面上に点の位置が固定されて表現されているグラフＧをリベテッドグラフと言う。Ｇをその平面上で、各枝を水平および垂直な線分のみで描いた図形を直交線分描画と呼ぶ。どの点の次数も４以下で同じ位置を占める２点が存在しないリベテッドグラフは常に直交線分描画をもつ。与えられたリベテッドグラフＧに対し、その直交線分描画には多様性があるが、本文では線分数の最小化を問題としている。主要な成果として、点の位置が一般である場合、任意のリベテッドグラフＧを３ｍ（ｍ：枝数）以下の線分数で描画する時間Ｏ（ｍ）のアルゴリズムを与えている。更に、４レギュラーなバイパータイトリベテッドグラフと呼ばれるグラフは実際に３ｍだけの線分数を要することを証明している。

RECTILINEAR DRAWING OF A GRAPH ON A PLANE WITH THE MINIMUM NUMBER OF SEGMENTS

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A rivetted graph G(V,E) is a graph whose vertices are fixed on a plane in position. Its rectilinear drawing D(G) is a configuration of G on the plane in such a way that each edge is composed of horizontal and vertical segments. This paper concerns with reduction of the number of segments. As a main result, it is proved by presenting a linear time and space construction algorithm that any graph has a rectilinear drawing such that the total number of segments is no more than $3|E|$. It is shown that the 4-regular bipartite graph needs this number of segments.

1. INTRODUCTION

A rivetted graph G(V,E) is a graph each vertex of which an XY coordinate (position) on an XY plane is assigned. A rectilinear drawing, or simply, a drawing, D(G) of G is a configuration of G on the plane such that each edge is an alternating sequence of segments parallel to the X axis (horizontal) or to the Y axis (vertical) and that two segments are allowed to have at most one point in common, except, possibly, the endpoints. It is clear that a rivetted graph, even if it is nonplanar, has a rectilinear drawing if and only if the vertices are assigned distinct positions and the degree of each vertex is at most four. Furthermore, when the vertices are in general position, i.e. no two vertices are assigned the same X-coordinate or Y-coordinate, at least two segments are necessary and five are enough for one edge.

For a given rivetted graph, its drawing is not unique. In Fig.1(A), two different drawings $D_1(G)$ and $D_2(G)$ are given for G in (B) and (C), respectively. Each drawing has its characteristics. As for the numbers of segments, we have: $D_1(G)$ is consisting of total 17 segments, of which 6 edges are realized with two segments, and 1 edge with five segments. The corresponding numbers of $D_2(G)$ are 16 and 5, and 0, respectively. It has been checked for all possible drawings of G that the minimum total number of segments is 16, that the maximum number of edges that are realized with two segments is 6, and that there is no drawing that attains both numbers.

Reduction of the segments in rectilinear drawing of a rivetted graph is our main concern. It may be significant in such a CAD area that manipulates graphical configurations. For example, it is related to a better layout and routing of a VLSI chip, and their data reduction in memory. Thus, in the above example, the drawing $D_2(G)$ is the desired one with respect to the total number of segments.

The main result of this paper proves an upper bound $3|E|$ for the total numbers of segments for general rivetted graphs. The proof is made constructive providing a linear time and space algorithm to draw a given rivetted graph with no more than $3|E|$ segments. The algorithm provides a drawing without edges composed of 5 segments. The bound is tight since it can be shown that any 4 regular bipartite rivetted graph needs $3|E|$ segments at least. Furthermore, it is shown that if the graph is not 4 regular, there always exists a drawing with less than $3|E|$ segments.

An interesting problem in this respect is a decision problem 2-SEG which is to decide if a given rivetted graph can be drawn all edges with 2 segments. Though it is not presented here for the space, the authors have proved that 2-SEG and 2-SAT (2-satisfiability problem; a standard decision problem in P) are equivalent in computational complexity in the sense that one is reducible from the other by an $O(|E|)$ time and space algorithms.

A similar problem, called the orthogonal representation of a plane graph with the minimum number of segments preserving the region, is found in Tamassia [1] who solved the problem providing an efficient algorithm to get the required representation. The problem was the one the authors also solved independently but too late for publication. While the problem in this paper is the one originated from the problem through the discussion with Drs. R. Tamassia and I. G. Tollis of University of Illinois, Urbana, USA.

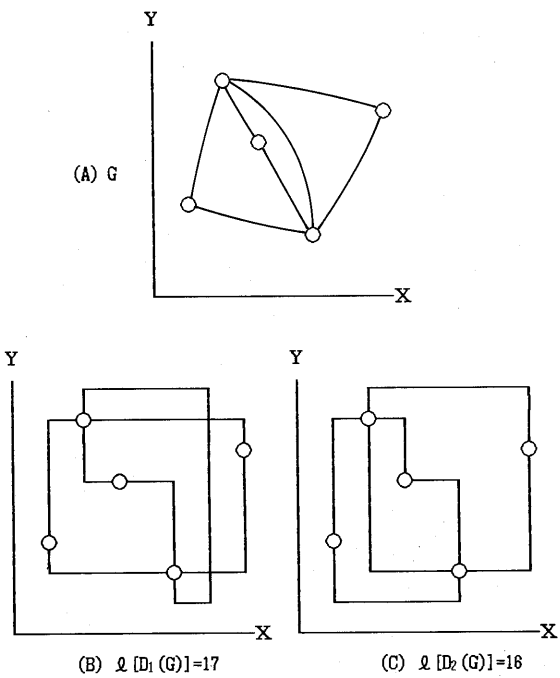

Fig.1 A rivetted graph G and its two rectilinear drawings

2.PRELIMINARIES

A rivetted graph G(V,E) is assumed to satisfy the condition that vertices are in general position and that the degree of each vertex is at most 4. Note that G is not assumed to be a planar graph at all. Let m=|E| and n=|V|. An edge e between two vertices u and v is denoted as e=(u,v). In a drawing D(G), $\ell[e]$ denotes the number of segments of e and $\ell[D(G)]$ the total number, and $\ell_m[G]$ the minimum of $\ell[D(G)]$ over all drawings D(G).

For a vertex v on the XY plane, the horizontal and vertical lines containing v are called the x axis and y axis of v, respectively. The right (left) half of the x axis of v is called the x^+ axis (x^- axis) of v. y^+ and y^- axes of v are defined analogously. They are called the half axes of v while the whole x and y axes are the full axes. One of the x^+ and x^- axes (y^+ and y^- axes) is called opposite of the other. In a drawing, a half axis without a segment is called empty. While, a half axis with a segment is called occupied.

LEMMA 1:

Let D(G) be a drawing in which e=(u,v) is composed of ℓ segments between half axes α of u and β of v. Then, ℓ is even (odd) if α and β are orthogonal (parallel). Furthermore, if $\ell \geq 6$, e can be redrawn to the one composed of 2, 3, 4, or 5 segments without affecting other configurations. ■

By this fact, drawings considerd here are assumed that every edge is composed of 2, 3, 4, or 5 segments. Thus, it is always that

$$2m \leq \ell[D(G)] \leq 5m.$$

LEMMA 2:

Let $e=(u,v) \in E$.

a. If a full axis λ of u is available, e can be realized within 4 segments.

b. Suppose that full axes λ of u and μ of v are available. If λ and μ are parallel, or orthogonal), e can be realized with 3 or 2 segments, respectively. ■

In a drawing, let an edge e=(u,v) be drawn using half axis α of v. To get another drawing by redrawing e so as to use another half axis β of v is called to switch e at vertex v from axis α to axis β. Consider two edges e and f incident to a vertex v. To switch e and f at v is to switch e and f at v from the axes of the ones to the axes of the other.

3. TOTAL NUMBER OF SEGMENTS

In this section, the following theorem will be proved.

THEOREM 1:

Any rivetted graph G has a drawing D(G) satisfying the properties:

I. every edge is composed of 2, 3, or 4 segments, and

II. $\ell[D(G)] \leq 3m$.

We prove the theorem providing an algorithm to get a desired D(G). The algorithm works in four phases. In Phase 1, E is partitioned into the minimum number of walks. In Phase 2, E is ordered following the partition. In Phase 3, an initial drawing which satisfies Property I is given. In Phase 4, the obtained initial drawing is modified to a drawing which satisfies both properties I and II. Since the procedures in Phases 2, 3, and 4 contain some essential difference depending on if G is Eulerian or not, they are described in two cases A and B separatedly.

Without loss of generality, we assume that G is connected. Futhermore, it is assumed that $n \geq 3$ and $m \geq 3$ since the theorem is trivial if $n \leq 2$ or $m \leq 2$.

A walk is an alternating sequence of vertices and edges beginning and terminating with vertices such that consecutive vertices and edges are in adjacency relation in G and that no edge appears more than once. If the first and the last vertices are identical, the walk is called a closed walk. If E itself is a closed walk, it is called an Euler walk, and the graph Eulerian.

Phase 1 (Partition of E):

Find a partition of E into the minimum number of walks.

The following lemma is a list of known properties of the walk partition which will be referred to later.

LEMMA 3:

Let $\pi = \{W_1, \ldots, W_\omega\}$ be a set of the minimum number of walks that partition E. Then,

1. Let θ be the number of odd degree vertices. Note that θ is even. Then, $\omega=1$ if $\theta=0$, and $\omega=\theta/2$ otherwise.
2. If $\theta\geqq 2$, every walk is not closed. (Thus, the biginning and terminating vertices of a walk are distinct.) ■

CASE A: G is Eulerian

Phase 2 (Ordering of E):
Find an Euler walk

$$W=(v_1,e_1,v_2,e_2,v_3, \ldots ,e_{m-1},v_t,e_m,v_1).$$

that satisfies the constraint;

C_1: The second vertex v_2 and the last vertex v_t are distinct.

Since $n\geqq 3$, it is possible to find such W.

Phase 3 (Initial drawing):

The construction of edges proceeds from one edge after another along W. The stage when the edges e_1, e_2, ... , e_{i-1} for $2\leqq i\leqq m$ have been constructed and not the rest (and we are going to add $e_i=(u,v)$ where u is the common vertex with e_{i-1}) is called the i-th stage. At this stage, e_i and v are called the front edge and front vertex, respectively.

We start with e_1 following the constraint;

C_2: $e_1=(v_1,v_2)$ is composed of 2 segments.

For the 2 nd edge and the following, we follow the constraint called the straight way principle (abbreviated as STW principle) which is:

C_3 (STW principle): Each pair of consecutive two edges (e_{i-1},e_i) for subscript pair (i-1,i) =(1,2),(2,3), ... ,(m,1), must be on the same full axis at the common vertex.

It is possible to follow C_3 all the stages since at the i-th stage, one of full axes of the front vertex is empty when i=2, 3, ... ,m-1, and the opposite axis of e_1 at v_1 is left empty when i=m.

These constraints are still able to be satisfied if we impose the following constraints.

C_4: When the front edge visits the last vertex v_t the first time (earliest at the 2-nd stage), choose a half axis that is parallel to the axis of v_1 on which e_1 lies.

C_5: Each front edge is constructed within 4 segments.

C_4 is possible to be satisfied because all the axes are empty when v_t is the first time to be a front vertex since $v_t\notin\{v_1,v_2\}$ by C_1. C_5 is also satisfied for e_2, ... ,e_{m-1} by Lemma 2a since at least one empty full axis remains at the front vertex. For the last edge e_m, it is probable that only one empty axis remains at the front vertex v_1. However, if C_4 is satisfied, there remains a half axis of v_1 which is orthogonal to the half axis of v_t on which e_m is. Therefore, e_m can be added with even number, which is 2 or 4, of segments.

Thus we get a D(G) as an initial drawing in which all the edges are composed of 2, 3, or 4 segments. However, it is very doubtful if the number of segments contained is no more than 3m. In fact, we need a modification.

Phase 4 (Modification):

The idea of eliminating consecutive 4's is applied which is described as follows. At the i th stage (i=3, ... ,m-1) in Phase 3, suppose that e_{i-1} has been realized with 4 segments and that we anticipate that the front edge $e_i=(u,v)$ needs 4 segments (if we proceed as in Phase 3). As illustrated in Fig.2, this situation indicates that front vertex v has once been visited and the axes parallel to the segment of e_i at u are occupied.

We come back to the i-1 th stage. As shown in the figure, consider to switch e_{i-1} at u from its half axis to the opposite one. Since e_{i-1} used 4 segments, this operation does not increase the number of segments of the edge. (Possibly, it decreases to 2.) This means that the current front edge $e_i=(u,v)$ retains its possibility to use any half axis contained in one full axis at u. Furthermore, since $i\leqq m-1$, e_i can use any half axis of a full axis at the front vertex v. And these axes at u and v are orthogonal from the fact that e_i once needed 4 segments. Then by Lemma 2b, e_i can be constructed with 2 segments on these axes of u and v. Construction of e_{i-1} after that can be done with 4 or 2 segments. Thus we require that the i-1 th and i th stages are revised as follows: Add e_i with 2 segments. Then, add e_{i-1} within 4 segments.

The above consideration is generalized to the following case. See an illustrative example shown in Fig.3. Let $i\geqq 3$, $i+p\leqq m-1$, and $p\geqq 0$. Suppose that we aknowledged at the i+p th stage in Phase 3 that e_{i-1} is composed of 4 segments and e_i, ..., e_{i+p-1} with 3 segments, and that we are forced to use 4 segments for the current front edge $e_{i+p}=(u,v)$. Then, it is not difficult to see that we can revise the procedure for e_{i-1}, ... ,e_{i+p} as follows.: Construct e_{i+p} with 2 segments first. Then, construct e_{i+p-1}, ... , and e_i in this order all with 3 segmentments. Finally realize e_{i-1} with 4 or possibly 2 segments. Note that the above idea cannot be applied always to eliminating the last two 4's if one exist.

Before estimating the number of segments thus obtained, we must note the following fact.

LEMMA 4: Any closed walk constructed following the STW principle consists of even number of edges that consist of even number of segments. ■

Now let N(W) be the sequence of the numbers of segments corresponding to W in the drawing obtained in the above. N'(W) denotes the sequence derived from N(W) by deleting all elements that are 3. Then, N'(W) satisfies the following properties:

a. The first element is 2.
b. Any consecutive two elements, except the last two, contains one 2.
c. The length is even.

It is clear that the average of the numbers of N(W) is 3 or less.

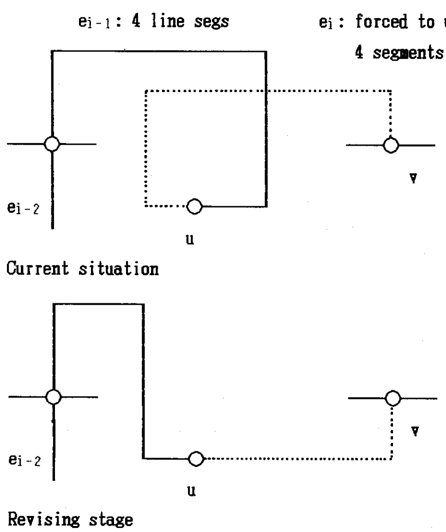

Fig.2 Switch of e_{i-1} at u which makes e_i realizable with 2 segments

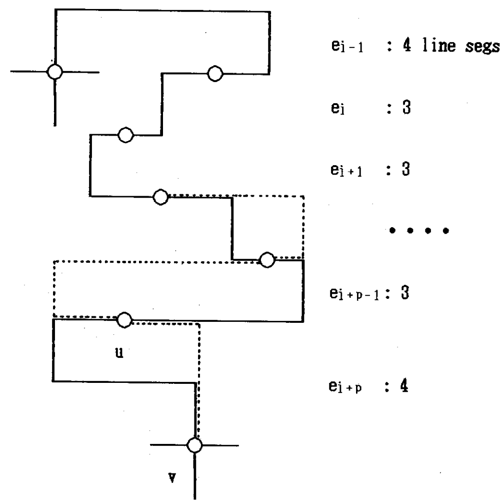

Fig.3 Eliminating consecutive 4's: Reconstruct e_{i+p} first with 2 line segs

CASE B:G is not Eulerian.

Phase 2 (Ordering of E):
For each walk, give an ordering arbitrarily. (We do not require such a constraint as C_1.)

Phase 3 (Initial drawing):
Take a walk one by one and construct the edges along each walk following only the constraints C_3 and C_5.

We do not require C_2. Note that it may be impossible to be followed for the second walk and after since some axes are already occupied by the previous walks. However, C_5 is always possible to be satisfied. This is by the same reason as in Case A for e_i, $i \leq m-1$.

Thus, we get a drawing which contains no edge with 5 segments.

Phase 4 (Modification):
Revisement of each walk is made by the idea of eliminating consecutive 4's the same as in Phase 4 of Case A. In a drawing D(G), let $N(W_i)$ and $N'(W_i)$ denote as before the sequence of the numbers of segments corresponding to W_i and the one obtained from this neglecting 3's, respectively. For the current D(G), $N'(W_i)$ contains no two consecutive 4's. But it is probable that the length is odd and the first and the last elements are both 4, contrary to the result in Case A. Thus we need a further modification, the idea of eliminating first 4.

Suppose that there is a walk W_i in the initial drawing D(G) such that the first element of $N'(W_i)$ is 4. In other words, if W_i is denoted as

$$W_i=(v_1,e_1,v_2,e_2, \ldots ,e_p,v_{p+1},e_{p+1}, \ldots),$$

all $e_1, \ldots, e_p$ are composed of 3 segments and e_{p+1} of 4 segments. Then, release all the edges of W_i, and reconstruct as follows. Construct e_{p+1} with 2 segments. Then, construct e_p, e_{p-1}, ..., and e_1 in this order all with 3 segments. Then, come back to e_{p+2} again and go further to the end.

Let the resultant drawing be a new initial drawing D(G) for which the first element of N'(W) is 2. Important is that in reconstruction, e_{p+1}, e_p, ... ,e_1 are excuted within the full axes whose half or full axes were occupied by W_i in D(G). This fact guarantees the possibility of the reconstruction independent from the other configuration.

Given an initial drawing, apply the operations of eliminating first 4 and then of eliminating consecutive 4's, the latter repeatedly to each W_i. Then, $N'(W_i)$ satisfies the following properties.

a. the first element is 2, if N'(W) is nonempty.
b. any two consecutive two elements, except the last two, containes one 2, and
c. the length is even.

It is clear that the average of the elements of $N(W_i)$ is 3 or less. Hence the proof of the theorem.

In the procedure, each edge is referred to at most three times. First in partitioning E into the walks, second as the front edge, and the last on the way back in the revised procedure. Note that the reisement searches one edge at most once because it is between two 4's or between the first element and the first 4.

Corollary 1: The algorithm in the proof of Th.1 runs in O(m) time and space. ■

4. GRAPHS THAT NEED 3m SEGMENTS

It will be shown here that the bound 3m is tight by showing a class of graphs that need 3m segments.

In a rivetted graph G, a vertex v is called a right end (left end, upper end, lower end) if all the adjacent vertices are inside the left (right, lower, upper) half of the XY plane with v as the origin. G is called bipartite if every vertex is either a right end or left end, or every vertex is either an upper end or a lower end.

THEOREM 2: If G is 4 regular and bipartite, $\ell_m[G]=3m$.

PROOF: Without loss of generality, we assume that G is the type of every vertex being left end or right end. Let L and R be the sets of left end vertices and right end vertices, respectively. Then, $|L|=|R|=n/2$.

The x^+ half axes of the vertices of L and the x^- half axes of the vertices of are called the inside axes. The x^- half axes of the vertices of L and the x^+ axes of the vertices of R are called the outside axes. Other half axes are called the vertical axes.

Let D[G] be a drawing. There, an edge e =(u,v) is called of type [inside,inside] if u and v are on inside axes. While e is of type [inside,vertical] if one of u, v is on an inside axis and the other on a vertical axis. Other types are analogously defined. Then, E is classified into six subclasses depending on the type. They are shown in Tab. 1, together with the number ℓ, the number of segments an edge of the subclass consists of. The last column denotes the numbers of the edges of each type.

Type	ℓ	number of edges
[inside,inside]	3	a
[inside,vertical]	2 or 4	b
[inside,outside]	3	c
[vertical,vertical]	3 or 5	d
[vertical,outside]	4	e
[outside,outside]	5	f

Tab. 1 Types of edges in a drawing D(G)

The total numbers of inside axes, vertical axes, and outside axes are n, 2n, and n, respectively. Therefore, we have

$$2a+b+c=n,$$
$$b+2d+e=2n,$$
$$c+e+2f=n.$$

From them, $a+b+c+d+e+f=2n$ and $2a+b-e-2f=0$. Thus,

$$\ell[D(G)] \geq 3a+2b+3c+3d+4e+5f$$
$$=6n-b+e+2f$$
$$=6n+2a \geq 3m.$$

QED.

From the proof, we see that a drawing of a 4-regular bipartite rivetted graph with the minimum number (=3m) of segments is attained only when D(G) includes no [inside,inside] type edge and all the [inside,vertical] edges are composed of 2 segments and all the [vertical,vertical] type edges of 3 segments.

THEOREM 3:

If G is not 4-regular, $\ell_m[G] < 3m$.

PROOF:

The theorem will be proved in two cases, G being Eulerian or not.

CASE 1: G is Eulerian

G contains a vertex of degree 2. Represent the Euler walk by

$$W=(v_1,e_1,v_2,e_2,\ldots,v_t,e_m)$$

such that v_1 is of degree 2, i.e. the edges incident to v_1 are e_1 and e_m. Construct W by the procedure CASE A, with an additional constraint which is a modification of C_4.

C_4': Construct e_m with 4 or 2 segments.

This is automatically satisfied if degree of v_t is 4 since we follow C_4. Otherwise, the last edge must be constructed as to satisfy C_4', which is trivially possible.

If $\ell[e_t]=4$, D(G) is modified as follows. Switch e_t at v_1 to y^+ or y axis so that e_m can be constructed with 3 segments. Clearly, this operation is possible since the switched e_m are on the parallel axes at v_t and at v_1.

The resultant drawing contains one less segments than D(G), and the proof.

If $\ell[e_t]=2$, the average of the elements of N(W) is less than 3 since N'(W) satisfies

a. the first element is 2,

b. two consecutive elements contains one 2 (without exceptions), and

c. the length is even.

CASE 2: G is not Eulerian

Of ω (>1) open walks, let any one be

$$W=(v_1,e_1,\ldots,v_{u-1},e_{u-1},v_u),\ (v_1 \neq v_u).$$

Construct W by the procedure in CASE B subject to the additional constraint C_4 and

C_5: e_1 at v_1 and e_{u-1} at v_u are orthogonal.

As mentioned before, we did not impose C_4 in CASE B because it is probable that the second and the following walks cannot satisfy the constraint since we do not know which axis is left at the front vertex. On the contrary, the constraint can be satisfied if the walk is the first one. C_5 can be attained trivially if v_u appears in W once. If v_u appears twice, C_5 is automatically satisfied since we follow constraint C_4.

After W, construct all the walks left exactly the same way as in CASE B. Let the resultant drawing be D(G). Note that the average of segment numbers for each walk is not more than 3. Thus, if we show that the average with respect to W is less than 3, the proof will be completed.

Now let us see N'(W). It satisfies

a. the first element is 2,

b. two consecutive elements contains one 2 (without exception), and

c. the length is odd.

Property c comes from C_5.

The sequence N'(W) is (2 4 2 4 ... 4 2) or the one some of its 4's are replaced by 2's. Hence, the average is less than 3.

QED.

5.EXAMPLE

In Fig.4, a rivetted graph G is shown. G has an Euler walk W. The edges are ordered along W. Fig.5 shows an initial drawing D(G) by the procedures up to Phase 3. Since e_5 is the first edge to visit v_n (n=6), we choose x^+ axis of v_n which is parallel to the axis of e_1 at v_1. Because of this, e_m (m=12) can be realized within 4 segments. Suppose otherwise, that is, if it were the case that at v_6, e_5 and e_6 use y axes and e_{11} uses x^+ axis, e_{12} would need 5 segments.

Now go to Phase 4. Line seg number sequence N(W) contains a subsequence (4,3,4) corresponding to (e_6,e_7,e_8). Then, we revise e_8, e_7, and e_6. Construct e_8 with 2 segments. Then e_7 with 3 segments, the number preserved, the configuration changed. And then e_6. Fortunately, the number of segments of e_5 reduces from 4 to 2. By these rearrangements, the axes used as a whole did not change. Restart the procedure from e_9 through the last e_{12}. The result is shown in Fig. 6.

6.CONCLUDING REMARKS

The concept of the rivetted graph and its rectilinear drawing is introcduced. The main concern in this paper is to reduce the number of segments contained. The main result proves that the upper bound is 3|E| providing a linear time and space algorithm to get a drawing satisfying the bound. It is also shown that the 4-regular bipartite rivetted graph needs 3|E| segments. However, it is open to determine, hopefully with a polynomial time construction algorithm, the minimum number of segments for each graph.

ACKNOWLEDGEMENT

The authors wish to thank Prof. S. Ueno and Mr. H. Miyano of Tokyo Institute of Technology for their helpful discussions.

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[1]R. Tamassia, On embedding a graph in the grid with the minimum number of bends, SIAM J. COMPUT., 3 (1987), pp. 421-444

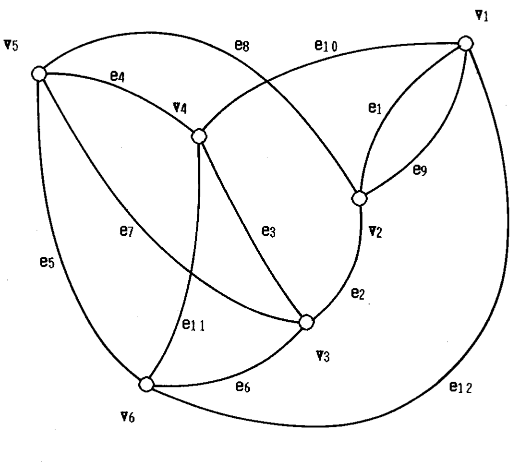

Fig.4 A rivetted graph G with m=12 and 3m=36

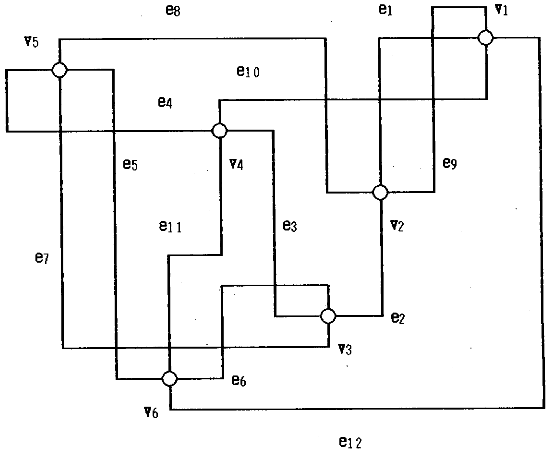

Fig.5 Initial drawing D(G) with N(W)=(223334344334) and ℓ[D(G)]=38

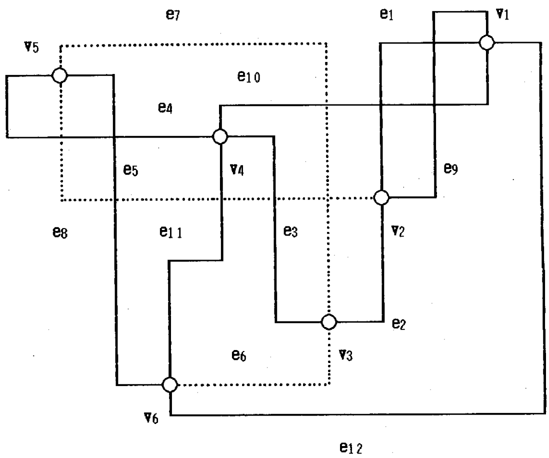

Fig.6 Revised drawing D(G) with N(W)=(223332324334) and ℓ[D(G)]=34