2-Edge-Connectivity Augmentation Problems for Directed Graphs

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Abstract

The paper discusses the 2-edge-connectivity augmentation problem for directed graphs, which is defined by "Given a directed graph G=(V,E) and a cost function c of ordered pairs of vertices of G into nonnegative integers each of which is the cost of an edge from the first vertex of the pair to the second one, find a set of directed edges, E', of minimum total cost such that the graph G+E'=(V,E∪E') is a 2-edge-connected directed graph", where edges of E' connect distinct vertices of V. We show that the unweighted version of the problem, that is, c(u,v)=c(u',v') for any two ordered pairs (u,v), (u',v') of vertices of G and we ask for a solution E' of minimum cardinality, can be solved in $O(|V|^2(|E|+|V|)^{3/2})$ time.

1. Introduction

The subject of the paper is the 2-edge-connectivity augmentation problem for directed graphs: "Given a directed graph G=(V,E) and a cost function c of ordered pairs of vertices of G into nonnegative integers each of which is the cost of an edge from the first vertex of the pair to the second one, find a set of directed edges, E', of minimum total cost such that the graph G+E'=(V,E∪E') is a 2-edge-connected directed graph", where edges of E' connect distinct vertices of V. A directed graph G=(V,E) is 2-edge-connected if and only if G has at least two directed disjoint paths from u to v for any pair of vertices u,v of V. The paper considers the unweighted version of the problem, that is, c(u,v)=c(u',v') for any two ordered pairs (u,v), (u',v') of vertices of G and we ask for a solution E' of minimum cardinality. We show that the problem can be solved in polynomial time.

The k-edge (k-vertex, respectively)-connectivity augmentation problem, k-ECA (k-VCA) for short, is similarly defined for both directed and undirected graphs.We restrict ourselves to k-ECA and k-VCA for directed graphs. (For those of undirected graphs, see [7-12].) An $O(|V|+|E'|)$ algorithm for 2-ECA with unity edge-costs and the NP-completeness of 2-ECA with G' having no edge have been given by Eswaran and Tarjan [1]. The NP-completeness of 2-ECA with G' restricted to a tree has been shown by Frederickson and Ja'ja [3], which also provides an $O(|V|^2)$ approximation algorithm with worst approximation no more than twice the optimal. A characterization of k-ECA with G' being a directed tree is given by Kajitani and Ueno [5]. An $O(|V|)$ algorithm for k-VCA with G' restricted to a rooted directed tree is given by Masuzawa, Hagihara and Tokura [6].

Time complexity of the unweighted k-ECA of directed graphs is an open problem, and our discussion is the first step showing that it is polynomial-time solvable if k=2. It is shown how to compute the cardinality of an optimal solution |E'| from a given directed graph G and that there is an $O(|V|^2(|E|+|V|)^{3/2})$ algorithm for solving the 2-ECA for directed graphs.

2. Preliminaries

Although some definitions are given in Section1, we describe some other definitions used in the following discussion. A *digraph* G=(V(G),E(G)) consists of a finite set V(G) of vertices and a finite set E(G) of directed edges; a directed edge from u to v is denoted by (u,v). u and v are called the starting vertex and the ending one, respectively. For simplicity a graph and an edge means a directed graph and a directed edge, respectively, unless otherwise stated. V(G) and E(G) are often denoted by V and E, respectively, if there is no confusion. If u=v then the edge is called a *self-loop*. Two edges e_1 and e_2 having the same ordered pair are called *multiple edges*. For disjoint subsets S, S'⊆V(G), S∩S'=∅, let $E_G(S,S')=\{(u,v)|u\in S, v\in S'\}$, where the subscript G is often omitted. $E_G(S,S')$ is called an *(S,S')-cut*. If S'=V(G)-S then it is called an *S-cut* or simply a *cut*. We denote $OE_G(S)=E_G(S,V-S)$ or $OE_G(u)=E_G(\{u\},V-\{u\})$. Similarly $IE_G(S)$ and $IE_G(u)$ are defined by replacing S and V-S with V-S and S, respectively. $od_G(S)=|OE_G(S)|$ ($id_G(S)=|IE_G(S)|$, respectively) is called the *outdegree* (*indegree*) of S in G. If S={u} then it is denoted by $od_G(u)$. Let $ON_G(u)$ ($IN_G(u)$, respectively) denote the set of all ending (starting) vertices of those in $OE_G(S)$ (in $IE_G(S)$). If S={u} then $OE_G(S)$ is denoted by $OE_G(u)$. A *directed path* P(u,v) from u to v is an alternating sequence of vertices and edges $u=v_0,e_1,v_1,\ldots,v_{n-1},e_n,v_n=v$ such that $v_0,\ldots,v_n$ are all distinct and each $e_i=(v_{i-1},v_i)$, $1\le i\le n$. The *length* of the path is n. A *semipath* from v_1 to v_n is similarly defined except that each e_i is (v_i,v_{i+1}) or (v_{i+1},v_i) for $1\le i\le n$. In particular, these paths are also called a (u,v)-path and a (u,v)-semipath, respectively. A *cycle* a directed path together with an edge (v_0,v_n). The length of the cycle is n+1. A cycle which contains a specified vertex v is also denoted by χ_v. A graph G is called *simple* if G contains neither self-loops nor multiple edges. A graph G is *complete* if it is simple and E(G) contains all possible edges except loops. A *subgraph* G' of a graph G is a graph with V(G')⊆V(G) and E(G')⊆E(G), where ⊆ denotes inclusion; ⊂ does proper inclusion. G' is a *spanning subgraph* of G if V(G')=V(G). A graph is *acyclic* if it contains no cycle. G is *weakly connected* if there is a (u,v)-semipath for any two vertices u and v. In G, a vertex v is reachable from u if there is a (u,v)-path. A vertex $r \in V(G)$ is called a *root* of G if and only if every vertex $v \in V(G)$ is reachable from r. G is strongly connected if any vertex is a root. An *arborescence* is a directed acyclic graph with only one root having no entering edge and all other vertices having exactly one entering edge. A *leaf* in an arborescence is a vertex without outgoing edges. An arborescence is often called a *rooted tree*. *Deletion* of a set S of vertices from G is to construct $G=(V-S,E-(IE_G(S)\cup OE_G(S)))$ which is denoted by G-S. If S={v} then we often denote as G-v. Deletion of a set Q of edges from G is also denoted by G-Q, and if Q={e} then it is denoted by G-e. If E' is a set of edges such that $E'\cap E=\emptyset$ then G+E' denotes the graph $(V,E\cup E')$. If E'={e} then we denote as G+e. *Shrinking* of a vertex set $S\subseteq V$ into a vertex v_s is to construct the graph, denoted as $G[S,v_s]$, with the vertex set $(V-S)\cup\{v_s\}$ and the edge set $E(G-S)\cup\{(v_s,v)|(u,v)\in E,u\in S,v\in V-S\}$. Two paths P,P' are said to be *edge-disjoint* (*vertex-disjoint*, respectively) if and only if they have no edge in common (they have no vertex except endvertices in common). For two vertices u,v of G, let $M_G(u,v)$ ($L_G(u,v)$, respectively) denote the maximum number of edge-disjoint (vertex-disjoint) (u,v)-paths of G. The *edge-connectivity* ec(G) (the *vertex-connectivity* vc(G)) of G is the minimum $M_G(u,v)$ ($L_G(u,v)$) over all ordered pairs u,v in G. A graph G is *k-edge-connected* (*k-vertex-connected*, respectively) if and only if $ec(G)\geq k$ ($vc(G)\geq k$). A *k-edge-component* (*k-vertex-component*, respectively) of G is a maximal subset of vertices such that, for any ordered pair u,v in the set, G has at least k edge-disjoint (vertex-disjoint) paths from u to v. A k-component means a k-edge-component unless otherwise stated, and a component often means a 1-component. Let $C_k(G)$ or simply C_k denote the set of all k-components of G. Distinct k-components are disjoint and, therefore, each k-component S is partitioned into at least two (k+1)-components if S is not a (k+1)-component. A subset $K\subseteq E(G)$ is called a (u,v)-separator if and only if every (u,v)-path of G has an edge of K, and a (u,v)-separator of minimum cardinality is called a (u,v)-cut. For any pair u, $v\in V(G)$ it is known that [Menger] $M_G(u,v)=k$ if and only if G has a (u,v)-cut of cardinality k and that the cardinality of this (u,v)-cut is equal to that of $E_G(S,V-S)$ for some S with $u\in S$ and $v\in V-S$.

A partial order $\angle$ is a binary relation on a set S defined by (1)~(3):

(1) $x \angle x$ for $\forall x\in S$. (Reflective)

(2) If $x \angle y$ and $y \angle x$ then x=y. (Antisymmetric)

(3) If $x \angle y$ and $y \angle z$ then $x \angle z$. (Transitive)

A pair $[S,\angle]$ is called a *partially ordered set* (or a *poset* for short). A poset $[S,\angle]$ is called a *totally ordered set* if and only if (4) holds:

(4) Either $x \angle y$ or $y \angle x$ holds for $\forall x,\ y\in S$.

For two elements x, y of a poset $[S,\angle]$, x *covers* y if and only if $y \angle x$ and there is no z such that $y \angle z \angle x$. x is called minimal (maximal, respectively) if and only if there is no $y\in S$ such that $y \angle x$ ($x \angle y$). For a subset $S'\subseteq S$, $x\in S$ is called a lower bound (an upper bound, respectively) of S' if $x \angle y$ ($y \angle x$) for $\forall y\in S'$. $x\in S$ is called a *greatest lower bound* (a *least upper bound*, respectively) if (1) x is a lower bound (an upper bound) of S', and (2) $x' \angle x$ ($x \angle x'$) for any other lower bound (upper bound) x' of S'. We denote x=glbS' (x=lubS').

3. Minimum 2-edge-connectivity augmentation

In this section we characterize the minimum number of directed edges whose addition to a given directed graph result in a 2-edge-connected graph. It is shown that the minimum number can be computed from a given graph, and we propose an algorithm for finding a minimum solution to the problem.

3.1. Augmentation numbers

First of all we need some definitions. A subset $S\subseteq V(G)$ is called a *weak k-component* of G if S is a maximal subset of a (k-1)-component such that, for any $u\in S$, there is $v\in S$ satisfying either $M(u,v)\geq k$ or $M(v,u)\geq k$. V(G) is partitioned into some weak k-components. Each weak k-component is partitioned into some k-components. We define a partial order $\angle_k$ on the set of k-components included in each weak k-component W of G as follows: for $S_1, S_2\in C_k(G)$ with $S_1, S_2\subseteq W$, $S_1 \angle_k S_2$ if and only if $M(u,v)\geq k$ and $M(v,u)=k-1$ for any $u\in S_2$ and $v\in S_1$. A maximal subset $H\subseteq W$ such that $\angle_k$ is a total ordering in the set is called a *k-chain*. The minimal (maximal, respectively) element of a k-chain H is called the *k-source* (*k-sink*) of H. Clearly a weak k-component W is a union of some k-chains. Any k-component $S'\subseteq W$ which is the k-source of some k-chain is called a k-source of W. A k-chain H is called a *source* (*sink*, respectively) *k-chain* if $E(V-W,H)=\phi$ ($E(H,V-W)=\phi$) and there is no other k-chain whose k-sink (k-source) is also S', where W is the weak k-component containing H, and S' is the k-sink (the k-source) of H. Let S_0 and T_0 be the k-source (k-sink, respectively) of a source (a sink) k-chain H_0. Then the *source* (*sink*) *palm* Q of S_0 (T_0) is the maximal union of those source (sink) k-chains H_0, ... H_m, $m\geq 1$, sharing S_0 as the k-source (T_0 as the k-sink). Note that if there is any $S'\in Z_1$ satisfying $T_i \angle_k S'$ ($S' \angle_k S_i$) then $S'\in Q$, where S_i (T_i) is the k-source (k-sink) of H_i, $1\leq i\leq m$. Each k-chain which is not a member of any source or sink palm is called a *finger*. A palm means a source palm, a sink palm or a finger. We provide some examples in the following.

Example 1. Consider the graph G_1 (excluding bold lines) of Figure 1. Then

Z_1={S={1,2,3,4,5,6,7,9}, {8}, {10}, {11}, {12}, {13}}

$=Z_3=Z_4$ (we set P=H and W=P),

Z_5={X=S∪{8,10}, {11}, {12}, {13}}.

Now we define the augmentation number D(G) of any given directed graph G. The sets of all 2-components, 2-chains, palms, weak 2-components, 1-components and weak 1-components of G are denoted by Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6, respectively, in the following discussion. Let sZ_3, tZ_3 and fZ_3 be the sets of source palms, sink palms and fingers, respectively.

For each $S \in Z_i$ ($1 \le i \le 6$), we call

$$iED_G(S)=\max(0,\ 2-id_G(S))$$

the *in-edge demand* of S. We define recursively the *in-demand* $iD_G(S)$ of $S \in Z_i$ ($1 \le i \le 6$) by first setting

(1) $iD(S)=iED(S)$ for each $S \in Z_1$.

For simplicity the subscript G is often omitted. Before defining iD(S) for $W \in Z_4$, we may need the following supplementary change of iD(S'), $S' \in Z_1$. For each $H \in Z_2$, let

$$iLD(H)=\Sigma_{S' \subseteq H,\ S' \in Z_1} iD(S').$$

Set

(2)

$$iD'(S')=\begin{cases} 1 & \text{if there is } H \in Z_2 \text{ such that } iLD(H)<iED(H) \text{ and } S' \text{ is the 2-source of } H, \\ iD(S') & \text{otherwise.} \end{cases}$$

We denote

$$iLD'(H)=\Sigma_{S' \subseteq H,\ S' \in Z_1} iD'(S'),$$

and

$$iLD'(P)=\Sigma_{S' \subseteq P,\ S' \in Z_2} iD'(S') \text{ for each } P \in Z_3.$$

Set

(3)

$$iD''(S')=\begin{cases} 1 & \text{if there is } P \in Z_3 \text{ such that } iLD'(P)<iED(P) \text{ and } S' \text{ is the 2-source of } P, \\ iD'(S') & \text{otherwise.} \end{cases}$$

Set

$$iD(S')=iD''(S') \text{ for each } S \in Z_1,$$

and we denote

$$iLD(S)=\Sigma_{S' \subseteq S,\ S' \in Z_j} iD(S') \text{ for each } S \in Z_m,\ 4 \le m \le 6,$$

where j=1 if m=4, j=4 if m=5 j=5 if m=6. iD'(S') or iD''(S') (iLD'(S), respectively) is also called the *in-edge demand* of S' (*in-local demand* of S). Then we define

$$iD(S)=\max\{iLD(S),\ iED(S)\} \text{ for each } S \in Z_m,\ 4 \le m \le 6.$$

We similarly compute the *out-local demand* oLD(V(G)) by setting oD(S)=oED(S) for each $S \in Z_1$ and replacing iLD, iD', iLD' or iD'' with oLD, oD', oLD' or oD'', respectively.

Note that the sets Z_j, $1 \le j \le 6$, constructed from G are identical to those defined from the reversed graph $\overline{G}$, where $V(\overline{G})=V(G)$, and $(u,v) \in E(\overline{G})$ if and only if $(v,u) \in E(G)$. Hence $oD_G(S)=iD_{\overline{G}}(S)$ for $S \in Z_1$, and similarly for oLD_G, oD'_G, oLD'_G, or oD''_G. Finally the *augmentation number* D(G) of G is defined by

$$D(G)=\begin{cases} 0 & \text{if } ec(G) \ge 2 \\ \max\{iLD(V(G)),\ oLD(V(G))\}, & \end{cases}$$

where $iLD(V(G))=\Sigma_{S \in Z_6}\ iD(S)$, and similarly for oLD(V(G)). We assume that $iLD(V(G)) \le oLD(V(G))$ throughout the paper. (If $iLD(V(G))<oLD(V(G))$ then we consider $\overline{G}$ (instead of G) for which $iLD(V(\overline{G})) \ge oLD(V(\overline{G}))$, and the following discussion assures that the solution A for G will be obtained from a solution $\overline{A}$ for $\overline{G}$ by reversing the direction of each edge of $\overline{A}$.

Example 2. We provide an example of computing iLD(V(G)) and D(G). Consider the graph G_1 (excluding bold lines) of Figure 1. Z_i, $1 \le i \le 5$, have been determined in Example 1. The computation of D(V(G)) is summarized in Table 1, where the computation is terminated at Z_5, since $Z_5=Z_6$. We obtain $iLD(V(G_1))=5$, $oLD(V(G_1))=7$ and D(G)=max(5,7)=7. A solution for G_1 is $A_1=\{(11,13),\ (12,1),\ (11,12),\ (13,11),\ (8,13),\ (10,1),\ (12,1)\}$ (denoted by bold lines), in which edges are determined one by one in this order ((11,13) is the first) by the algorithm to be proposed in Section 3.3.

Proposition 1. Let H be any 2-chain with the 2-source S and the sink T ($\neq$S). Then the following (1) through (5) hold.

(1) $|E(H-S,S)|=|E(T,H-T)|=1$.

(2) $iD(S)=\begin{cases} 1 & \text{if } E(V-H,S)=\phi, \\ 0 & \text{otherewise,} \end{cases}$

and

$$oD(S)=\begin{cases} 1 & \text{if } E(T,V-H)=\phi, \\ 0 & \text{otherewise.} \end{cases}$$

(3) $iD(S')=iD'(S')=iD''(S')=oD(S')=oD'(S')=oD''(S')=0$ for any 2-component $S' \subseteq H-(S \cup T)$.

(4) iED(H)=0 if there is another 2-chain H' sharing T as the sink.

(5) H is a source 2-chain if iED(H)=1.

(6) If P is a palm with iED(P)=1 then P is a source palm.

Proposition 2. Suppose that $iLD(V(G)) \ge oLD(V(G))$ and that V(G) is a weak 2-component containing at least two 2-components. Then V(G) contains a 2-component S satisfying iD(S)=1, iD'(S)=1 or iD''(S)=1.

Proof. Suppose that any 2-component $S \subset V$ (=V(G)) has iD(S)=iD'(S)=iD''(S)=0. Then

$$iLD(V(G))=\Sigma_{S \in Z_1} iD''(S)=0.$$

If V is a 2-chain then the 2-source $S \subset V$ has iED(S)=iD(S)=1. Hence V contains distinct 2-chains each of which has the 2-source and the 2-sink that is distinct from its 2-source. If there is a 2-source $S \subset W$ which is contained in only one 2-chain then $id_G(S)=1$ or iED(S)=iD(S)=1. Hence any 2-source $S \subset W$ is shared by at least two 2-chains contained in W. It follows that W contains a source 2-chain and, therefore, a source palm is included in W. If some 2-chain H has iED(H)>0 then iLD(H)=0<iED(H) and, therefore, the 2-source $S \subset H$ gets iD'(S)=1. This implies that every 2-chain $H \subset W$ has iED(H)=0. Similarly every palm $P \subseteq W$ has iED(P)=0. Let $P=H_1 \cup \cdots \cup H_m \subseteq W$, $m \ge 1$, be a source palm of a 2-source S, where each H_i is a source 2-chain of S. Since $iED(H_i)=0$, $1 \le i \le m$, and iED(P)=0, we have $m \ge 3$ and there are at least two non-source 2-chains sharing S as the 2-source. Each H_i contains the 2-sink T_i with $oD(T_i)=1$, we obtain oLD(V(G))>iLD(V(G))=0, a contradiction.

Q.E.D.

The augmentation number D(G) has the following important property.

Proposition 3. For any given directed graph G, $ec(G)\geq 2$ if and only if $D(G)=0$.

Proof. The definition of D(G) clearly means that $D(G)=0$ if $ec(G)\geq 2$. Now suppose that $ec(G)<2$. If $|Z_5|>1$ then each $S\in Z_5$ has $iED(S)\geq 2$, or $D(G)>0$. Assume that $ec(G)\geq 1$ and $|Z_1|\geq 2$. If $|Z_4|>1$ then each $S\in Z_4$ has $iED(S)\geq 1$, or $D(G)>0$. Thus we assume that $V(G)\in Z_4$. Then, by Proposition 2, there is a 2-component S with $iD(S)=1$, $iD'(S)=1$ or $iD''(S)=1$, showing that $D(G)>0$.

Q.E.D.

3.3. Proof of the lower bound

Let $\alpha(G)$ be minimum cardinarity of solutions for the unweighted version of 2-ECA. We are going to prove that $\alpha(G)=D(G)$ for any given directed graph G. Proposition 3 shows that $D(G)=0=\alpha(G)$ if $ec(G)\geq 2$. Hence we consider only the case with $ec(G)<2$, or $D(G)>0$. First we show that $\alpha(G)\geq D(G)$ in this section, and the converse, $\alpha(G)\leq D(G)$, will be shown in the following section.

First, we investigate some graph structures related to in-demands of 2-components, 2-chains and palms. Given a directed graph G, V(G) is the disjoint union of one or more 1-components, each 1-component is the disjoint union of one or more weak 2-components, each weak 2-component is the union of one or more palms, each palms is the union of one or more 2-chains, and each 2-chain is the disjoint union of one or more 2-components including a 2-source and a 2-sink, where they may be identical. The next proposition follows from the definition of 2-sources and 2-sinks.

Proposition 4. Suppose that S and S' are distinct 2-components contained in a weak 2-component of G. Then $0\leq iD(S)\leq 1$ and $0\leq oD(S)\leq 1$, and if $iD(S)=1$ ($oD(S)=1$, respectively) then S is a 2-source (a 2-sink).

We consider a 2-chain $H\in Z_2$ with $iLD(H)<iED(H)$ and a palm $P\in Z_3$ with $iLD'(P)<iED(P)$ in the following two propositions. We can easily prove the next proposition by case analysis and, therefore, the proof is omitted.

Proposition 5. Let H be and 2-chain, S be the 2-source of H, W be the weak 2-component containing H, X be the 1-component containing W and Y be the weak 1-component containing X. Suppose that $iLD(H)<iED(H)$ and put $iLD(H)=p$ and $iED(H)=q$. Then the following (1) and (2) hold.

(1) H does not satisfy ($p=0$ and $q=2$).

(2) We have $p=1-m$ and $q=2-m$ if and only if the following (a) and (b) hold, where $0\leq m\leq 1$.

(a) $id_G(H)=m$.

(b) $|E(W-H,S)|=1$ if $m=1$, and $H=W=X$ if $m=0$.

Remark 1. Similar results hold for the 2-sink S of a 2-chain H with $oLD(H)=p$ and $oED(H)=q$.

Proposition 6. Let P be any palm, S be the 2-source of P, W be the weak 2-component containing P, X be the 1-component containing W, and Y be the weak 1-component containing X. Suppose that $iLD'(P)<iED(P)$. Put $iLD'(P)=p$, $iED(P)=q$, $id_G(S)=r$, and y be the told number of source 2-chains having S in common. Then the following (1) through (3) hold.

(1) P is a source palm.

(2) $P=W=X$ if $q=2$, and $|E_G(W-P,P)|=1$ if $q=1$.

(3) $\begin{cases} \text{(i) } r\geq 2 \text{ and } y\geq 3 \text{ if } p=0 \text{ and } q=2, \\ \text{(ii) either } (r=1 \text{ and } y\geq 1) \text{ or } (r=y=2) \text{ if } p=1 \text{ and } q=2, \\ \text{(iii) } r\geq 3,\ y\geq 2 \text{ and } |E(W-P,S)|=1 \text{ if } p=0 \text{ and } q=1. \end{cases}$

Proof. Since $iED(P)>0$, P is a source palm and (1) holds. If $q=2$ then, clearly, $P=W=X$. If $q=1$ then either $P=W\subset X$ or $P\subset W\subseteq X$. If $P=W$ then $E(V(G)-W,P)=\phi$, $iD(S)=1=iD'(S)$ and $iLD'(P)=1$, contradicting that $iLD'(P)=P=0$. Hence $|E_G(W-P,P)|=1$ and (2) holds. Let H be any 2-chain having S as the 2-source. First assume that $p=1$ and $q=2$. Then $P=W=X$. If $id_G(S)=1$ then $iD(S)=1$ and $y\geq 1$. If $id_G(S)=2$ then $iD(S)=0$ and $y\geq 2$. If $y=2$ then we get $iD'(S)=1$ since $iLD(H)=0<iED(H)=1$, showing that $p=1$ and $q=2$. If $y>2$ then $iD'(S)=iD(S)=0$ since $iLD(H)=0=iED(H)$. Hence $p=0$, a contradiction. If $id_G(S)>2$ then $y\geq 3$ and, similarly, we have $p=0$, a contradiction. Thus (3)(ii) holds.

Next assume that $p=0$ and $q=2$. The discussion similar to (3)(i) shows that if $p=0$ then $id_G(S)\geq 2$ and $y\geq 3$. Thus (3)(i) holds.

Finally assume that $p=0$ and $q=1$. Then $P\subset W$ and $iLD(H)=iED(H)=0$, where H is any 2-chain having S as the 2-source. Hence $id_G(S)\geq 2$ and $y\geq 2$. $|E(W-P,S)|=1$ since $E(V(G)-W,P)=\phi$ with $q=1$. If $id_G(S)=2$ then $|E(W-H,S)|=1$ and $y=2$. That is, $id_G(H)=1$ and $iED(H)=1$, a contradiction. Thus $id_G(S)\geq 3$ and (3)(iii) holds. Q.E.D.

Remark 2. Note that If $P=W=X$ is a source palm then $id_G(X)=0$. Results similar to Proposition 6 hold on the 2-sink S with $od_G(S)=r$ and a palm P with $oLD'(P)=p$, $oED(P)=q$.

Now we prove that $\alpha(G)\geq D(G)$.

Lemma 1. $\alpha(G)\geq D(G)$.

Proof. Let A be any edge set with $|A|=\alpha(G)$ such that $G'=G+A$ is 2-edge-connected. By assuming that $|A|<D(G)$ we show a contradiction that $ec(G')<2$. Since $D(G)>0$, we have $ec(G)<2$ by Proposition 3. For simplicity put $K(S)=E_{G'}(V(G),S)\cap A$ for any subset $S\subset V(G)$. Since $|A|<\Sigma_{Y'\in Z_6} iD(Y')$, there is a weak 1-component $Y\in Z_6$ with $|K(Y)|<iD(Y)$. If $iLD(Y)\leq iED(Y)$ then $id_{G'}(Y)\leq 1$ and $ec(G')<2$. Hence $iLD(Y)>iED(Y)$, or $iD(Y)=iLD(Y)$. Since $|K(Y)|<iD(Y)=\Sigma_{X'\in Z_5} iD(X')$, there is a 1-component $X\in Z_5$ with $|K(X)|<iD(X)$. If $iLD(X)\leq iED(X)$ then $id_{G'}(X)\leq 1$ and $ec(G')<2$. Therefore $iLD(X)>iED(X)$. Since $|K(X)|<iD(X)=iLD(X)=\Sigma_{W'\in Z_4, W'\subseteq X} iD(W')$, there is a weak 2-component $W\subseteq X$ with $|K(W)|<iD(W)$. If $iLD(W)\leq iED(W)$ then $|K(W)|<iED(W)$, $id_{G'}(W)\leq 1$ and $ec(G')<2$. Hence $iLD(W)>iED(W)$. Then

$|K(W)|<iD(W)=iLD(W)=\Sigma_{S'\in Z_1, S'\subseteq W}$ iD"(S'). If $W\in Z_1$ then $|K(W)|<iLD(W)=iD''(S')=iD(S)=iED(S)$, $id_{G'}(W)<2$ and $ec(G')<2$. Now suppose that W contains at least two 2-components. Then $0\le iD''(S')\le 1$ for any 2-component $S'\subset W$. For each 2-component $S\subset W$ with $iD''(S)=1$, S is a 2-source of W and there are three cases (1) $iD(S)=iD'(S)=iD''(S)$, (2) $iD(S)=0$ and $iD'(S)=iD''(S)=1$, (3) $iD(S)=iD'(S)=0$ and $iD''(S)=1$.

Since each 2-component $S\subset W$ with $iD''(S)=1$ satisfies only one of the three cases, we can partition $C_W=\{S\in Z_1 | S\subset W\}$ into three sets C_1, C_2 and C_3, where each 2-component in C_i satisfies the case (i), $1\le i\le 3$. Clearly $iLD(W)=|C_1|+|C_2|+|C_3|$. For each $S\in C_1$, we have $|K(S)|\ge 1$ since $ec(G')\ge 2$. For each $S\in C_2$ ($S\in C_3$, respectively), there is exactly one source 2-chain H_S (exactly one source palm P_S) having S as the 2-source and such that $iLD(H_S)<iED(H_S)$ ($iLD'(P_S)<iED(P_S)$) and such that $S\ne S'$ with S, $S'\in Z_1$ implies $H_S\ne H_{S'}$ ($P_S\ne P_{S'}$). Then $iLD(H)=0$ since $iD(S)=0$. Proposition 5 shows that $id_G(H)=1$ and $|E(W-H,S)|=1$. Since $ec(G')\ge 2$, we have $|K(H)|\ge 1$. Now consider P, for which $iLD'(P)=0$ since $iD'(S)=0$. If $iED(P)=1$ then $W-P\ne\phi$ and $|E_G(W-P,P)|=1$ by Proposition 6(2). Then $|K(P)|\ge 1$ since $ec(G')\ge 2$. Now assume that $iED(P)=2$. Then $P=W=X\subseteq Y$ and if $X\subset Y$ then $id_G(X)=0$ by Proposition 6(2). Proposition 6(3)(i) shows that $Y=id_G(S)\ge 2$ and y (the total number of source 2-chains in P) ≥ 3. If $P=V(G)$ then $iD''(S)=1$ and $iLD(V(G))=1$. We have $oLD(V(G))=oLD(P)\ge 3$ since each source 2-chain in P has the 2-sink T with $oD(T)=1$. Hence $iLD(V(G))=1<oLD(V(G))$ contradicting our assumption that $iLD(V(G))\ge oLD(V(G))$. If $P=Y\subset V(G)$ then $|K(P)|\ge 2$ since $ec(G')\ge 2$. If $P=W=X\subset Y$ then $id_G(X)=0$ and, therefore, $|K(P)|\ge 2$.

The discussion so far shows that there is one-to-one correspondence of 2-components $S\subset W$ with iD"(S) into edges of K(W), meaning that $iD(W)\le|K(W)|$, a contradiction. Q.E.D.

3.3. Admissible pairs and an algorithmic proof of the upper bound

We prove that $\alpha(G)\le D(G)$ in this section. The proof is by induction on augmentation numbers of directed graphs. It is shown that if $D(G)>0$ then G has a pair u, $v\in V(G)$ such that $G'=G+(u,v)$ has $D(G')=D(G)-1$. Hence, by induction, we have $\alpha(G)-1=\alpha(G')\le D(G')$ and $\alpha(G)\le D(G)$. First we define an admissible pair u, v of G. For notational simplicity we denote $B_1=Z_6$ (weak 1-components), $B_2=Z_5$ (1-components), $B_3=Z_4$ (weak 2-components), $B_4=Z_1$ (2-components), and let

$$m=\begin{cases} 1 & \text{if } |B_1|\ge 2,\\ 2 & \text{if } |B_1|=1 \text{ and } |B_2|\ge 2,\\ 3 & \text{if } |B_2|=1 \text{ and } |B_3|\ge 2,\\ 4 & \text{if } |B_3|=1,\end{cases}$$

where $|B_4|\ge 2$ since we are assuming that $ec(G)<2$. Note that if $2\le m\le 4$ then $V(G)\in B_{m-1}$. A pair of vertices u_1, u_2 of G with $D(G)>0$ is called an *admissible pair* if and only if they satisfy the following (1) through (4).

(1) There are two sequences $S_{4i},\cdots,S_{mi}$, $1\le i\le 2$, such that $u_i\in S_{4i}\subseteq\cdots\subseteq S_{mi}$, where $S_{ij}\in B_i$ and $S_{i1}\ne S_{i2}$, $m\le i\le 4$.

(2) If $S_{21}\ne S_{22}$ then S_{21} (S_{22}, respectively) is a 1-sink (1-source) and $oD(S_{21})$ ($iD(S_{22})$) is maximum among those 1-sinks (1-sources) in S_{11} (S_{12}).

(3) If $S_{31}\ne S_{32}$ and $S_{3j}\subset S_{2j}$ (j=1 or 2) then S_{31} (S_{32}, respectively) has $iED(S_{31})\ge 1$ ($oED(S_{32})\ge 1$), and $oD(S_{31})$ ($iD(S_{32})$) is maximum among such weak 2-components in S_{2j}, $1\le j\le 2$. If $|B_3|\ge 4$ then $id_G(S_{31}\cup S_{32})\ge 2$ ($od_G(S_{31}\cup S_{32})\ge 2$).

(4) If $S_{41}\ne S_{42}$ and $S_{4j}\subset S_{3j}$ (j=1 or 2) then S_{41} (S_{42}, respectively) is a 2-sink (2-source) of S_{31} (S_{32}) such that $oD(S_{41})$ ($iD(S_{42})$) is maximum among those 2-sinks (2-sources) in S_{31} (S_{32}).

We show an example of an admissible pair u_1, u_2 by using the graph G_1 of Figure 1. The computation of $D(G_1')$ for $G_1'=G_1+(u_1,u_2)$ is also given.

Example 3. The graph G_1 (excluding bold lines) of Figure 1 has an admissible pair $u_1=11$ and $u_2=13$. The two sequences are $u_j\in S_{4j}=S_{3j}=S_{2j}=S_{1j}$, $1\le j\le 2$, where $S_{41}=\{11\}$ with $oD(S_{41})=2$ and $S_{42}=\{13\}$ with $iD(S_{42})=2$. For $G_1'=G_1+(11,13)$, let Z_i' denote the set corresponding to Z_i of G_1, $1\le i\le 5$.

$Z_1'=\{S'=\{1,2,3,4,5,6,7,9\}, \{8\}, \{10\}, \{11\}, \{12\}, \{13\}\}$,

$Z_2'=\{H_1=S\cup\{8\}, H_2=S\cup\{10\}, \{11\}, \{12\}, \{13\}\}$

$Z_3'=Z_2'$ (we set $P_i=H_i$, $1\le i\le 2$).

$Z_4'=\{W=S\cup\{8,10\}, \{11\}, \{12\}, \{13\}\}$

$Z_5'=\{X=W\cup\{10,11,13\}, \{12\}\}$

Table 2 summarizes the computation of $iLD(V(G_1'))=4$, $oLD(V(G_1'))=6$ and $D(G_1')=6$. Repeating this procedure determines a solution A_1 with $|A_1|=D(G_1)$ as shown by bold lines in Figure 1.

Now we proceed to formal discussion to prove that $\alpha(G)\le D(G)$. We can easily prove the following proposition, and the proof is omitted.

Proposition 7. If $D(G)>0$ then G has an admissible pair.

Let u_1, u_2 be any fixed admissible pair of G with $D(G)>0$, and we denote $G'=G+e$, where $e=(u_1,u_2)$. Let Z_1', Z_2', Z_3', Z_4', Z_5' and Z_6' denote the sets of 2-components, 2-chains, palms, weak 2-components, 1-components and weak 1-components of G', respectively. Each $S\in Z_j'$ with $S\notin Z_j$ is called a *Z_j-augmenting set* for each j, $1\le j\le 6$.

Proposition 8. Let F be any Z_j-augmenting set for some j, $1\le j\le 6$. Then the following (1) through (5) hold.

(1) If j=6 then (i) and (ii) hold.

(i) $F=S_{11}\cup S_{12}$, and $F\in Z_6$ for $\forall F'\in Z_6'-\{F\}$.

(ii) $F''\in Z_j$ for $\forall F''\in Z_j'$ and each j, $1\le j\le 5$.

(2) If $j\le 5$ then $|Z_6|=1$.

(3) If j=5 then F is a 1-chain containing S_{21} and

S_{22} as the 1-sink and 1-source, respectively.

(4) If j=4 then F is a union of at least two weak 2-components.

(5) If j=1 then F is a subchain of a 2-chain.

(The proof is omitted.)

Proposition 8 allows us to extend the definitions of iED_G, iLD_G, oED_G and oLD_G onto F which is a subset when we consider it on G. For example, $iLD_G(F)=\Sigma_{S'\subseteq F, S'\in Z_1} iD_G(S')$, $iLD'_G(F)=\Sigma_{S'\subseteq F, S'\in Z_1} iD'_G(S')$ and $iLD''_G(F)=\Sigma_{S'\subseteq F, S'\in Z_1} iD''_G(S')$ for $F\in Z_1'$.

First we prove the inductive basis of our proof.

Proposition 9. If D(G)=1 then $\alpha(G)=1$.

Proof. If $|Z_j|\geq 2$ for some j, $4\leq j\leq 6$, then we can easily show that $D(G)\geq 2$. Hence $V(G)\in Z_4$. Suppose that W=V(G) contains at least two 2-chains. If W has no source 2-chain then W has at least two 2-sources S with iD(S)=1. If W has a source 2-chain then W contains a source palm P. If some source palm P contains at least two source 2-chains then each 2-chain has the 2-sinks T with oD(T)=1. If any source palm P is identical to a source 2-chain then the 2-source S of P has iD'(S)=1. Hence there is either two 2-sinks T with iD(T)=1 or two 2-sources S with iD'(S)=1. Hence W is a 2-chain having S and T as the 2-source and the 2-sink, respectively. Since $u_1\in T$ and $u_2\in S$, we have ec(G')=2, showing that $\alpha(G)=1=D(G)$. Q.E.D.

Now assume that $D(G)\geq 2$, and we will prove that D(G)-D(G')=1.

Proposition 10. If $|Z_6|\geq 2$ then D(G)-D(G')=1.

Proof. If $|Z_6|\geq 2$ then $F=S_{11}\cup S_{12}$ is the only Z_6-augmenting set, $iLD(V(G))\geq 2$ and $oLD(V(G))\geq 2$. For any $F'\in Z_6'-\{F\}$ or $F'\in Z_j'$ ($1\leq j\leq 5$), we have $F'\in Z_6$ or $F'\in Z_j$, respectively, and it can be proved that

$$iD_{G'}(F')=\begin{cases} iD_G(F')-1 & \text{if } u_2\in F', \\ iD_G(F') & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned} &iLD(G)-iLD(G') \\ &=iD_G(S_{11})+iD_G(S_{12})-iD_{G'}(F) \\ &=iD_G(S_{11})+iD_G(S_{12})-iLD_{G'}(F) \\ &=iD_G(S_{11})+iD_G(S_{12})-(iD_{G'}(S_{11})+iD_{G'}(S_{12}))=1. \end{aligned}$$

Similarly oLD(G)-oLD(G')=1, and D(G)-D(G')=1 if $|Z_6|\leq 2$. Q.E.D.

In the following we assume that $|Z_6|=1$ unless otherwise stated.

Let $H_1, R_1, H_2, R_2, \cdots, H_{m-1}, R_{m-1}, H_m$ ($m\geq 1$) be a sequence of 2-chains in a weak 2-component W satisfying the following (1) through (3) (Figure 2).

(1) Each H_i (R_i, respectively) has the 2-source S_i (S_{i+1}) and the 2-sink T_i, where $1\leq i\leq m$ for H_i and $1\leq i\leq m-1$ for R_i.

(2) There are (u_1',u_1)-path P_1 and (u_2,u_2')-path P_2 in G, where $u_1'\in T_m$, $u_2'\in S_1$, u_j' may be indentical to u_j and if $u_j\neq u_j'$ then $(V(P_j)-\{u_j'\})\cap W=\phi$ for each j, $1\leq j\leq 2$.

(3) There may be other 2-chains having S_i as the 2-source or having T_j as the 2-sink, and there may exists a longer sequence having the above sequence as a subsequence.

We call this sequence a *candidate chain* (with respect to u_1 and u_2). Any Z_1-augmenting set is a subchain of some H_i, and H_i contains at most one Z_1-augmenting set. Let $Q=H_1\cup R_1\cup\cdots\cup H_m$. For simplicity we also call Q a candidate chain. If there is any 2-component $S\subset Q$ with $iD''_G(S)=1$ then $S=S_1$ or $S=S_m$. Let $F\subset H_i$ be any 2-component or Z_1-augmenting set of G. Then $iLD_G(F)=\Sigma_{S'\subseteq F, S'\in Z_1} iD''(S')=0=iD''_{G'}(F)$ if $1<i<m$. Suppose that $F\subset H_1$. If $S_1\cap F=\phi$ then $iLD_G(F)=0=iD''_{G'}(F)$. If $S_1\subseteq F$ then $iD''_{G'}(F)=0$, while $iLD_G(F)=iD''_G(S_1)\in\{0,1\}$. Next suppose that $F\subset H_m$. If $F\cap S_m=\phi$ then $iLD_G(F)=0=iD''_{G'}(F)$. We have

$iLD_G(F)=iD''_G(S_m)=iD''_{G'}(F)\in\{0,1\}$ if $S_m\subseteq F$.

Thus we obtain the next proposition.

Proposition 11. Let F_1 (F_m, respectively) be the 2-component of G' with $S_1\subseteq F_1\subset H_1$ ($S_m\subseteq F_m\subset H_m$). Then $iD''_{G'}(F_1)=0$ and $iLD_G(F_m)=iD''_G(S_m)=iD''_{G'}(F_m)$.

Proposition 11 shows that there are four possible combinations as shown in Table 3. Let $F_i\subset H_i$, $1\leq i\leq m$, be any Z_1-augmenting set if it exists. Suppose that $V(G)\in Z_4$. Then $u_2=u_2'\in S_1$ and $u_1=u_1'\in T_m$. Hence either (i) or (ii) is possible:

(i) $iD''_G(S_1)=iD''_G(S_m)=1$.

(ii) $iD''_G(S_1)=1$ and $iD''_G(S_m)=0$.

Since any other $S\in Z_2'$, $S\neq F_i$ ($1\leq i\leq m$), is a 2-component of G, we can easily show that $iD''_{G'}(S)=iD''_G(S)$. Thus

$$\begin{aligned} &iLD_G(V(G))-iLD_{G'}(V(G')) \\ &=iD''_G(S_1)+iD''_G(S_m)-\{iD''_{G'}(F_1)+iD''_{G'}(F_m)\}=1, \end{aligned}$$

and we obtain the following proposition.

Proposition 12. If $V(G)\in Z_4$ then $iLD_G(V(G))-iLD_{G'}(V(G'))=1$.

Next suppose that $V(G)\in Z_5$ and $V(G)\neq Z_4$. Let $W_1,\cdots,W_n$ ($n\geq 2$) be a sequence of weak 2-components satisfying the following (1) and (2) (Figure 3).

(1) $u_2\in W_1$ and $u_1\in W_n$.

(2) There are n-1 edges $e_i=(v_i,w_i)$, where $v_i\in W_i$, $w_i\in W_{i+1}$ for $1\leq i\leq n-1$, and we set $u_2=w_0$ and $u_1=v_n$.

Shrink each W_i into x_i for $1\leq i\leq n$, where $x_1=u_2$ and $x_n=u_1$, and denote the resulting graph by Γ_W. Let $C_W=\{W_i | x_i$ is a cutpoint separating u_1 from u_2 in $\Gamma_W\}$, and let Q_W denote the union of all members of C_W. Q_W is a weak 2-component of G' and any other $W'\in Z_4'$, $W'\subseteq V(G)-Q_W$, is in Z_4. Let $Q_i\subseteq W_i$ be the cardidate chain with respect to w_{i-1} and v_i. Then, by Propositions 11 and 12,

$iLD_G(W_1)-iLD_{G'}(w_1)=1$ and $iLD_G(W_j)=iLD_{G'}(W_j)$

for each j, $2\leq j\leq n$. Any 2-component S of G' is a 2 -

component of G and $iD''_G(S)=iD''_{G'}(S)$. It is easy to see that $iD_{G'}(W)=iD_G(W)$ for any weak 2-component W of G with $W \notin C_W$. If $1<i<n$, that is, x_i is a cutpoint of Γ_W then

$iED(W_i)=0 \le iLD(W_i)$ and, therefore,

$iD_G(W_i) = iLD_G(W_i) = iLD_{G'}(W_i)$.

For W_i,

$iLD_G(W_1) \ge 1 = iED_G(W_1)$ and, therefore,

$iD_G(W_1)-iD_{G'}(W_1)=iLD_G(W_1)-iD_{G'}(W_1)=1$.

For W_n, $iED_G(W_n)=1$. Then we can show that $iLD_G(W_n)\ge 1$ and, therefore,

$iD_G(W_n)-iD_{G'}(W_n)=iLD_G(W_n)-iD_{G'}(W_n)=0$.

If $|Z_4|=2$ then $Q_W=V(G)$. If $|Z_4|>3$ then $iED_{G'}(Q_W)=0$. If $|Z_4|=3$ then

$iLD_G(W_1)+iLD_G(W_n)\ge 2$,

$iLD_G(W_1)+iLD_G(W_n)-iLD_{G'}(Q_W)=1$ and

$iLD_{G'}(Q_W)\ge 1=iED_{G'}(Q_W)$.

Therefore $iD_{G'}(Q_W) = iLD_{G'}(Q_W)$ in any case. Hence

$iLD_G(V(G))-iLD_{G'}(V(G'))=\Sigma_{w\in C_W} iD_G(W)-iD_{G'}(Q_W)$

$=\Sigma_{w\in C_W}\max\{iLD_G(W),iED(W)\}-\Sigma_{w\in C_W} iLD_{G'}(W)$

$=\Sigma_{w\in C_W}(iLD_G(W)-iLD_{G'}(W))=iLD_G(W_1)-iLD_{G'}(W_1)=1$.

Thus we obtain the next proposition.

Proposition 13. If $V(G)\in Z_5$ and $V(G)\notin Z_4$ then $iLD_G(V(G))-iLD_{G'}(V(G))=1$.

Finally suppose That $V(G)\in Z_6$ and $V(G)\notin Z_5$. Then the discussion is analogous to the previous cases, and we can prove the following proposition.

Proposition 14. If $V(G) \in Z_6$ and $V(G) \notin Z_5$ then $iLD_G(V(G))-iLD_{G'}(V_{G'}))=1$.

We summarize the discussion so far in the next lemma.

Lemma 2. $\alpha(G)\le D(G)$.

Proof. if $D(G)=1$ then $\alpha(G)=D(G)=1$ by Proposition 9. Assume that if $D(G)=k\ge 1$ then the lemma hold. Now consider any graph G with $D(G)=k+1$. By Proposition 14, there is an adimissible pair u_1,u_2 such that $D(G)-D(G')=1$, meaning that $\alpha(G')\le D(G')$. Hence $D(G)=D(G')+1\ge\alpha(G')+1=\alpha(G)$. Q.E.D.

`Combining Lemmas 1 and 2 shows our main theorem.

Theorem 1. $\alpha(G)=D(G)$.

4. Concluding remarks

We briefly mention time complexity of the proposed algorithm as follows.

(1) $M_G(u,v)$ for all pairs u, v can be computed in $O(|V||E|^{3/2})$ time [2,4].

(2) Elements of Z_j ($1\le j\le 6$) can be determined and the *component tree* T(G) can be constructed in $O(|V|^2)$ time, where vertices of T(G) represent elements of Z_j, $1\le j\le 6$, and vertices of V(G) with edges representing inclusion among those elements. The first level is the vertex for V(G), its sons represent elements of Z_6 and their sons do those of Z_5, and so on, until each vertices of V(G).

(3) D(G) is obtained in $O(|E||V|)$ time and an admissible pair can be found in $O(|V|)$ time.

Therefore G' can be constructed in $O(|V|(|V|+|E|^{3/2}))$. Repeating this procedure D(G) times determines a solution. Since D(G) is $O(|V|)$,

$|V|(|V|+|E|^{3/2})+|V|(|V|+(|E|+1)^{3/2})+\cdots$

$+|V|(|V|+(|E|+D(G))^{3/2})\le D(G)\cdot|V|^2+|V|^2(|E|+D(G))^{3/2}$,

which is $O(|V|^2(|E|+|V|)^{3/2})$.

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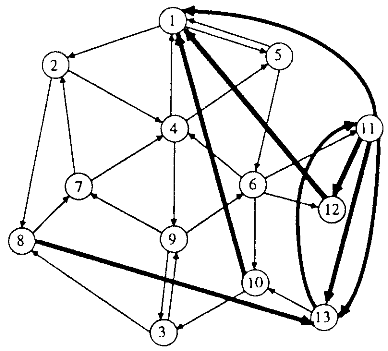

Figure 1. A directed graph G_1 and a solution (bold lines) for G_1.

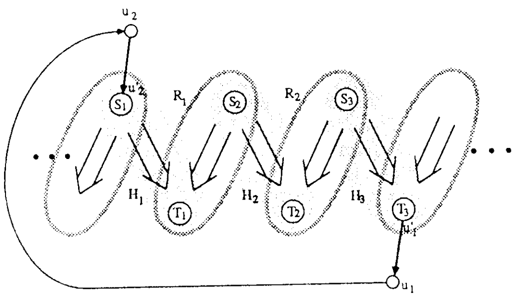

Figure 2. A candidate chain.

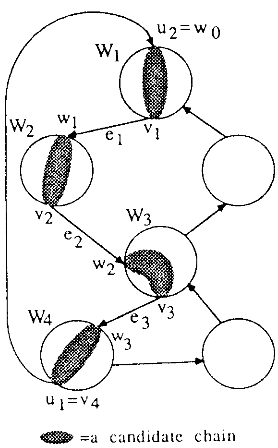

Figure 3. An illustration of the situation of the case where $V(G) \in Z_5$ and $V(G) \notin Z_4$.

Table 1. Computation of $D(G_1)$ for G_1 (excluding bold lines) of Figure 1, where $iLD(V(G_1))=5$ and $oLD(V(G_1))=7$.

Z_1	S	{8}	{10}	{11}	{12}	{13}
iD	0	0	0	1	1	2
iD'	1	0	0	1	1	2
iD"	1	0	0	1	1	2
Z_2	H	{10}	{11}	{12}	{13}	
iLD	1	0	1	1	2	
iED	1	0	1	1	2	
Z_3	P	{10}	{11}	{12}	{13}	
iLD'	1	0	1	1	2	
iED	1	0	1	1	2	
Z_4	W	{10}	{11}	{12}	{13}	
iLD	1	0	1	1	2	
iED	1	0	1	1	2	
iD	1	0	1	1	2	
Z_5	X	{11}	{12}	{13}		
iLD	1	1	1	2		
iED	1	1	1	2		
iD	1	1	1	2		

Z_1	S	{8}	{10}	{11}	{12}	{13}
oD	0	1	1	2	2	1
OD'	0	1	1	2	2	1
oD"	0	1	1	2	2	1
Z_2	H	{10}	{11}	{12}	{13}	
oLD	1	1	2	2	1	
oED	1	1	2	2	1	
Z_3	P	{10}	{11}	{12}	{13}	
oLD'	1	1	2	2	1	
oED	1	1	2	2	1	
Z_4	W	{10}	{11}	{12}	{13}	
oLD	1	1	2	2	1	
oED	1	1	2	2	1	
oD	1	1	2	2	1	
Z_5	X	{11}	{12}	{13}		
oLD	2	2	2	1		
oED	0	2	2	1		
oD	2	2	2	1		

Table 2. Computation of $D(G_1')$ for $G_1'=G_1+(11,13)$.

Z_1	S'	{8}	{10}	{11}	{12}	{13}
iD	0	0	0	1	1	1
iD'	1	0	0	1	1	1
iD"	1	0	0	1	1	1
Z_2	H_1H_2	{11}	{12}	{13}		
iLD	0	0	1	1	1	
iED	1	0	1	1	1	
Z_3	P_1P_2	{11}	{12}	{13}		
iLD'	1	0	1	1	1	
iED	1	0	1	1	1	
Z_4	W	{11}	{12}	{13}		
iLD	1	1	1	1		
iED	1	1	1	1		
iD	1	1	1	1		
Z_5	X	{12}				
iLD	3	1				
iED	2	1				
iD	3	1				

Z_1	S'	{8}	{10}	{11}	{12}	{13}
oD	0	1	1	1	2	1
oD'	0	1	1	1	2	1
oD"	0	1	1	1	2	1
Z_2	H_1H_2	{11}	{12}	{13}		
oLD	1	0	1	2	1	
oED	0	0	1	2	1	
Z_3	P_1P_2	{11}	{12}	{13}		
oLD'	1	1	1	2	1	
oED	0	0	1	2	1	
Z_4	W	{11}	{12}	{13}		
oLD	2	1	2	1		
oED	0	1	2	1		
oD	2	1	2	1		
Z_5	X	{12}				
oLD	4	2				
oED	1	2				
oD	4	2				

Table 3. Four combinations of $iD"_G(S_1)$, $iD"_G(S_m)$, $iD"_{G'}(F_1)$ and $iD"_{G'}(F_m)$.

$iD"_G(S_1)$	$iD"_G(S_m)$	$iD"_{G'}(F_1)$	$iD"_{G'}(F_m)$
1	1	0	1
1	0	0	0
0	1	0	1
0	0	0	0