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Hypercubes and Oriented Matroids

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ランクが3以下の有向マトロイドのトープグラフを特徴づける。このようなグラフは、平面上のpseudolinesのarrangementの2次元領域の隣接関係を表すグラフと本質的に同じものである。その特徴づけを使うと、与えられたグラフが上記のようなグラフに同型であるか否かを多項式時間で判定できる。一般の（ランク4以上の）有向マトロイドのトープグラフを特徴づけるという問題は未解決であるが、本論文で得られたトープグラフの諸性質、特に、どの点に対しても対極点があるという性質(antipodality)やハイパーキューブの中に距離を保存するように埋め込めるという性質は、特徴づけ問題を解くのに有用であろう。

Hypercubes and Oriented Matroids

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In this note we give a characterization of tope graphs of oriented matroids of rank at most three, which are essentially equivalent to graphs representing adjacent relations of regions of an arrangement of pseudolines in the real plane. This characterization enables us to test in a polynomial time whether a given graph is isomorphic to such a graph. The problem to characterize tope graphs of oriented matroids of any higher rank is still open. The properties of tope graphs shown in this note, in particular, antiopdality and isometrical-embeddability in hypercube, will be probably useful for the characterization.

1. Introduction

Through this note, we assume graphs have neither loops nor multiple edges. We denote the vertex-set, the edge-set and the distance function of a graph G by $V(G), E(G)$ and d_G, respectively.

A graph G is *embeddable* in a graph G' if there exists an injection $f : V(G) \to V(G')$, called an *embedding* of G into G', such that $[u,v] \in E(G)$ implies $[f(u), f(v)] \in E(G')$. For two connected graphs G and G', an embedding f of G into G' is *isometric* if $d_G(u,v) = d_{G'}(f(u), f(v))$ for all $u, v \in V(G)$. If there exists such an isometric embedding, G is called *isometrically embeddable* in G'.

The *hypercube* $Q(E)$ on a finite set E is the graph such that the vertex-set is $\{-,+\}^E$ and such that $X, Y \in \{-,+\}^E$ are adjacent if they differ in exactly one component. We will be concerned with graphs isometrically embeddable in some hypercube. Such graphs can be considered as the graphs G such that the vertices of G can be addressed with elements of $\{0,1\}^E$ for some E, such that the Hamming distance between addresses of any two vertices is one. The tope graphs of oriented matroids discussed in this note constitute a broad class of graphs isometrically embeddable in some hypercube.

Here the "tope graph" of an oriented matroid M represents a natural adjacent relation among the topes (maximal elements of the cocircuit span) of M, see [4,7,9,10,17]. To characterize tope graphs of oriented matroids is our main theme in this note. This characterization problem is fundamental because

(1) the tope graph of an oriented matroid determines the oriented matroid uniquely up to reorientation and a complete answer to the problem will lead to an axiomatization of oriented matroids using only the graph language;

(2) in the case of linear oriented matroids, the problem is equivalent to the open problem to characterize 1-skeletons of zonotopes, for zonotopes, see e.g. [8]; and

(3) in the case of oriented matroids (linear oriented matroids, resp.) of rank at most three, the problem is essentially equivalent to the open problem to characterize graphs representing adjacent relations of regions of an arrangement of pseudolines (lines, resp.) in $\mathbf{R}^2$.

We will first consider the characterization problem in more general systems than oriented matroid, i.e., L^1-embeddable system (L^1-system) and acycloid, and then we will characterize tope graphs of oriented matroids of rank at most three.

L^1-systems, introduced in Section 2, are defined by the reorientation property of topes of oriented matroids. The tope graphs of L^1-systems are exactly the graphs isometrically embeddable in some hypercube. For such graphs two characterizations are known, see [1,3,6,16]. In Section 3, we present a constructive characterization of L^1-systems.

Acycloids [20] satisfy the negativity closedness property in addition to the reorientation property. In Section 4, we present two characterizations of tope graphs of acycloids.

The characterizations obtained in Sections 3 and 4 will be useful to characterize tope graphs of oriented matroids. Indeed, in Section 5 we characterize such graphs in the case of rank at most three. The characterization theorem says : a graph G is isomorphic to the tope graph of an oriented matroid of rank at most three if and only if G is antipodal, planar and isometrically embeddable in some hypercube (Corollary 5.3). Using this theorem, we can test in a polynomial time whether a given graph is isomorphic to the graph representing adjacent relation of regions of an arrangement of pseudolines in $\mathbf{R}^2$.

For the case of any higher rank and for the linear case even if the rank is three, the characterization problem is still open.

The proofs of theorems in this note can be found in [21].

2. Definitions

In this section we introduce L^1-embeddable systems and review definitions of acycloids and oriented matroids, and then we define the tope graphs of their three systems.

Let E be a finite set. A *signed vector* on E is an element of $\{-,0,+\}^E$. We will begin with some notations on signed vectors. For a signed vector $X = (X_e : e \in E)$ on E, the *support* of X is the set $\{e \in E : X_e \neq 0\}$. The *negative* $-X$ of X is defined in the trivial way. For $e \in E$, we denote by ${}_{\bar{e}}X$ the signed vector on E obtained from X by replacing X_e by $-X_e$. For $X, Y \in \{-,0,+\}^E$, define $D(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}$.

Let $A = (E, \mathcal{T})$ be a pair with a finite set E and $\mathcal{T} \subseteq \{-,0,+\}^E$. An element e in E is a *loop* of A if $X_e = 0$ for all $X \in \mathcal{T}$, and the set of loops of A is denoted by E_0. Define $E_1 = \{e \in E : X_e = Y_e \neq 0 \text{ for all } X, Y \in \mathcal{T}\}$. Two distinct elements e and $f \in E - (E_0 \cup E_1)$ are *parallel* if either $X_e = X_f$ for all $X \in \mathcal{T}$ or $X_e = -X_f$ for all $X \in \mathcal{T}$. A is called *simple* if it has no loops and no parallel elements.

An *L^1-embeddable system* (*L^1-system*) is a pair $A = (E, \mathcal{T})$, where E is a finite set and $\emptyset \neq \mathcal{T} \subseteq \{-,+\}^E$, satisfying:

(A1) (*reorientation property*) if $X, Y \in \mathcal{T}$ and $X \neq Y$, there exists $f \in D(X,Y)$ such that ${}_{\bar{f}}X \in \mathcal{T}$.

Note that every L^1-system is simple, but can have nonempty E_1. A (simple) *acycloid* is a pair $A = (E, \mathcal{T})$, where E is a finite set and $\emptyset \neq \mathcal{T} \subseteq \{-,+\}^E$, satisfying the reorientation property (A1) and

(A2) $X \in \mathcal{T}$ implies $-X \in \mathcal{T}$.

A general acycloid may have loops or parallel elements, but in this note, we consider only simple cases. In either definitions of L^1-system or acycloid, we call an element of $\mathcal{T}$ a *tope* of A.

Let $A = (E, \mathcal{T})$ be a pair such that $\mathcal{T} \subseteq \{-,0,+\}^E$. The *simplification* $sim(A)$ of A is the pair obtained by excluding all loops and by identifying any parallel class (i.e., class of elements parallel each other) with its representative element.

For an acycloid $A = (E, \mathcal{T})$ and $e \in E$, let $\mathcal{T}/e = \{$the restriction of X to $E - e : X, {}_{\bar{e}}X \in \mathcal{T}\}$, and call $A/e = (E - e, \mathcal{T}/e)$ the *elementary contraction* of A by e. For an ordered subset $S = \{e_1, e_2, ..., e_n\}$ of E, the *contraction* A/S of A by S is defined inductively by $A/S = (sim(A/(S - e_n)))/e_n$.

The next theorem presents a characterization of oriented matroids in terms of topes:

Theorem 2.1([15]). An oriented matroid is a pair $M = (E, \mathcal{T})$ where E is a finite set and $\emptyset \neq \mathcal{T} \subseteq \{-,0,+\}^E$ satisfying:

(1) all elements of $\mathcal{T}$ have the same support;
(2) the simplification $sim(M)$ is an acycloid; and
(3) the simplification of every contraction of M has the reorientation property.

We denote by $()$ the signed vector on the empty set $\emptyset$, and define for convenience that $A = (\emptyset, \{()\})$ is an oriented matroid, and hence is also an acycloid and an L^1-system.

In the above-mentioned three systems, that is, an L^1-system, an acycloid and an oriented matroid, we will be concerned with graphs formed by topes.

We will define the "tope graph" for a general system $A = (E, \mathcal{T})$, $\mathcal{T} \subseteq \{-,0,+\}^E$, satisfying that all elements of $\mathcal{T}$ have the same support: the *tope graph* G_A of A is a graph such that $V(G_A) = \mathcal{T}$ and such that $X, Y \in V(G_A)$ are adjacent if and only if there exists no $Z \in \mathcal{T} - \{X, Y\}$ with $D(X,Z) \subseteq D(X,Y)$. When A has no parallel elements, $X, Y \in V(G_A)$ are adjacent if and only if $|D(X,Y)| = 1$.

We give examples of tope graphs of an L^1-system, an acycloid and an oriented matroid, in Fig.1 (a), (b) and (c), respectively. Here we note that the acycloid of Fig.1 (b) is not an oriented matroid. For such acycloids, see [11, 15, 20].

3. Tope graphs of L^1-systems

The class of L^1-systems is closely related to the class of graphs isometrically embeddable in some hypercube. In this section we present a constructive characterization of L^1-systems, which will then yield a similar characterization of graphs isometrically embeddable in some hypercube.

Proposition 3.1. A graph G is isomorphic to the tope graph of an L^1-system if and only if G is isometrically embeddable in some hypercube.

Note. Let G be a graph isometrically embeddable in some hypercube $Q(E)$, and let f_1 and f_2 be isometric embeddings of G into $Q(E)$. We call f_1 and f_2 *isomorphic* if $f_1(V(G))$ and $f_2(V(G))$ coincide under an automorphism of $Q(E)$. Now let f be an isometric embedding of G into $Q(E)$, $[x,y] \in E(G), f(x) = X, f(y) = Y$ and $D(X,Y) = \{e\}$. Then we have $\{Z \in f(V(G)) : Z_e = X_e\} = \{z \in V(G) : d_G(x,z) < d_G(y,z)\}$. Hence f is uniquely determined up to isomorphism.

Let G be a graph. For $X, Y \subseteq V(G), [X,Y]$ denotes the set of edges with one endpoint in X and the other in Y. When G is connected, for $[a,b] \in E(G)$ we define $C(a,b) = \{x \in V(G) : d_G(a,x) < d_G(b,x)\}$. A subset X of $V(G)$ is called *convex* in G if the subgraph induced by X is connected and if for all $u, v \in X$ all shortest (u,v)-paths are contained in the subgraph.

For graphs isometrically embeddable in some hypercube, the following theorem is well known:

Theorem 3.2 (Djoković[6]). A graph G is isometrically embeddable in some hypercube if and only if G satisfies

(1) G is connected bipartite, and

(2) $C(a,b)$ is convex for all $[a,b] \in E(G)$.

Let $A = (E, \mathcal{T})$ be a pair with a finite set E and $\emptyset \neq \mathcal{T} \subseteq \{-,+\}^E$. Let $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ be such that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$ and $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$, and such that for any $X \in \mathcal{T}_1 - \mathcal{T}_2$ there exists no $e \in E$ with ${}_{\bar{e}}X \in \mathcal{T}_2 - \mathcal{T}_1$. Let $p \notin E$, and put $\mathcal{T}' = \{X + p^+ : X \in \mathcal{T}_1\} \cup \{X + p^- : X \in \mathcal{T}_2\}$, where $X + p^i$ ($i \in \{-,+\}$) denotes the signed vector Z on $E \cup \{p\}$ with $Z_e = X_e$ for all $e \in E$ and with $Z_p = i$. Then we call the pair $A' = (E \cup \{p\}, \mathcal{T}')$ the *expansion* of A with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. A' is called the L^1-*expansion* if $(E, \mathcal{T}_1)$ and $(E, \mathcal{T}_2)$ are L^1-systems.

Lemma 3.3. If $A = (E, \mathcal{T})$ is an L^1-system, any L^1-expansion A' of A is also an L^1-system.

Theorem 3.4. A pair $A = (E, \mathcal{T})$ is an L^1-system with $E_1 = \emptyset$ if and only if A can be obtained from the smallest L^1-system $(\emptyset, \{()\})$ by a sequence of L^1-expansions.

Let G be a graph, and let $W_1, W_2 \subseteq V(G)$ be such that $W_1 \cup W_2 = V(G), W_1 \cap W_2 \neq \emptyset$ and $[W_1 - W_2, W_2 - W_1] = \emptyset$. The *expansion* of G with respect to W_1 and W_2 is the graph G' constructed as follows:

(i) replace each vertex $v \in W_1 \cap W_2$ by two vertices u_v, u'_v, which are joined by an edge;
(ii) join u_v to the neighbours of v in $W_1 - W_2$ and u'_v to those in $W_2 - W_1$;
(iii) if $v, w \in W_1 \cap W_2$ and $[v, w] \in E(G)$, then join u_v to u_w and u'_v to u'_w.

A subset X of $V(G)$ is called L^1 in G if for all $u, v \in X$ there exists at least one shortest (u, v)-path of G in the subgraph induced by X. The expansion G' is called L^1 if W_1 and W_2 are L^1 -subsets of $V(G)$.

The next theorem is equivalent to Theorem 3.4:

Theorem 3.5. A graph G is isometrically embeddable in some hypercube if and only if G can be obtained from K_1 by a sequence of L^1-expansions.

Note that the above theorem is similar to the constructive characterization of median graphs by Mulder [18]. The graph in Fig.2 shows the L^1-expansion of the graph in Fig.1 (a) with respect to W_1 and W_2, where $W_1(= \mathcal{T}_1) = V(G)$ and $W_2(= \mathcal{T}_2) = \{(+++), (+-+), (+--), (++-)\}$.

4. Tope graphs of acycloids

In this section we present two characterizations of tope graphs of acycloids. One is in Theorem 4.2 (Corollary 4.3) and the other is in Theorem 4.6.

A graph G, which contains at least one edge, is *antipodal* if it has a central symmetry, that is, for any $v \in V(G)$ there exists a unique $\overline{v} \in V(G)$, the *antipode* of v, such that $d_G(v, u) \leq d_G(v, \overline{v})$ for all neighbourhoods u of $\overline{v}$, see [2,13,19]. For convenience, we define a one-vertex graph K_1 is antipodal.

A connected graph G is *even* if for any vertex v of G there exists a unique vertex v' such that $d_G(v, v') = diam(G)$, the diameter of G. An even graph G is called *harmonic* if $[u', v'] \in E(G)$ whenever $[u, v] \in E(G)$, and is called *symmetric* if $d_G(u, v) + d_G(u, v') = diam(G)$ for all $u, v \in V(G)$. It is easy to see that every symmetric even graph is harmonic, cf. [14, Proposition 13]. Also we can easily check that a graph is symmetric even if and only if it is antipodal, cf. [2, p.107]; in the symmetric case, the above-mentioned vertex v' coincides with the antipode of v. In the following, we will extend the meaning of antipode by calling v' the *antipode* of v in any even graph and use the same notation $\overline{v}$ for v'.

Proposition 4.1. If a graph G is isomorphic to the tope graph of an acycloid, G is symmetric even, or equivalently, G is antipodal.

Theorem 4.2. A graph G is isomorphic to the tope graph of an acycloid if and only if G is a harmonic even graph isometrically embeddable in some hypercube.

Corollary 4.3. A graph G is isomorphic to the tope graph of an acycloid if and only if G is an antipodal graph isometrically embeddable in some hypercube.

Next we will present a constructive characterization of acycloids similar to that of L^1-systems, and a constructive characterization of tope graphs of acycloids.

Let $A = (E, \mathcal{T})$ be a pair with a finite set E and $\emptyset \neq \mathcal{T} \subseteq \{-, +\}^E$. The L^1-expansion A', which we defined in Section 3, is *acycloidal* if $\mathcal{T}_2 - \mathcal{T}_1 = \{-X : X \in \mathcal{T}_1 - \mathcal{T}_2\}$ and if $\mathcal{T}_1 \cap \mathcal{T}_2$ is closed under negation.

Lemma 4.4. If $A = (E, \mathcal{T})$ is an acycloid, any acycloidal expansion A' of A is also an acycloid.

The next theorem is similar to Theorem 3.4 and the proof is also similar:

Theorem 4.5. A pair $A = (E, \mathcal{T})$ is an acycloid if and only if A can be obtained from the smallest acycloid $(\emptyset, \{()\})$ by a sequence of acycloidal expansions.

Let G be an even graph, and let $W_1, W_2 \subseteq V(G)$ be such that $W_1 \cup W_2 = V(G), W_1 \cap W_2 \neq \emptyset$ and $[W_1 - W_2, W_2 - W_1] = \emptyset$. The L^1-expansion G' of G with respect to W_1 and W_2 is called *acycloidal* if $W_2 - W_1 = \{\overline{v} : v \in W_1 - W_2\}$ and if $v \in W_1 \cap W_2$ implies $\overline{v} \in W_1 \cap W_2$. It is easy to see that if G is harmonic even, then so is G'.

The next theorem is equivalent to Theorem 4.5 and similar to Theorem 3.5:

Theorem 4.6. A graph G is isomorphic to the tope graph of an acycloid if and only if G can be obtained from K_1 by a sequence of acycloidal expansions.

The graph in Fig.1 (b) can be obtained from $Q(E = \{1, 2, 3, 4\})$ by the acycloidal expansion with respect to W_1 and W_2, where

$W_1(= \mathcal{T}_1) = \{X \in \{-, +\}^E :$ at most two components of X are plus $\}$ and

$W_2(= \mathcal{T}_2) = \{X \in \{-, +\}^E :$ at least two components of X are plus $\}$.

5. Tope graphs of rank-three oriented matroids

In this section we characterize tope graphs of oriented matroids of rank at most three.

Let $\mathbf{S}^2$ be a 2-dimensional unit sphere, that is, $\mathbf{S}^2 = \{x \in \mathbf{R}^3 : \| x \| = 1\}$. A *Jordan curve* in $\mathbf{S}^2$ is the image of a unit circle, under a self-homeomorphism of $\mathbf{S}^2$. For a Jordan curve J in $\mathbf{S}^2$, the two components of $\mathbf{S}^2 - J$ are called the *sides* of J and denoted by J^+ and J^-. For convenience, put $J^0 = J$.

A pair $(E, \mathcal{J})$ is an *arrangement* of Jordan curves in $\mathbf{S}^2$ if E is a finite set, and $\mathcal{J}$ is a collection $\{J_e^i : e \in E, i \in \{-, 0, +\}\}$ of subsets of $\mathbf{S}^2$ satisfying

(i) for any $e \in E$, J_e^0 is a Jordan curve of $\mathbf{S}^2$ with sides J_e^- and J_e^+; and

(ii) for any two e and $f \in E$, $|J_e^0 \cap J_f^0| = 2$ and $J_e^i \cap J_f^j \neq \emptyset$ for all $i, j \in \{-, +\}$.

Let σ be the mapping: $\mathbf{S}^2 \to \{-, 0, +\}^E$ defined by $\sigma(x)_e = i$ if and only if $x \in J_e^i$.

By Cordovil's result in [5], we can easily lead the following theorem:

Theorem 5.1. Let $(E, \mathcal{J})$ be an arrangement of Jordan curves in $\mathbf{S}^2$. Then $\sigma(\mathbf{S}^2) \cap \{-, +\}^E$ is the set of topes of an oriented matroid of rank at most three. Conversely, every oriented matroid without loops of rank at most three can be obtained this way.

Theorem 5.2. A graph G is isomorphic to the tope graph of an oriented matroid of rank at most three if and only if G is harmonic even, planar and isometrically embeddable in some hypercube.

Corollary 5.3. A graph G is isomorphic to the tope graph of an oriented matroid of rank at most three if and only if G is antipodal, planar and isometrically embeddable in some hypercube.

6. Concluding remarks

1. One can easily see that Djoković's theorem, Theorem 3.2, is a good characterization of graphs isometrically embeddable in some hypercube, i.e., one can verify the conditions (1), (2) of Theorem 3.2 in time bounded by a polynomial in $V(G)$. Similarly, Theorem 5.2 is a good characterization of tope graphs of an oriented matroid of rank at most three. (Note that even though we don't know any good characterization of tope graphs of oriented matroids of general rank, one can test in a polynomial time whether a given graph is isomorphic to the tope graph of some oriented matroid, see [12].)

2. In relation to Theorem 4.2, we present the next problem:

Let G be an even graph isometrically embeddable in some hypercube. Then, is G harmonic even ?

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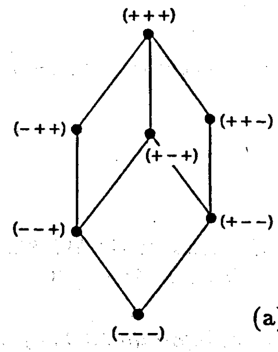
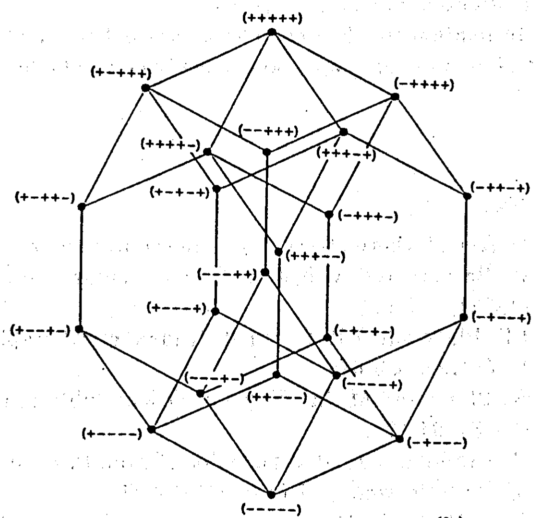


(a)



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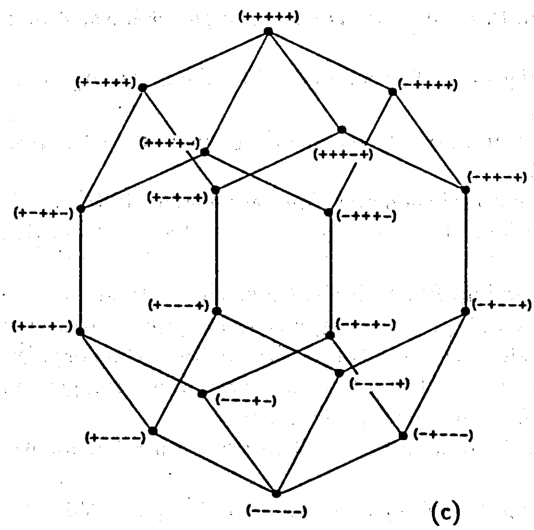

(c)

Fig.1

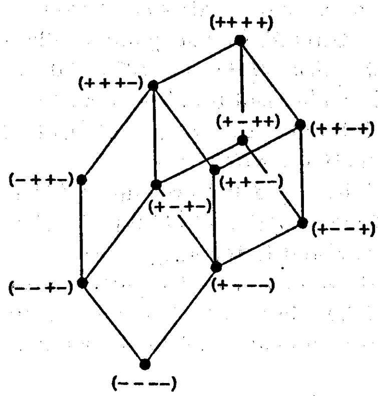

Fig.2