3-連結グラフに対する点被覆及び連結点被覆問題について

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概要：本論文では，3-連結グラフに対する点被覆及び連結点被覆問題について考える．但し，3-連結グラフの部分族である準車輪の族及び車輪拡大グラフの族に対象を制限して問題を考える．本稿では，まず車輪拡大グラフに対する点被覆問題が NP 完全問題であることを示す．これと既存の結果とを合わせれば，車輪拡大グラフに対する連結点被覆問題もまた NP 完全問題であることが示される．次に，準車輪グラフに対する連結点被覆問題を線形マトロイドマッチング問題に帰着させ，線形マトロイドマッチング問題の解法を利用することにより，準車輪グラフに対する最小連結点被覆が $O(|V|^3)$ で定められることを示す．
キーワード：点被覆，連結点被覆，3-連結グラフ，準車輪グラフ，車輪拡大グラフ

On the Complexity of Vertex Cover and Connected Vertex Cover Problems for 3-Connected Graphs

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Abstract : The subject of paper is the vertex cover problem (VCP) and the connected vertex cover problem (CVCP) for 3-connected graphs. More specifically, VCP and CVCP for the two classes of 3-connected graphs, called quasi-wheels and super-wheels, are considered. First we prove that VCP for super-wheels is NP-complete. This result, combined with the known result on the relationship between VCP and CVCP for super-wheels, implies that CVCP for super-wheels is NP-complete. By reducing CVCP for quasi-wheels to a linear matroid matching problem, it is shown that a minimum connected vertex cover for any given quasi-wheel can be obtained in polynomial time.
Key word : Vertex covers, Connected vertex covers, 3-connected graphs, Quasi-wheels, Super-wheels

1 Introduction

In this paper we consider the vertex cover problem and the connected vertex cover problem for 3–connected graphs. A *vertex cover* of a graph $G = (V, E)$ is a set $N \subseteq V$ such that each element of E is incident upon some element of N, where V and E are the sets of vertices and edges, respectively, of G. A *connected vertex cover* of G is a vertex cover N of G such that the subgraph $G[N]$ induced by N is a connected graph. *The vertex cover problem* (VCP for short) is the problem of finding a vertex cover of minimum cardinality, and *the connected vertex cover problem* (CVCP for short) is similarly defined. The *recognition problem* (RP for short) of a class of graphs is the problem of deciding whether or not a given graph is in the class. For example, RP for 3–connected graphs is solvable in time $O(|V| + |E|)$ [8].

VCP is one of basic *NP*–complete problems [10]. It is also known to remain *NP*–complete under various restrictions on graphs, e.g., for cubic planar graphs [18] and for cubic planar 3–connected graphs [17]. On the other hand polynomially solvable classes of graphs include series–parallel graphs (in time $O(|V| + |E|)$ [1, 14]), bipartite graphs (in time $O(|E|\sqrt{|V|})$ [7] by graph matching), and perfect graphs [5].

CVCP is also known to be *NP*–complete and remains so for planar graphs with maximum vertex degree at most 4 [3], and for 3–connected planar graphs [20]. When maximum degree is bounded by 3, however, CVCP becomes polynomially solvable [16]. For further details, refer to [9].

Tutte gave a complete characterization of 3–connected graphs [15] : any 3–connected graph can be constructed from a wheel by repeating edge addition and/or vertex splitting operations (the details will be given later). Two subclasses of 3–connected graphs, called *quasi–wheels* and *super–wheels*, were introduced in [19, 20], based on the characterization above of 3–connected graphs. These are our target subclasses of 3–connected graphs for which we consider VCP, CVCP, and RP. It is already known that VCP is solvable in time $O(|V|)$ when quasi–wheels are planar [2] (which are called Halin graphs), and that RP for super–wheels is *NP*–complete [19]. We prove in this paper that RP for quasi–wheels and VCP for super–wheels are both *NP*–complete. The *NP*–completeness of CVCP for super–wheels is also obtained using the results in [19]. As a sole positive result CVCP for quasi–wheels is shown to be solvable in time $O(|V|^3)$. Thus, quasi–wheels form a rare subclass of graphs for which, in spite of *NP*–completeness of RP, CVCP is solvable in polynomial time. Table 1 summarizes known results along with our results to be given in the paper.

2 Preliminaries

2.1 Basic definitions

We suppose that $G = (V, E)$ is a graph with a vertex set V and an edge set E. For any vertex set $S \subseteq V$, $G' = (S, E')$ is called a subgraph of G *induced by* S and is denoted as $G[S]$, where $E' = E \cap S \times S$. The *degree* $\delta_G(v)$ of a vertex v is a total number of edges (v, v'), $v' \neq v$, incident upon v in G.

A vertex set $S \subseteq V$ is an *independent set* of $G = (V, E)$ if for any vertex pair $\{v, w\} \subseteq S$, $e = (v, w) \notin E$. S is a *nonseparating independent set* of G if it is an independent set whose removal does not increase the number of connected components. The *nonseparating independent set problem* (NISP for short) is the problem of finding a nonseparating independent set of maximum cardinality. Note that for any connected graph $S \subseteq V$ is a nonseparating independent set if and only if $V - S$ is a connected vertex cover.

2.2 3–connected graphs

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose deletion from G disconnected it or result in a single vertex. A graph G is called a *k–connected graph* if and only if $\kappa(G) \geq k$. We denote an elementary cycle of length n by C_n. A wheel of order $n + 1$ ($n \geq 3$), is the graph formed by a cycle C_n and one new vertex v_0 joined to each vertex of C_n with an edge. We denote this wheel by $W_{n+1} = K_1 + C_n$. We refer to C_n and v_0 as the *rim* and the *hub* of W_{n+1}, respectively. The edges joining v_0 and vertices of C_n are called the *spokes*. Fig.1 (a) shows a wheel W_6. A graph G is a 3–connected graph if and only if G is either a wheel or a graph obtained from a wheel by repeating the following operation 1 and/or 2 (Tutte's theorem [15]):

Operation 1 : Join non-adjacent vertices u, v with an edge. (Fig.1 (b))

(This operation is called an *edge addition*.)

Operation 2 : For a vertex v with $\delta_G(v) \geq 4$, replace v with a pair of adjacent vertices v', v'' and join each vertex that was adjacent to v to exactly one of v' and v'', by means an edge in such a way that $\delta_{G'}(v') \geq 3$ and $\delta_{G'}(v'') \geq 3$, where G' is a graph constructed by this operation. (Fig.1 (c))

(This operation is called a *vertex splitting.*)

Note that each of an edge addition and a vertex splitting can break planarity of graphs. A graph obtained from a wheel by repeated application of only vertex splittings is called a *quasi-wheel* [20]. We call a graph obtained from a wheel by repeated application of only edge additions a *super-wheel*.

2.3 2–polymatroids

A pair $M^\bullet = (S, r^\bullet)$ of a set S and a function $r^\bullet$ is called a *2-polymatroid* if $r^\bullet$ associates a nonnegative integer to each subset of S and satisfies the following conditions.

(P1) $r^\bullet(\emptyset) = 0$;

(P2) $r^\bullet(X'^\bullet) \leq r^\bullet(X^\bullet)$ if $X'^\bullet \subset X^\bullet$, for any $X'^\bullet, X^\bullet \subseteq S$;

(P3) $r^\bullet(X^\bullet \cup X'^\bullet) + r^\bullet(X^\bullet \cap X'^\bullet) \leq r^\bullet(X^\bullet) + r^\bullet(X'^\bullet)$ for any $X^\bullet, X'^\bullet \subseteq S$;

(P4) $r^\bullet(\{x\}) \leq 2$ for every $x \in S$.

S and $r^\bullet$ are called the *underlying set* and the *rank function* of $M^\bullet$, respectively. $X^\bullet \subseteq S$ is a *matching* of $M^\bullet$ if $r^\bullet(X^\bullet) = 2|X^\bullet|$. The *matroid matching problem* for a 2-polymatroid $M^\bullet$ is the problem of finding a maximum cardinality matching of $M^\bullet$.

It is easy to seen that a 2-polymatroid is a proper generalization of a matroid. In fact $M = (S, r)$ is a *matroid* if $r : 2^S \rightarrow Z^+$ satisfies the conditions P1,P2,P3 and that $r(\{x\}) \leq 1$ for each $x \in S$, instead of P4. $X \subseteq S$ is called *independent* if $r(X) = |X|$, and otherwise it is called *dependent*. A maximal independent set of a matroid is called a *base*.

For a graph $G = (V, E)$ an edge set $X \subseteq E$ is a *cutset* if the number of connected components of $G' = (V, E - X)$ is more than that of G. An edge set $X \subseteq E$ is *cutset-free* if X contains no *cutset* of G. A matroid $M(G) = (E, r)$ is called a *cographic matroid* of G if $r(X)$ is the size of a largest cutset-free subset of X for any $X \subseteq E$. In other words X is an independent set of $M(G)$ if and only if X is cutset-free.

From a matroid $M = (S, r)$, we can construct a 2-polymatroid $M^\bullet = (T^\bullet, r^\bullet)$, where $T^\bullet$ is a set of pairs of elements from S, and $r^\bullet$ is an extension of r to $T^\bullet$ s.t. $r^\bullet(X^\bullet) = r(\cup_{e \in X^\bullet} e)$ for $X^\bullet \subseteq T^\bullet$.

3 Characterization

3.1 Quasi–wheels

Let $G = (V, E)$ be a quasi-wheel. We denote sets of vertices and edges of C_n by $V(C_n)$ and $E(C_n)$, respectively. If G is derived from a wheel $W_{n+1} = K_1 + C_n$ then C_n remains in G and it is called the *rim* of G. Its vertices V is partitioned into two sets $V_R = V(C_n)$ and $V_T = V - V_R$, where V_T consists of vertices introduced by vertex splittings. $G[V_T]$ is a tree and is called the *hub tree.* A subgraph $G - E(C_n)$ is a tree and is called the *inner tree.* Vertices in V_R are called *rim vertices* and those in V_T are called *inner vertices.* We can easily prove the following proposition.

Proposition 3.1.1 *G is a quasi-wheel if and only if G has a spanning tree T such that*

(i) *any non-leaf v of T has degree $\delta_T(v) = \delta_G(v)$ (≥ 3),*

and

(ii) *$E(G) - E(T)$ form a cycle containing all leaves of T.*

The proposition is restated as follows. (Note that such a cycle C is a rim of G if G is a quasi-wheel.)

Corollary 3.1 *G is a quasi-wheel if and only if G has a cycle C such that*

(i) *any vertex v of C has $\delta_G(v) = 3$,*

and

(ii) *$E(G) - E(C)$ form a spanning tree whose leaves are those of $V(C)$.*

We define *the rim identification problem* (RIP for short) as the problem of identifying the rim of a given quasi-wheel (or a given super-wheel). We can identyfy the rim of a planar quasi-wheel in time $O(|V|^2)$ [20]. We propose the procedure *MQW(G)* that constructs a graph $G' = (V', E')$ from any given cubic graph G with $|E| \geq 4$.

MQW(G) : (Make Quasi Wheel)

Step1. Let $G = (V, E)$ be any given graph and v_0 be a new vertex. Set $V' \leftarrow V \cup \{v_0\}$, $E' \leftarrow \emptyset$.

Step2. Construct a graph $G' = (V', E')$ by repeating the following (1),(2) for each $e_i = (v, w) \in E$.

(1) $V' \leftarrow V' \cup \{x_i\}$, where x_i is a new vertex.
(2) $E' \leftarrow E' \cup \{(v, x_i), (x_i, w), (x_i, v_0)\}$, where $(v, x_i), (x_i, w), (x_i, v_0)$ are new edges.

Note that $\delta_{G'}(v_0) = |V|$ and $\delta_{G'}(v) = 3$ for any other vertex $v \in V'$. (See Fig.2 for examples of G and G'.) We obtain the following two lemmas concerning the NP-completeness of RP and RIP for quasi-wheels.

Lemma 3.1 *Let G be any cubic graph, and G' be constructed from G by $MQW(G)$. Then G contains a Hamilton cycle if and only if G' is a quasi-wheel.*

Proof. Suppose G contains a Hamilton cycle C. Let $Y = \{(v, x_i), (x_i, w) \in E' \mid e_i = (v, w) \in E(C)\}$. Then Y forms a cycle C' in G'. It is easy to see that C' satisfies Corollary 3.1. That is, G' is a quasi-wheel and C' is a rim of G'.

Suppose that G' is a quasi-wheel. Then G' has a cycle C' satisfying Corollary 3.1. Hence C' does not contain v_0. That is, for any $e_i = (v, w) \in E$, $\{(v, x_i), (x_i, w)\} \cap E(C') = \emptyset$ or $\{(v, x_i), (x_i, w)\} \subseteq E(C')$. Let $Z = \{e_i = (v, w) \in E \mid \{(v, x_i), (x_i, w)\} \subseteq E(C')\}$. Clearly, C' is a rim of G' and Z form a simple cycle C of G. Suppose that there is $y \in V(G) - V(C)$. Then $G' - E(C')$ has a simple cycle consisting of four vertices v_0, y, x_i, x_j, where $e_i, e_j \in E$ both of which are incident upon v_0 in G. (Fig.3) This contradicts Corollary 3.1, showing that C is a Hamilton cycle of G. □

Theorem 3.1 *RP and RIP for quasi-wheels are NP-complete.*

Proof. The problem of deciding whether or not a given cubic graph contains a Hamilton cycle is NP-complete [4]. The proof of Lemma 3.1 shown that RP and RIP for quasi-wheels are NP-hard.

We can verify that a simple cycle C satisfies Corollary 3.1 in polynomial time. Therefore both RP and RIP for quasi-wheels belong to NP. □

3.2 Super–wheels

Let $G = (V, E)$ be a super-wheel. If a super-wheel G is derived from a wheel $W_{n+1} = K_1 + C_n$ then $V = V(C_n) \cup \{v_0\}$, where v_0 is the hub of W_{n+1}. We also call v_0 the *hub* of G. C_n remains in G and it is called a *rim* of G. Put $V_R = V(C_n)$, and vertices in V_R are called *rim vertices*. $G - v_0$ contains a Hamilton cycle C such that $C = C_n$. We prove following corollary and lemma.

Proposition 3.2.1 *We can identify a hub of a given super-wheel $G = (V, E)$ in time $O(|E|)$.*

Proof. We can assume that $G = (V, E)$ is a super-wheel without multiple edges. Any hub is a vertex of degree $|V| - 1$. Suppose that G has at least two vertices v_0 and v of degree $|V| - 1$. Suppose that v_0 is a hub of G. There is a simple cycle C such that $V(C) = V - v_0$. And there is a simple cycle C' in $G - v$ such that $V(C') = V - \{v\}$ since v is adjacent to any vertex of $V - v_0$. G is a super-wheel with the hub v and the rim C', since v is adjacent to any vertex of $V - v$. Hence, any vertex of degree $|V| - 1$ of a super-wheel can be its hub.

Hence, we can identify a hub of a given super-wheel $G = (V, E)$ in time $O(|E|)$. □

4 Vertex covers of super–wheels

We first prove the next lemma. A graph is called Hamiltonian if and only if it has a Hamilton cycle.

Lemma 4.1 *VCP for Hamiltonian graphs is NP-complete.*

Proof. Let $G = (V, E)$ be a given connected graph, where $V = \{v(1), v(2), v(3), ..., v(n-1), v(n)\}$ and $n \geq 4$. Put $v(n+1) \equiv v(1)$, and let

$$V_h = V \cup \{x_i, y_i, z_i \mid i = 1, ..., n\},$$

$$E_h = E \cup E_T \cup E_J,$$

where x_i, y_i, z_i are new vertices.

$$E_T = \{(x_i, y_i), (y_i, z_i), (z_i, x_i) \mid i = 1, ..., n\},$$

$$E_J = \{(x_i, v(i)), (y_i, v(i+1)) \mid i = 1, ..., n\}.$$

We denote $J(v(i), v(i+1)) = \{x_i, y_i\}, i = 1, ..., n$.

Let $G_h = (V_h, E_h)$ be the graph constructed from G (Fig.4). Note that G_h contains a Hamilton cycle. Clearly VCP for Hamiltonian graphs belongs to NP. First suppose that $N \subseteq V$ is a vertex cover of G with $|N| \leq k$. Let

$$N_k = N \cup \{x_i, y_i \mid i = 1, ..., n\}.$$

Then N_k is a vertex cover of G_h and $|N_h| \leq |N| + 2n \leq k + 2n$. Conversely, suppose that $N_h \subseteq V_h$ be a vertex cover of G_h with $|N_h| \leq k + 2n$. Then there is a vertex cover $N'_h \subseteq V_h$ with $|N'_h| \leq |N_h|$ such that

$\{x_i, y_i\} \subseteq N'_h$ and $z_i \notin N'_h$ for $i = 1, ..., n$. Put $N' = N - \{x_i, y_i \mid i = 1, ..., n\}$. Then N' is a vertex cover of G with $|N'| \leq k$. Since VCP for connected graphs is NP-complete, so is VCP for Hamiltonian graphs. □

We obtain the following theorem.

Theorem 4.1 *VCP for super-wheels is NP-complete.*

Proof. Let G_h be the Hamiltonian graph constructed in the proof of Lemma 4.1. Let $G' = (V', E')$ be a graph defined by

$$V' = V_h \cup \{v_0\},$$

$$E' = E_h \cup \{(v_0, v) \mid v \in V_h\},$$

where v_0 is a new vertex. Clearly, G' is a super-wheel with a hub v_0. Suppose that N is a vertex cover of G with $|N| \leq k$. Define N_h as above, and let $N' = N_h \cup \{v_0\}$. Then N' is a vertex cover of G' with $|N'| = |N_h| + 1 \leq k + 2n + 1$. Conversely, suppose N' is a vertex cover of G' with $|N'| \leq k + 2n + 1$. The assumption $v_0 \notin N'$ means that $|N'| = 4n > |N'|$, a contradiction. Hence $v_0 \in N'$ and, therefore, there is a vertex cover N'' of G' with $|N''| \leq |N'|$ such that $\{x_i, y_i\} \subseteq N''$ and $z_i \notin N''$ for $i = 1, ..., n$. That is, $N'' - \{x_i, y_i \mid i = 1, ..., n\}$ is a vertex cover of G and its cardinality is no greater than n, and the theorem follows. □

5 Connected vertex covers of quasi–wheels

5.1 Standard connected vertex covers of quasi–wheels

Let $G = (V, E)$ be a quasi–wheel with a rim C_n (a simple cycle of n vertices) and $N \subseteq V$ be a connected vertex cover of G. For simplicity, let $V(C_n) = \{1, ..., n\}$, where numbering in clockwise. An edge $e = (v, w) \in E(C_n)$ is called a *B-edge* if and only if $v, w \in N$. We denote the set of B–edges of N by $B(N)$. Let $H = (V_H, E_H)$ and $I = (V_I, E_I)$ be the hub tree and the inner tree with respect to C_n, respectively. N is a *standard connected vertex cover* of G (Fig.5 (a)) if and only if $N = V_H \cup N_{C_n}$, where N_{C_n} is defined as in (i) or (ii) :

(i) $N_{C_n} = \{2i \mid 1 \leq i \leq n/2\}$

if n is even ;

(ii) $N_{C_n} = \{2i \mid 1 \leq i \leq (n-1)/2\} \cup \{n\}$

otherwise.

Lemma 5.1 *Let N be any connected vertex cover of a quasi-wheel G, and C_n be a rim of G. Put $|N \cap V(C_n)| - \lceil n/2 \rceil = x$. Then*

$$|B(N)| = \begin{cases} 2x & \text{if } n \text{ is even;} \\ 2x+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $|N \cap V(C_n)| \geq \lceil n/2 \rceil$ since N is a vertex cover of G. If $|N \cap V(C_n)| = \lceil n/2 \rceil$ then

$$|B(N)| = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

We obtain following equations.

$$(|E(C_n)| - |B(N)|) + 2|B(N)| = 2|N \cap V(C_n)|,$$

$$\begin{aligned} |B(N)| &= 2|N \cap V(C_n)| - n \\ &= 2(|N \cap V(C_n)| - \lceil n/2 \rceil) + \alpha \\ &= 2x + \alpha, \end{aligned}$$

where

$$\alpha = 2\lceil n/2 \rceil - n,$$

$$\alpha = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

□

Given a rim C_n of G, let $H = (V_H, E_H)$ be the hub tree of G with respect to C_n, and let $S(N) = \{v \mid v \in V_H \text{ and } v \notin N\}$. We can easily prove the following lemma by induction on $|S(N)|$.

Lemma 5.2 *For any connected vertex cover N of G, the induced subgraph $I[N]$ of I contains at least $2k+1$ connected components, where $k = |S(N)|$.*

We obtain the following theorem.

Theorem 5.1 *A standard connected vertex cover of a quasi-wheel G is a minimum connected vertex cover.*

Proof. Let N and N_S be any connected vertex cover and a standard connected vertex cover of G, respectively. We have $|N_S| = |V_H| + \lceil n/2 \rceil$, and by Lemma 5.2, $I[N]$ contains at least $2k+1$ connected components, where $k = |S(N)|$. Therefore $B(N)$ contains at least $2k$ edges, since N is a connected vertex cover of G.

First suppose that n is even. By Lemma 5.1,

$$x = |N \cap V(C_n)| - \lceil n/2 \rceil = |B(N)|/2 \geq k,$$

or

$$|N \cap V(C_n)| \geq k + \lceil n/2 \rceil.$$

since

$$|N \cap V_H| = |V_H| - k,$$

We have

$$|N| = |N \cap V(C_n)| + |N \cap V_H| \geq |V_H| + \lceil n/2 \rceil = |Ns|.$$

Next suppose that n is odd. Then, by Lemma 5.1,

$$2x + 1 = 2(|N \cap V(C_n)| - \lceil n/2 \rceil) + 1 \geq 2k.$$

Since x is a nonnegative integer,

$$|N \cap V(C_n)| - \lceil n/2 \rceil \geq k.$$

Therefore,

$$\begin{aligned} |N \cap V(C_n)| &\geq k + \lceil n/2 \rceil, \\ |N \cap V_H| &= |V_H| - k, \end{aligned}$$

and

$$|N| = |N \cap V(C_n)| + |N \cap V_H| \geq |V_H| + \lceil n/2 \rceil = |Ns|.$$

It follows that $|N| \geq |N_s|$, and the theorem follows. □

Corollary 5.1 *For any quasi-wheel G, there is a maximum nonseparating independent set consisting of only vertices of degree 3.*

Proof. Let Ns be a standard connected vertex cover of $G = (V, E)$. For any $v \in V - Ns$, we have $\delta_G(v) = 3$. Hence, by Theorem 5.1, $V - Ns$ is a maximum nonseparating independent set of G. □

5.2 Restricted nonseparating independent sets of graphs

It was shown by Ueno et al. [16] that NISP for graphs with maximum vertex degree at most 3 is polynomially solvable. We shall consider the following extension of their problem and show analogously that it is also polynomially solvable; namely, the problem of finding a maximum cardinality nonseparating independent set consisting of only degree 3 vertices in an arbitrary graph (NISP-3 for short). As was done in [16] we reduces this problem to the linear matroid matching problem (in fact, to the cographic matroid matching problem), for which some polynomial time algorithms have been already proposed [6, 12, 13]. The *cographic matroid matching problem* is a matroid matching problem where a 2-polymatroid is constructed from a cographic matroid.

Let $M(G) = (E, r)$ be a cographic matroid of $G = (V, E)$ and $V_3 = \{v \in V \mid \delta_G(v) = 3\}$. From $M(G)$ and V_3, we construct a 2-polymatroid $M_3^*(G) = (V_3^*, r^*)$ as follows. For each $v \in V_3$, let v^* be any one pair of edges among those incident upon v in G. Let $X^* = \{v^* \mid v \in X\}$ for $X \subseteq V_3$. We denote $V_3^* = \{v^* \mid v \in V_3\}$ and $E_{X^*} = \{e, e' \in E \mid v^* = \{e, e'\} \in X^*\}$ for any $X \subseteq V_3$. Define the rank function r^* by $r^*(Y^*) = r(E_{Y^*})$ for any $Y^* \subseteq V_3^*$. Then it is easy to see that $M_3^*(G) = (V_3^*, r^*)$ is a 2-polymatroid. Fig.7 show an example of $M_3^*(G)$. If G is the graph of Fig.7, then $V_3 = \{v_1, v_2, v_3, v_4\}$ and $V_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}$. For a maximum independent set of G, we have $X = \{v_1, v_4\}$ as a maximum independent set of G consisting of only vertices of degree 3, and a maximum matching $X^* = \{v_1^*, v_4^*\}$ of $M_3^*(G)$.

Lemma 5.3 *$X \subseteq V_3$ is a nonseparating independent set of G if and only if E_{X^*} is a matching of $M_3^*(G)$.*

Proof. We can assume that G is connected. Note that X^* is a matching if and only if $u^* \cap v^* = \emptyset$ for any $u^*, v^* \in X^*$ and that G remains connected even after E_{X^*} is removed from G.

First suppose that $X \subseteq V_3$ is a nonseparating independent set of G. Since X is an independent set, every edge incident to $v \in X$ connects v and a vertex in a connected subgraph $G[V - X]$. Since $\delta_G(v) = 3 = |v^*| + 1$, $(V, E - E_{X^*})$ is connected and contains a spanning tree of G, meaning that E_{X^*} is an independent set of $M(G)$. Clearly $u^* \cap v^* = \emptyset$ for any $u^*, v^* \in X^*$, and $r^*(X^*) = 2|X^*|$. That is, X^* is a matching in $M_3^*(G)$.

Next suppose that $X \subseteq V_3$ is not a nonseparating independent set of G. Suppose that G has two adjacent vertices v_1 and v_2 in X. Then there are three situations shown in Fig.8 (a)-(c) : either $v_1^* \cap v_2^* \neq \emptyset$ (Fig.8 (a)) or removal of $v_1^* \cup v_2^*$ from G result in a disconnected graph (Fig.8 (b),(c)). In either case, $r^*(X^*) < 2|X^*|$ and, therefore, X^* is not a matching. Now we assume that X is an independent set of G such that $G[V - X]$ contains two connected components C_1 and C_2. Put $G' = (V, E - E_{X^*})$. Since $\delta_{G'}(v) = 1$ for any $v \in X$, G' has no edges connecting C_1 and C_2. Hence E_{X^*} is not an independent set of $M(G)$, implying that X^* is not a matching of $M_3^*(G)$. □

Thus, NISP-3 can be reduced to solving the cographic matroid matching problem for $M_3^*(G)$ (Fig.6). Gabow and Stallmann's algorithm solves the graphic and cographic matroid matching problems in time $O(|V|^2|E|)$ [6]. Hence we obtain the following theorem.

Theorem 5.2 *NISP-3 can be solved in time* $O(|V|^2|E|)$.

Hence, from Theorem 5.1, We have the following corollary, since any quasi-wheel has $O(|V|)$ edges.

Corollary 5.2 *CVCP for quasi-wheels can be solved in time* $O(|V|^3)$.

6 Acknowledgments

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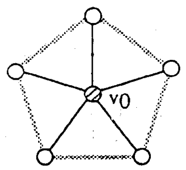
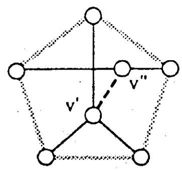
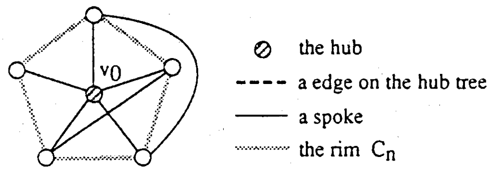


(a) A wheel W_6.



(b) A quasi-wheel.



(c) A super-wheel.

Fig.1 Examples of a wheel, a quasi-wheel and a super-wheel.

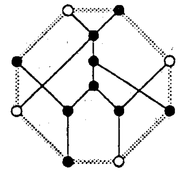
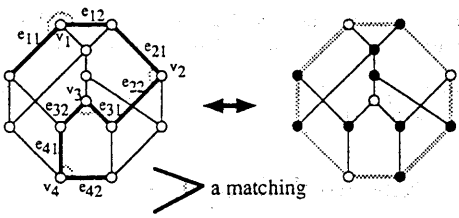


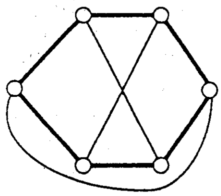
Fig.5 A quasi-wheel and a standard connected vertex cover (denoted as black vertices).



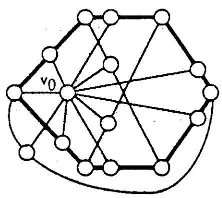
(a) A matching $\{v_1^*, v_2^*, v_3^*, v_4^*\}$ (denoted by four pairs of bold edges) of a 2-polymatroid $M_3^*(V_3^*, r^*)$ constructed from a quasi-wheel G.

(b) The corresponding mimimun connected vertex cover (denoted as black vertices).

Fig.6 Connected vertex covers for a quasi-wheel.



(a) A graph G=(V,E) which contains a Hamilton cycle.



(b) The G'=(V',E') constructed from G by *MQW(G)*.

Fig.2 An example of a cubic graph G and another graph G' constructed from G by *MQW(G)*.

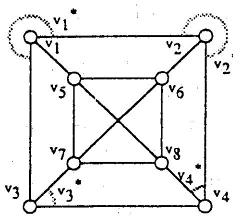
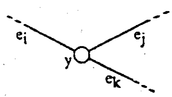
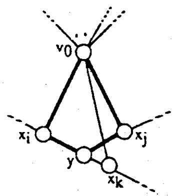


Fig.7 An example of a graph G from which a cographic matroid M(G) and a 2-polymatroid $M_3^*(G)$ are constructed.

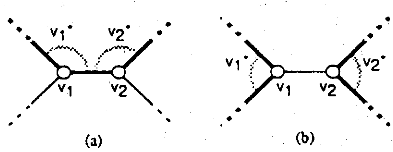


(a) The vertex y and the three edges e_i, e_j, e_k incident upon y of G.



(b) The two vertices v_0 and y on a simple cycle (denoted as bold lines) of G'.

Fig.3 An vertex $y \in V(G)-V(C)$ of G' and the vertex y of G.



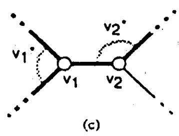
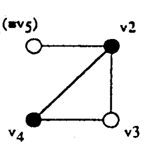
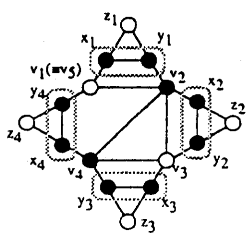


Fig.8 $v_1^*, v_2^* \in V_3^*$ corresponding to adjacent vertices $v_1, v_2 \in X$ of G.



(a) A graph G without Hamilton cycles, and a minimum vertex cover (denoted as black vertices).



(b) A Hamiltonian graph G_h constructed from G, and a minimum vetex cover (denoted as black vertices).

Fig.4 Construction of a Hamiltonian graph G_h from a given connected graph G.

Table.1 A summary of results.

	QUASI-WHEELS	SUPER-WHEELS
RP	NP-complete *(this paper)*	NP-complete [19]
planar VCP	$O(\|V\|)$ [2]	$O(\|E\|)$ [19]
VCP	Open	NP-complete *(this paper)*
CVCP	$O(\|V\|^3)$ *(this paper)*	NP-complete *(this paper)* and [19]