平面上の点集合の平衡分割問題

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Abstract

平面上に mq個の白点集合 Sと nq個の赤点集合 Tがあり、$S \cup T$のどの 3 点も同一直線上にはないものとする。すると、$S \cup T$を下記の 2 つの条件を満たすような q 個の集合 $P_1, P_2, \cdots, P_q$に分割できるという予想を与え、これが $n = 1$ のとき成り立つことを示したが、今回は $n = 2$ のときにもこの予想が成り立つことを証明する。またその証明から、所望の分割を得る $O(N^2 \log N)$ のアルゴリズムもえられる、ただし $N = mq + nq$である。

(1) すべてのの $1 \leq i < j \leq q$に対して、$\mathrm{conv}\,(P_i) \cap \mathrm{conv}\,(P_j) = \phi$である。

(2) すべての $1 \leq i \leq q$に対して、$|P_i \cap S| = n$ and $|P_i \cap T| = m$ がなりたつ。

Balanced partitions of two sets of points in the plane

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Abstract

We prove the following theorem: Let $n = 1$, $m \geq 2$ and $q \geq 1$ be integers and let S and T be two disjoint sets of points in the plane such that no three points of $S \cup T$ are on the same line, $|S| = nq$ and $|T| = mq$. Then $S \cup T$ can be partitioned into q disjoint subsets $P_1, P_2, \cdots, P_q$ satisfying the following two conditions: (i) $\mathrm{conv}\,(P_i) \cap \mathrm{conv}\,(P_j) = \phi$ for all $1 \leq i < j \leq q$; and (ii) $|P_i \cap S| = n$ and $|P_i \cap T| = m$ for all $1 \leq i \leq q$.

We can obtain an $O(N \log N)$ time algorithm for finding the above desired partition by our proof, where $N = mq + nq$. We also proved that the above theorem holds for $n = 1$, and give a conjecture which says that the above theorem holds for all integer $n \geq 3$. We don't give a complete proof of this theorem because of lack of pages, and a complete proof can be seen in [3].

Key Works: balanced partition, point set, plane,

1 Introduction

For a set P of points in the plane, we denote by $\mathrm{conv}(P)$ the *convex hull* of P, which is the smallest convex set containing P. In [2], we proved the following theorem.

Theorem A *Let m be a positive integer, and let S_1, S_2 and T be three disjoint sets of points in the plane such that no three points of $S_1 \cup S_2 \cup T$ are collinear (i.e., no three points of it are on the same line) and $|T| = (m-1)|S_1| + m|S_2|$. Put $q = |S_1 \cup S_2|$. Then $S_1 \cup S_2 \cup T$ can be partitioned into q subsets $P_1, P_2, \cdots, P_q$ which satisfy the following three conditions: (i) $\mathrm{conv}(P_i) \cap \mathrm{conv}(P_j) = \phi$ for all $1 \le i < j \le q$; (ii) $|P_i \cap (S_1 \cup S_2)| = 1$ for all $1 \le i \le q$; and (iii) $|P_i \cap T| = m-1$ if $|P_i \cap S_1| = 1$, and $|P_i \cap T| = m$ if $|P_i \cap S_2| = 1$.*

In view of theorem A with $S_1 = \emptyset$, we gave the following conjecture in [2].

Conjecture B *Let $m \ge 2$, $n \ge 2$ and q be positive integers. Let S and T be two disjoint sets of points in the plane such that no three points of $S \cup T$ are collinear, $|S| = nq$ and $|T| = mq$. Then $S \cup T$ can be partitioned into q subsets $P_1, P_2, \cdots, P_q$ satsifying the following two conditions: (i) $\mathrm{conv}(P_i) \cap \mathrm{conv}(P_j) = \phi$ for all $1 \le i < j \le q$; and (ii) $|P_i \cap S| = n$ and $|P_i \cap T| = m$ for all $1 \le i \le q$.*

The above conjecture is true when $q = 2$ because the conjecture with $q = 2$ is equivalent to well-known discrete Ham Sandwich Theorem on the plane ([1] p.212). In this paper we show that the conjecture is true in the case of $n = 2$.

Theorem 1 *Let $m \ge 2$ and $q \ge 1$ be integers and let S and T be two disjoint sets of points in the plane such that no three points of $S \cup T$ are collinear, $|S| = 2q$ and $|T| = mq$. Then $S \cup T$ can be partitioned into q disjoint subsets $P_1, P_2, \cdots, P_q$ satisfying the following two conditions:*
(i) $\mathrm{conv}(P_i) \cap \mathrm{conv}(P_j) = \phi$ for all $1 \le i < j \le q$; and
(ii) $|P_i \cap S| = 2$ and $|P_i \cap T| = m$ for all $1 \le i \le q$.

Let us note that from the proof of the above theorem, we can obtain a polynomial time algorithm for finding such a partition given in the theorem.

2 Proof of Theorem 1

In this paper, we deal with only *directed lines* in order to define the right side of a line and the left side of it. Thus a *line* means a directed line. A line l dissects the plane into three pieces: l and two open half-planes $R(l)$ and $L(l)$, where $R(l)$ and $L(l)$ denote the *open half-planes* which are on the right side and on the left side of l, respectively (see Figure 1). Let r_1 and r_2 be two rays emanating from the same point p. Then $r_1 \cup r_2$ dissects the plane into three pieces: $r_1 \cup r_2$ and two open regions $R(r_1) \cap L(r_2)$ and $L(r_1) \cap R(r_2)$, where $R(r_1) \cap L(r_2)$ denotes the open region which is on the right side of r_1 and on the left side of r_2, and $L(r_1) \cap R(r_2)$ denotes the other open region (see Figure 1). Namely, $R(r_1) \cap L(r_2)$ denotes the open region that is sweepted by the ray being rotated clockwise around p from r_1 to r_2. If the internal angle $\angle r_1 p r_2 = \angle r_1 r_2$ of $R(r_1) \cap L(r_2)$ is less than π, then we call $R(r_1) \cap L(r_2)$ the *wedge* defined by r_1 and r_2, and denote it by $\mathrm{wdg}(r_1 p r_2)$ or $\mathrm{wdg}(r_2 p r_1)$. Let us note that $p \notin R(r_1) \cap L(r_2)$ and $p \notin \mathrm{wdg}(r_1 p r_2)$ since they are open regions and do not contain their boundaries.

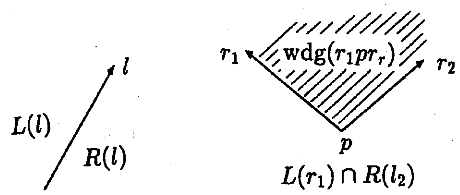

Figure 1: Open regions $R(l)$, $L(l)$ and $L(r_1) \cap R(r_2)$ and a wedge $\mathrm{wdg}(r_1 p r_2) = R(r_1) \cap L(r_2)$.

Let l_i be a line with suffix i, and p be a point on l_i. Then we denote by l_i^* the line which is obtained from l_i by reversing its direction. Moreover, we define the two rays r_i and r_i^* lying on the line l_i and having the same starting point p such that r_i has the same direction as l_i and r_i^* has the opposite direction of l_i. In particular, $l_i = r_i \cup r_i^*$ (see Figure 2). Conversely, given a ray r_i, we can similarly define the ray r_i^*, whose direction is opposite to r_i, and the two lines l_i and l_i^*.

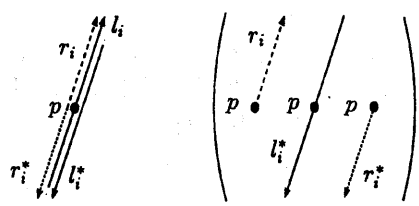

Figure 2: Lines l_i and l_i^*, and rays r_i and r_i^*.

For a region W in the plane, we define the integer-valued function f of W with respect to S and T by

$$f(W) := m|S \cap W| - 2|T \cap W|,$$

where S and T are the two disjoint sets of points in the plane given in Theorem 1. Hereafter f always denotes this function. A region W is said to be *balanced* if $f(W) = 0$. For example, $\mathrm{conv}(S \cup T)$ and $\mathrm{conv}(P_i)$ are balanced, where P_i is a subset of $S \cup T$ given in Theorem 1. In order to prove Theorem 1, we need the following lemma.

Lemma 2 *Let S and T be two disjoint sets of points in the plane given in Theorem 1. If there exist two lines l_1 and l_2 such that $|R(l_1) \cap S| = |R(l_2) \cap S|$ and $|R(l_1) \cap T| < |R(l_2) \cap T|$, then for every integer i, $|R(l_1) \cap T| \leq i \leq |R(l_2) \cap T|$, there exists a line l_3 such that $|R(l_3) \cap S| = |R(l_1) \cap S|$ and $|R(l_3) \cap T| = i$.*

Proof We first assume that $R(l_1) \cap S = R(l_2) \cap S$ (see Figure 3). Then we can continuously move a line l from l_1 to l_2 in such a way that each line l passes through at most one point of T but no point of S. Then $R(l) \cap S = R(l_1) \cap S$, and the number $|R(l) \cap T|$ changes ± 1 when l hits or passes a point of T. Therefore we can find the desired line l_3.

We next assume $R(l_1) \cap S \neq R(l_2) \cap S$. Consider two convex hulls $\text{conv}(S \cap R(l_1))$ and $\text{conv}(S \setminus R(l_1))$. Then we can find two vertices $x \in \text{conv}(S \cap R(l_1))$ and $y \in \text{conv}(S \setminus R(l_1))$ such that a line l_4 passing through x and y satisfies $\angle l_2 l_4 < \angle l_2 l_1$ (see Figure 3). Let l'_4 denote a line very close to l_4 such that $R(l'_4)$ contains x but not y, and l''_4 denote a line very close to l_4 such that $R(l''_4)$ contains y but not x. We may assume that no point of $(S \cup T) \setminus \{x, y\}$ lies between l'_4 and l''_4. We can continuously move a line l from l_1 to l'_4 in such a way that l passes no point of S and the number $|R(l) \cap T|$ changes ± 1. Moreover it follows that $R(l''_4) \cap T = R(l'_4) \cap T$ and $R(l''_4) \cap S = ((R(l'_4) \cap S) \setminus \{x\}) \cup \{y\}$.

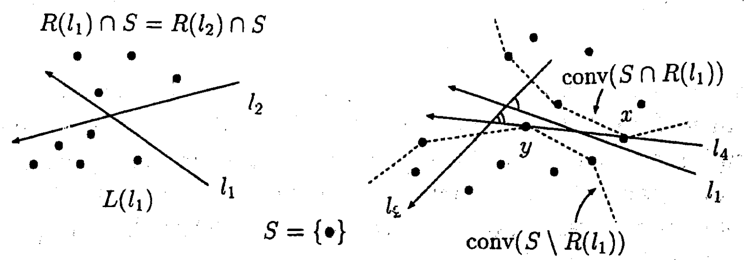

Figure 3: Two cases of the proof of Lemma 3.

Since $|R(l_1) \cap S| = |R(l_2) \cap S|$, by repeating this procedure we can obtain a line l_5 possessing the property that $R(l_5) \cap S = R(l_2) \cap S$ and that for every integer j between $|R(l_1) \cap T|$ and $|R(l_5) \cap T|$, there exists a line l_6 such that $|R(l_6) \cap S| = |R(l_1) \cap S|$ and $|R(l_6) \cap T| = j$. Since l_2 and l_5 satisfy the condition of the previous case, the lemma is proved. □

Proof of Theorem 1 We now prove Theorem 1 by induction on $|S|$. Unless otherwise stated, except when it moves, we always consider lines that pass through no points of $S \cup T$. We begin with the following Claim.

Claim 3 *For every integer i, $0 \leq i \leq q-1$, there exist a line l that passes through two distinct points of S and satisfies $|R(l) \cap S| = i$.*

Proof Let i be an integer such that $0 \leq i \leq q-1$. Let x be a vertex of $\text{conv}(S)$ which lies on the bottom of $\text{conv}(S)$, and let l_1 be the line that passes through x and the rightmost edge incident with x and goes upward. Then $|R(l_1) \cap S| = 0$ and $|L(l_1) \cap S| = 2q - 2$. By a suitable counterclockwise rotation of l_1 around x, we can find a line l which passes through x and one more point of S and satisfies $|R(l) \cap S| = i$ since no three points of S are collinear. □

Claim 4 *If q is even, then the theorem holds. Thus we may assume that q is odd. In particular, we can put $q = 2k + 1$, $k \geq 1$.*

Proof Suppose that q is even. Then by Ham Sandwich Theorem, there exists a line l such that $|R(l) \cap S| = |L(l) \cap S| = |S|/2$ and $|R(l) \cap T| = |L(l) \cap T| = |T|/2$, which imply that both $R(l)$ and $L(l)$ are balanced regions. By the inductive hypotheses on $R(l)$ and on $L(l)$, we can obtain the required partition of $S \cup T$. □

By the same argument in the above proof, if there exists a line l such that $f(R(l)) = 0$ and $0 < |R(l) \cap S| < |S|$, then both $R(l)$ and $L(l)$ are balanced, and thus we can obtain the desired partition of $S \cup T$ by the inductive hypotheses on $R(l)$ and on $L(l)$. Therefore we may assume that

$$f(R(l)) \neq 0 \quad \text{for every line } l \text{ with } 0 < |R(l) \cap S| < |S|. \tag{1}$$

We put

$$q = 2k + 1, \quad |S| = 4k + 2 \quad \text{and} \quad |T| = m(2k + 1).$$

Claim 5 *We may assume that for every line l for which $2 \leq |R(l) \cap S| = 2j \leq 2k$, we have $|R(l) \cap T| > mj$, in particular, $f(R(l)) < 0$ because otherwise the theorem holds.*

Proof If there exist two lines l_1 and l_2 such that $2 \leq |R(l_1) \cap S| = |R(l_2) \cap S| = 2j \leq 2k$ and $|R(l_1) \cap T| < mj < |R(l_2) \cap T|$, then by Lemma 2 we can find a line l_3 for which $|R(l_3) \cap S| = 2j$ and $|R(l_3) \cap T| = mj$, which contradicts (1). Therefore the existence of a line l such that $|R(l) \cap S| = 2j$ and $|R(l) \cap T| > mj$ (or $< mj$) is equivalent to the assertion that for every line l with $|R(l) \cap S| = 2j$, we have $|R(l) \cap T| > mj$ (or $< mj$). Thus it is enough to prove that for every $1 \leq j \leq k$, there exists a line l for which $|R(l) \cap S| = 2j$ and $|R(l) \cap T| > mj$.

We prove this by induction on j from $j = k$ to $j = 1$. By Claim 3, there exists a line l_4 such that l_4 passes through two points of S and $|R(l_4) \cap S| = 2k$, which implies $|L(l_4) \cap S| = 2k$. Since l_4 does not pass through any point of T and by the equality $|R(l_4) \cap S| = |L(l_4) \cap S|$, we may assume that $|R(l_4) \cap T| \geq |T|/2 > mk$, and thus the statement holds when $j = k$.

Suppose that the claim holds for $j + 1$ but does not for j, that is, assume that there exists a line l_1 such that $|R(l_1) \cap S| = 2j$ and $|R(l_1) \cap T| < mj$. Then for every line l_2 with $|R(l_2) \cap S| = 2j$, we have $|R(l_2) \cap T| < mj$. By Claim 3, there exists a line l_3 such that l_3 passes through two points, say x and y, of S and $|R(l_3) \cap S| = 2j$. Since no three points of $S \cup T$ are collinear, we can move l_3 leftward very slightly so that the resulting line l_4 satisfies that $R(l_4) \cap S = (R(l_3) \cap S) \cup \{x, y\}$ and $R(l_4) \cap T = R(l_3) \cap T$. Thus $|R(l_4) \cap S| = 2(j + 1)$ and $|R(l_4) \cap T| < mj < m(j + 1)$, which contradicts the fact that the claim holds for $j + 1$. Consequently the claim is proved. □

Let l_1 be a line which passes through two points of S, say x and y, and satisfies $R(l_1) \cap S = \emptyset$. By a suitable rotation of the plane, we may assume that l_1 lies horizontal and goes from right to left (see Figure 4). By considering a line l_1' lying very little below l_1, we have $|R(l_1) \cap T| = |R(l_1') \cap T| > m$ by $|R(l_1') \cap S| = 2$ and Claim 5. As it is easily seen, there exists a point z in $L(l_1) \cap S$ such that letting l_2 be the line passing through x and z, $|R(l_2) \cap S| = |L(l_2) \cap S| = 2k$. We assume that l_2 is directed from x to z, which means that l_2 goes downward (see Figure 4). Then by Claim 5, we have $|R(l_2) \cap T| > mk$ and $|L(l_2) \cap T| > mk$. Let

$$a := |R(l_2) \cap T| - mk \quad \text{and} \quad b := |L(l_2) \cap T| - mk.$$

Since $|R(l_2) \cap T| + |L(l_2) \cap T| = |T| = m(2k+1)$, we have

$$a > 0, \;\; b > 0, \;\; a + b = m, \;\; f(R(l_2)) = -2a \;\; \text{and} \;\; f(L(l_2)) = -2b. \qquad (2)$$

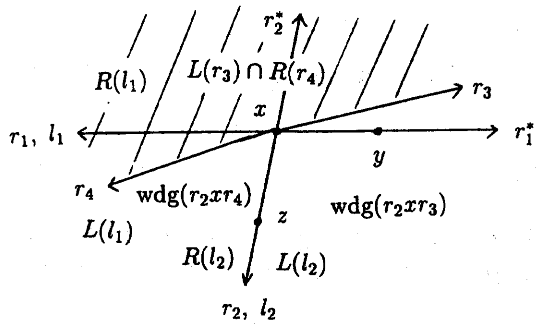

Figure 4: Lines l_1, l_2 and rays r_3 and r_4.

Hereafter we consider rays emanating from x, and so, unless otherwise stated, a *ray* means such a ray.

Claim 6 *Let r_2 denote the ray lying on l_2. We may assume that there exists two rays r_3 in $L(l_2)$ and r_4 in $R(l_2)$ such that both wedges* $\text{wdg}(r_2xr_3)$ *and* $\text{wdg}(r_2xr_4)$ *are balanced and $L(r_3) \cap R(r_4)$ contains exactly m points of T but no point of S. Of course, r_3 must lie in* $\text{wdg}(r_1^*xr_2^*)$ *and it may happend that r_4 lies below r_1 (see* Figure 4*).*

Proof We shall prove only the existence of r_3 which satisfies that $\text{wdg}(r_2xr_3)$ is balanced and $\text{wdg}(r_3xr_2^*)$ contains exactly a points of T but no point of S because we can show the existence of r_4 satisfying the similar conditions by the same argument, and the existence of these two rays implies Claim 6 by (2).

Recall that unless otherwise stated, we consider lines and rays which pass through no point of $S \cup T$. Note that an empty wedge $\text{wdg}(r_2xr_2)$ has no point of $S \cup T$ and is clearly balanced, that is, $f(\text{wdg}(r_2xr_2)) = 0$. We choose a ray r_3 in $L(l_2)$ so that

(a) $|\text{wdg}(r_2xr_3) \cap S|$ is even,

(b) $f(\text{wdg}(r_2xr_3)) \geq 0$, and

(c) $|\text{wdg}(r_2xr_3) \cap (S \cup T)|$ is maximum subject to (a) and (b).

We begin with a observation that the value $f(W)$ of a region W is always even when W contains even number of points in S, and that $|L(l_2) \cap S| = 2k$. We consider two cases.

Case 1 $\text{wdg}(r_3xr_2^*)$ contains at most $m-1$ points of $S \cup T$.

Since $|\text{wdg}(r_3xr_2^*) \cap S|$ is even, if $\text{wdg}(r_3xr_2^*)$ contains at least one point of S, then it contains at least two points of S, and so $f(\text{wdg}(r_3xr_2^*)) \geq 2m - 2(m-3) > 0$. Hence $f(L(l_2)) = f(\text{wdg}(r_2xr_3)) + f(\text{wdg}(r_3xr_2^*)) > 0$, contradicting the fact that $f(L(l_2)) <$

0. Therefore $\text{wdg}(r_3xr_2^*)$ contains no point in S, and hence $f(\text{wdg}(r_2xr_3)) = 0$ by the maximality (c) of $|\text{wdg}(r_2xr_3) \cap (S \cup T)|$. Consequently $\text{wdg}(r_2xr_3))$ is balanced, and moreover $\text{wdg}(r_3xr_2^*)$ contains exactly a points of T since $f(L(l_2)) = f(\text{wdg}(r_2xr_3)) + f(\text{wdg}(r_3xr_2^*)) = 0 - 2|\text{wdg}(r_3xr_2^*) \cap T| = -2a$.

Case 2 $\text{wdg}(r_3xr_2^*)$ contains at least m points of $S \cup T$.

In this case we shall prove that the theorem holds. Let r_5 be a ray in $\text{wdg}(r_3xr_2^*)$ such that $\text{wdg}(r_3xr_5)$ contains exactly m points of $S \cup T$ (see Figure 5). We distinguish two cases.

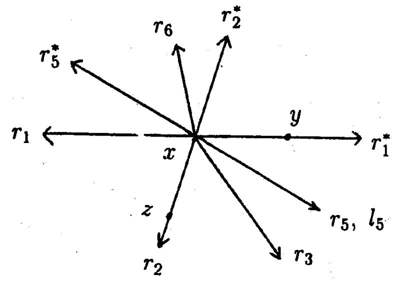

Figure 5: Lines l_5, l_6 and rays r_5 and r_6.

Subcase 2.1 $\text{wdg}(r_3xr_5)$ contains no point of S.

We omit the proof of this case.

Subcase 2.2 $\text{wdg}(r_3xr_5)$ contains at least one point of S.

We omit the proof of this case. $\square$

We turn our attention to the proof of the theorem. Choose rays r_3 and r_4 according to Claim 6. Then it is obvious that $(L(r_3) \cap R(r_4)) \cup \{x, z\}$ is balanced. In order to deal with a set of points of $S \cup T$ contained in $(L(r_3) \cap R(r_4)) \cup \{x, z\}$, we consider a point x' on l_2 and two rays r_3' and r_4' whose starting points are x'. First let x', r_3' and r_4' and are x, r_3 and r_4, respectively. Then we continuously move x' on l_2 toward the point z together with the rays r_3' and r_4' in such a way that both rays r_3' and r_4' pass through no point of $S \cup T$, and we stop moving if x' reaches z or at least one of rays r_3' and r_4' meets a point of $S \cup T$ and x' cannot move more down. we consider two cases.

Case 1 x' arrives at z (see Figure 6).

We omit the proof of this case.

Case 2 x' cannot arrive at z.

In this case, x' stops above z since at least one of r_3' and r_4' meets a point of $S \cup T$. Without loss of generality, we may assume that r_3' meets a point of $S \cup T$. Then r_3' is tangent to $\text{conv}(\text{wdg}(r_3xr_2^*) \cap T)$ at a vertex a and to $\text{conv}(\text{wdg}(r_2xr_3) \cap (S \cup T))$ at a vertex b (see Figure 7).

Suppose that b is a point of S. Then $(L(r_3') \cap R(r_4')) \cup \{a, b\}$, $\text{wdg}(r_2xr_3') \cup \{z\}$ and $\text{wdg}(r_2xr_4')$ are balanced regions, where $L(r_3') \cap R(r_4') \ni x$ and $\text{wdg}(r_2xr_3') \cup \{z\} \not\ni b$.

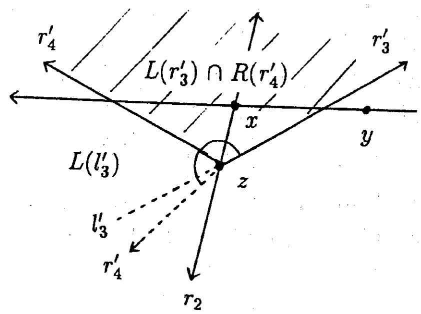

Figure 6: x' arrives at z.

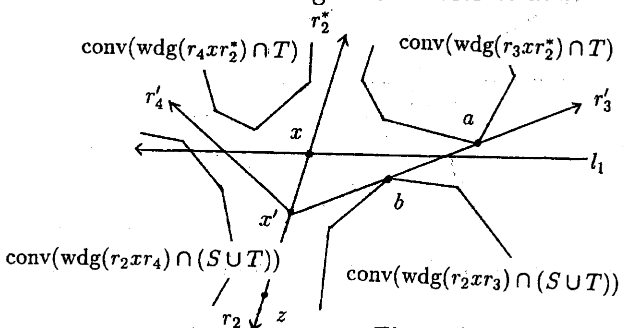

Figure 7:

Moreover, we can show that $L(r'_3) \cap R(r'_4)$ is convex, (i.e., $\angle r'_3x'r'_4 \leq \pi$) by applying Claim 5 to a line l'_3 as in Case 1. Hence we can get the desired partition of $S \cup T$ by inductive hypotheses. Therefore we may assume that b is a point of T.

It is celar that a is a point T. Then we move x' very little down and define a new r'_3 to be a ray that is very close to an old r'_3 and passes through below b and above a. Then a new region $L(r'_3) \cap R(r'_4)$ contains exactly m points of T and both new regions, wdg($r_2x'r'_3$) and wdg($r_2x'r'_4$) are balanced. Thus we can move x' toward to z again. By repeating this procedure, we can get the desired partition. Consequently the proof is complete. □

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