単調線形システムにおけるすべての極小な整数解について

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あらまし 本論文では，単調線形システムのすべての極小な整数解を列挙する問題について考える．我々は，まず，r 個の線形不等式から成るどんな n 変数単調線形システムにおいても，極大な実行不可能整数解の数が極小な実行可能整数解の数の高々rn 倍であることを示す．このことにより，単調線形システムにおける極小整数解列挙問題が有名なハイパーグラフ双対化問題の自然な拡張に多項式時間還元可能であることが導かれる．ここで，ハイパーグラフ双対化問題の拡張とは，ハイパーグラフの双対ペアを整数ベクトルの双対族に置き換えるという意味での拡張である．我々はこの拡張された双対化問題に対する擬多項式時間アルゴリズムを構成する．これらの結果は，特に，単調線形システムにおけるすべての極小な整数解が逐次擬多項式時間で列挙可能であることを意味する．

和文キーワード: 整数計画，逐次列挙，ハイパーグラフの双対化，ヒルベルト基底，単調不等式，正則離散関数

All Minimal Integer Solutions for a Monotone System of Linear Inequalities

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abstract We consider the problem of enumerating all minimal integer solutions of a monotone system of linear inequalities. We first show that for any monotone system of r linear inequalities in n variables, the number of maximal infeasible integer vectors is at most rn times the number of minimal integer solutions to the system. This bound leads to a polynomial-time reduction of the enumeration problem to a natural generalization of the well-known dualization problem for hypergraphs, in which dual pairs of hypergraphs are replaced by dual collections of integer vectors in a box. We provide a quasi-polynomial algorithm for the latter dualization problem. These results imply, in particular, that the problem of incrementally generating all minimal integer solutions to a monotone system of linear inequalities can be done in quasi-polynomial time.

英文 key words: Integer programming, incremental generation, hypergraph dualization, Hilbert basis, monotone inequalities, regular discrete functions.

1 Introduction

Consider a system of r linear inequalities in n integer variables

$$Ax \geq b, \quad x \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}, \tag{1}$$

where A is a rational $r \times n$-matrix, b is a rational r-vector, and c is a non-negative integral n-vector some or all of whose components may be infinite. We assume that (1) is a monotone system of inequalities: if $x \in \mathcal{C}$ satisfies (1) then any vector $y \in \mathcal{C}$ such that $y \geq x$ is also feasible. For instance, (1) is monotone if the matrix A is non-negative. Let us denote by $\mathcal{F} = \mathcal{F}_{A,b,c}$ the set of all minimal feasible integral vectors for (1), i.e., $y \in \mathcal{F}$ if there is no solution x of (1) such that $x \leq y,\ x \neq y$. Then, we have

$$\{x \in \mathcal{C} \mid Ax \geq b\} = \bigcup_{y \in \mathcal{F}} \{\, x \in \mathcal{C} \mid x \geq y\}.$$

In this paper, we are concerned with the problem of incrementally generating $\mathcal{F}$:

$GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$: *Given a monotone system (1) and a set $\mathcal{X} \subseteq \mathcal{F}_{A,b,c}$ of minimal feasible vectors for (1), either find a new minimal integral vector satisfying (1), or show that $\mathcal{X} = \mathcal{F}_{A,b,c}$.*

The entire set $\mathcal{F} = \mathcal{F}_{A,b,c}$ can be constructed by initializing $\mathcal{X} = \emptyset$ and iteratively solving the above problem $|\mathcal{F}| + 1$ times.

If A is a binary matrix, and b, c are vectors of all ones, then $\mathcal{F}$ is the set of (characteristic vectors of) all minimal transversals to the hypergraph defined by the rows of A. In this case, problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ turns into the well-known *hypergraph dualization problem*: incrementally enumerate all the minimal transversals (equivalently, all the maximal independent sets) for a given hypergraph The case where A is binary, c is the vector of all ones and b is arbitrary, is equivalent with the generation of so-called *multiple* transversals [5]. If A is integral and $c = +\infty$, the generation of $\mathcal{F}$ can also be regarded as the computation of the Hilbert basis for the ideal $\{x \in \mathbb{Z}^n \mid Ax \geq b,\ x \geq 0\}$. One more application of problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ is related to stochastic programming, more precisely to the generation of minimal *p-efficient* points for a given probability distribution of a discrete random variable $\xi \in \mathbb{Z}^n$. An integer vector $y \in \mathbb{Z}^n$ is called p-efficient, if $Prob(\xi \leq y) \geq p$. It is known that for every probability distribution and every $p > 0$ there are finitely many minimal p-efficient points and furthermore, for r-concave probability distributions these points are exactly the minimal integral points of a corresponding convex monotone system (see, e.g., [16]).

Let $J^* = \{j \mid c_j = \infty\}$ and $J_* = \{1, \ldots, n\} \setminus J^*$ be, respectively, the sets of unbounded and bounded integer variables in (1). Consider an arbitrary vector $x = (x_1, \ldots, x_n) \in \mathcal{F}_{A,b,c}$ such that $x_j > 0$ for some $j \in J^*$. Then it is easy to see that

$$x_j \leq \max_{i: a_{ij} > 0} \left\lceil \frac{b_i - \sum_{k \in J_*} \min\{0, a_{ik}\} c_k}{a_{ij}} \right\rceil < +\infty. \tag{2}$$

[Indeed, let x' be the vector obtained by decreasing the j^{th} component of x by 1, then $x' \in \mathcal{C}$ is infeasible for (1) and hence $b_i - a_{ij} \leq a_i x - a_{ij} = a_i x' < b_i$ for some $i \in \{1, \ldots, r\}$, implying $a_{ij} > 0$. Thus $a_i x < b_i + a_{ij}$. Since $a_{ij} x_j + \sum_{k \in J_*} \min\{0, a_{ik}\} c_k \leq a_{ij} x_j + \sum_{k \in J_*} \min\{0, a_{ik}\} x_k \leq a_{ij} x_j + \sum_{k \in J_*} a_{ik} x_k \leq a_i x$, where the last inequality follows from the non-negativity of the restriction of A on J^*, we have $a_{ij} x_j + \sum_{k \in J_*} \min\{0, a_{ik}\} c_k < b_i + a_{ij}$. This implies (2).]

Since the bounds of (2) are easy to compute, and since appending these bounds to (1) does not change the set $\mathcal{F}_{A,b,c}$, we shall assume in the sequel that all components of the non-negative vector c are finite, even though this may not be the case for the original system. This assumption does not entail any loss of generality and allows us to consider $\mathcal{F}_{A,b,c}$ as a system of integral vectors in a finite box. We shall also assume that the input monotone system (1) is feasible, i.e., $\mathcal{F}_{A,b,c} \neq \emptyset$. For a finite and non-negative c this is equivalent to $Ac \geq b$. In addition, we say that system (1) is non-trivial if $\mathcal{F}_{A,b,c} \neq \mathcal{C}$, i.e., $0 \notin \mathcal{F}_{A,b,c}$.

Let $\mathcal{A}$ be a collection of integral vectors in $\mathcal{C}$ and let $\mathcal{A}^+ = \{x \in \mathcal{C} \mid x \geq a \text{ for some } a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{C} \mid x \leq a \text{ for some } a \in \mathcal{A}\}$ denote the ideal and filter generated by $\mathcal{A}$. Any element in $\mathcal{C} \setminus \mathcal{A}^+$ is called *independent of* $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A})$ be the set of all maximal independent elements for $\mathcal{A}$, then for any finite box $\mathcal{C}$ we have the decomposition:

$$\mathcal{A}^+ \cap \mathcal{I}(\mathcal{A})^- = \emptyset, \qquad \mathcal{A}^+ \cup \mathcal{I}(\mathcal{A})^- = \mathcal{C}. \tag{3}$$

In particular, if $\mathcal{A}$ is the set $\mathcal{F} = \mathcal{F}_{A,b,c}$ of all minimal feasible integral vectors for (1), then the ideal $\mathcal{F}^+$ is the solution set of (1), while the filter $\mathcal{C} \setminus \mathcal{F}^+$ is generated by the set $\mathcal{I}(\mathcal{F})$ of all maximal infeasible integral vectors for (1):

$$\{x \in \mathcal{C} \mid Ax \not\geq b\} = \bigcup_{y \in \mathcal{I}(\mathcal{F})} \{y\}^-.$$

It is known that the problem of incrementally generating all maximal infeasible vectors for (1) is NP-hard even if c is the vector of all ones and the matrix A is binary:

Proposition 1 (c.f. [11]) *Given a binary matrix A and a set $\mathcal{X} \subseteq \mathcal{I}(\mathcal{F}_{A,b,1})$ of maximal infeasible Boolean vectors for $Ax \geq b$, $x \in \{0,1\}^n$, it is NP-complete to decide if the set $\mathcal{X}$ can be extended, that is if $\mathcal{I}(\mathcal{F}_{A,b,1}) \setminus \mathcal{X} \neq \emptyset$.*

It was conjectured in [10] that problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ cannot be solved in polynomial time either, unless P=NP. In this paper we show, however, that the latter problem can be solved in quasi-polynomial time.

Theorem 1 *Problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ can be solved in time $poly(|input|) + t^{o(\log t)}$, where $t = \max\{n, r, |\mathcal{X}|\}$.*

To prove this result, we first bound the number of maximal infeasible vectors for (1) in terms of the dimension of the system and the number of minimal feasible vectors.

Theorem 2 *Suppose that the monotone system (1) is feasible, i.e., $Ac \geq b$. Then for any non-empty set $\mathcal{X} \subseteq \mathcal{F}_{A,b,c}$ we have*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{A,b,c})| \leq r \sum_{x \in \mathcal{X}} p(x), \tag{4}$$

where $p(x)$ is the number of positive components of x. In particular, $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{A,b,c})| \leq rn|\mathcal{X}|$, which for $\mathcal{X} = \mathcal{F}_{A,b,c}$ implies the inequality $|\mathcal{I}(\mathcal{F}_{A,b,c})| \leq rn|\mathcal{F}_{A,b,c}|$.

It should be mentioned that the bounds of Theorem 2 are sharp for $r = 1$, e.g., for the inequality $x_1 + \ldots + x_n \geq n$ in binary variables. For large r, these bounds are accurate up to a factor poly-logarithmic in r. To see this, let $n = 2k$ and consider the monotone system of $r = 2^k$ inequalities of the form

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1,\ i_1 \in \{1,2\},\ i_2 \in \{3,4\}, \ldots, i_k \in \{2k-1, 2k\},$$

where $x = (x_1, \ldots, x_n) \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \le x \le c\}$. For any positive integral vector c, this system has 2^k maximal infeasible integral vectors and only k minimal feasible integral vectors, i.e.,

$$|\mathcal{I}(\mathcal{F}_{A,b,c})| = \frac{rn}{2(\log r)^2}|\mathcal{F}_{A,b,c}|.$$

Needless to say that in general, $|\mathcal{F}_{A,b,c}|$ cannot be bounded by a polynomial in r, n, and $|\mathcal{I}(\mathcal{F}_{A,b,c})|$. For instance, for $n = 2k$ the system of k inequalities $x_1 + x_2 \ge 1$, $\quad x_3 + x_4 \ge 1, \ldots, x_{2k-1} + x_{2k} \ge 1$ has 2^k minimal feasible binary vectors and only k maximal infeasible binary vectors.

We prove Theorem 2 in Section 2, and then use this theorem in the next section to reduce problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ to a natural generalization of the hypergraph dualization problem. Our generalized dualization problem replaces hypergraphs by collections of integer vectors in a box.

Theorem 3 *$GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ is polynomial-time reducible to the following problem:*

$DUAL(\mathcal{C}, \mathcal{A}, \mathcal{B})$**:** *Given an integral box $\mathcal{C}$, a family of vectors $\mathcal{A} \subseteq \mathcal{C}$, and a collection of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that $\mathcal{B} = \mathcal{I}(\mathcal{A})$.*

Note that for $\mathcal{C} = \{0,1\}^n$, problem $DUAL(\mathcal{C}, \mathcal{A}, \mathcal{B})$ turns into the hypergraph dualization problem. Other applications of the dualization problem on boxes can be found in [1, 8, 13]. We can extend the hypergraph dualization algorithms of [9] to problem $DUAL(\mathcal{C}, \mathcal{A}, \mathcal{B})$ and show that the latter problem can be solved in quasi-polynomial time:

Theorem 4 *Given two sets $\mathcal{A}$, and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$ in an integral box $\mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \le x \le c\}$, problem $DUAL(\mathcal{C}, \mathcal{A}, \mathcal{B})$ can be solved in $poly(n, m) + m^{o(\log m)}$ time, where $m = |\mathcal{A}| + |\mathcal{B}|$.*

Clearly, Theorem 1 follows from Theorems 3 and 4. The special cases of Theorems 2 and 3 for Boolean systems $c = (1, \ldots, 1)$ can be found in [5]. If c is bounded and the number of non-zero coefficients per inequality in (1) is fixed, the results of [3] also imply that problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ can be efficiently solved in parallel.

Let us add finally that even though by Proposition 1 generating all maximal infeasible vectors for (1) is NP-hard, this problem can be solved efficiently if the number of inequalities in (1) is fixed. Specifically, for $r = const$ the size of $\mathcal{F}_{A,b,c}$ can be bounded by a polynomial in n and $|\mathcal{I}(\mathcal{F}_{A,b,c})|$ and consequently, all elements of $\mathcal{I}(\mathcal{F}_{A,b,c})$ can be generated in quasi-polynomial time. In fact, for $r = const$ the problem of generating $\mathcal{I}(\mathcal{F}_{A,b,c})$ as well as the problem of generating $\mathcal{F}_{A,b,c}$ can be solved separately in incremental polynomial time.

Theorem 5 *Suppose that the monotone system (1) is nontrivial, i.e., $0 \notin \mathcal{F}_{A,b,c}$. Then for any non-empty subset $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_{A,b,c})$ we have*

$$\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}\right| \le \Big(\sum_{y \in \mathcal{Y}} q(y)\Big)^r, \tag{5}$$

where $\mathcal{I}^{-1}(\mathcal{Y})$ is the set of all minimal integral vectors of the ideal $\mathcal{C} \setminus \mathcal{Y}^-$ and $q(y)$ is the number of components y_l such that $y_l < c_l$. In particular, $\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}\right| \le \Big(n|\mathcal{Y}|\Big)^r$, which for $\mathcal{Y} = \mathcal{I}(\mathcal{F}_{A,b,c})$ implies $|\mathcal{F}_{A,b,c}| \le \Big(n|\mathcal{I}(\mathcal{F}_{A,b,c})|\Big)^r$.

Theorem 6 *If the number of rows in (1) is bounded, problems $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ and GEN $(\mathcal{I}(\mathcal{F}_{A,b,c}), \mathcal{Y})$ can be solved in polynomial time.*

Due to the space limitation, we only give the proofs of Theorems 2 and 5 in the next sections. The other proofs can be found in [4].

2 Bounding the number of maximal infeasible vectors

In this section we prove Theorem 2. We first need some notations and definitions.

Let $\mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}$ be a box and let $f : \mathcal{C} \to \{0, 1\}$ be a discrete binary function. The function f is called *monotone* if $f(x) \geq f(y)$ whenever $x \geq y$ and $x, y \in \mathcal{C}$. We denote by $T(f)$ and $F(f)$ the sets of all true and all false vectors of f, i.e.,

$$T(f) = \{x \in \mathcal{C} | f(x) = 1\} = (\min[f])^+, \quad F(f) = \{x \in \mathcal{C} | f(x) = 0\} = (\max[f])^-,$$

where $\min[f]$ and $max[f]$ are the sets of all minimal true and all maximal false vectors of f, respectively.

Let $\sigma \in \mathbb{S}_n$ be a permutation of the coordinates and let x, y be two n-vectors. We say that y is a *left-shift* of x and write $y \succeq_\sigma x$ if the inequalities

$$\sum_{j=1}^{k} y_{\sigma_j} \geq \sum_{j=1}^{k} x_{\sigma_j}$$

hold for all $k = 1, \ldots, n$. A discrete binary function $f : \mathcal{C} \to \{0, 1\}$ is called *2-monotonic with respect to* σ if $f(y) \geq f(x)$ whenever $y \succeq_\sigma x$ and $x, y \in \mathcal{C}$. Clearly, $y \geq x$ implies $y \succeq_\sigma x$ for any $\sigma \in \mathbb{S}_n$, so that any 2-monotonic function is monotone.

The function f will be called *regular* if it is 2-monotonic with respect to the identity permutation $\sigma = (1, 2, ..., n)$. Any 2-monotonic function can be transformed into a regular one by appropriately re-indexing its variables. To simplify notations, we shall state Lemma 1 below for regular functions, i.e., we fix $\sigma = (1, 2, ..., n)$ in this lemma.

For a given subset $\mathcal{A} \subseteq \mathcal{C}$ let us denote by $\mathcal{A}^*$ all the vectors which are left-shifts of some vectors of $\mathcal{A}$, i.e., $\mathcal{A}^* = \{y \in \mathcal{C} \mid y \succeq x \text{ for some } x \in \mathcal{A}\}$. Clearly, $T(f) = (\min[f])^*$ for a regular function f (in fact, the subfamily of *right-most* vectors of $\min[f]$ would be enough to use here.)

Given monotone discrete functions f and g, we call g a *regular majorant* of f, if $g(x) \geq f(x)$ for all $x \in \mathcal{C}$, and g is regular. Clearly, $T(g) \supseteq (\min[f])^*$ must hold in this case, and the discrete function h defined by $T(h) = (\min[f])^*$ is the unique minimal regular majorant of f.

For a vector $x \in \mathcal{C}$, and for an index $1 \leq k \leq n$, let the vectors $x^{(k]}$ and $x^{[k)}$ be defined by

$$x_j^{(k]} = \begin{cases} x_j & \text{for } j \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_j^{[k)} = \begin{cases} x_j & \text{for } j \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by e the n-vector of all 1's, let e_j denote the j^{th} unit vector, $j = 1, ..., n$, and let $p(x)$ denote the number of positive components of the vector $x \in \mathcal{C}$.

Lemma 1 *Given a monotone discrete binary function $f : \mathcal{C} \to \{0, 1\}$ such that $f \not\equiv 0$, and a regular majorant $g \geq f$, we have the inequality*

$$|F(g) \cap \max[f]| \leq \sum_{x \in \min[f]} p(x). \tag{6}$$

Proof. Let us denote by h the unique minimal regular majorant of f. Then we have $F(g) \cap \max[f] \subseteq F(h) \cap \max[f]$, and hence it is enough to show the statement for $g = h$, i.e. when $T(g) = (\min[f])^*$.

For a vector $y \in \mathcal{C} \setminus \{c\}$ let us denote by $l = l_y$ the index of the last component which is less than c_l, i.e., $l = \max\{j \mid y_j < c_j\} \in \{1, \ldots, n\}$. We claim that for every $y \in F(h) \cap \max[f]$ there exists an $x \in \min[f]$ such that

$$y = x^{(l-1]} + (x_l - 1)e_l + c^{[l+1)}, \tag{7}$$

where $l = l_y$. To see this claim, first observe that $y \neq c$ because $y \in F(f)$ and $f \not\equiv 0$. Second, for any j with $y_j < c_j$ we have $y + e_j \in T(f)$, by the definition of a maximal false point. Hence there exists a minimal true-vector $x \in \min[f]$ such that $x \leq y + e_l$ for $l = l_y$. We must have $x^{(l-1]} = y^{(l-1]}$, since if $x_i < y_i$ for some $i < l$, then $y \geq x + e_i - e_l \succeq x$ would hold, i.e. $y \succeq x$ would follow, implying $y \in (\min[f])^*$ and yielding a contradiction with $y \in F(h) = \mathcal{C} \setminus (\min[f])^*$. Finally, the definition of $l = l_y$ implies that $y^{[l+1)} = c^{[l+1)}$. Hence, our claim and the equality (7) follow.

The above claim implies that

$$F(h) \cap \max[f] \subseteq \{x^{(l-1]} + (x_l - 1)e_l + c^{[l+1)} \mid x \in \min[f],\ x_l > 0\},$$

and hence (6) and thus the lemma follow. $\square$

Lemma 2 *Let $f : \mathcal{C} \to \{0, 1\}$ be a monotone discrete binary function such that $f \not\equiv 0$ and*

$$x \in T(f) \quad \Rightarrow \quad \alpha x \stackrel{\text{def}}{=} \alpha_1 x_1 + \ldots \alpha_n x_n \geq \beta, \tag{8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a given real vector and β is a real threshold. Then

$$|\{x \in \mathcal{C} \mid \alpha x < \beta\} \cap \max[f]| \leq \sum_{x \in \min[f]} p(x).$$

Proof. Suppose that some of the weights $\alpha_1, \ldots, \alpha_n$ are negative, say $\alpha_1 < 0, \ldots, \alpha_k < 0$ and $\alpha^{[k+1)} \geq 0$. Since $\alpha x \geq \beta$ for any $x \in T(f)$ and since f is monotone, we have $x \in T(f) \Rightarrow \alpha^{[k+1)} x \geq \beta - \alpha^{(k]} c^{(k]}$. For any $x \in \mathcal{C}$ we also have $\{x \mid \alpha x < \beta\} \subseteq \{x \mid \alpha^{[k+1)} x < \beta - \alpha^{(k]} c^{(k]}\}$. Hence it suffices to prove the lemma for the non-negative weight vector $\alpha^{[k+1)}$ and the threshold $\beta - \alpha^{(k]} c^{(k]}$. In other words, we can assume without loss of generality that the original weight vector α is non-negative.

Let $\sigma \in \mathbb{S}^n$ be a permutation such that $\alpha_{\sigma_1} \geq \alpha_{\sigma_2} \geq \cdots \geq \alpha_{\sigma_n} \geq 0$. Then the threshold function

$$g(x) = \begin{cases} 1 & \text{if } \alpha x \geq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

is 2-monotonic with respect to σ. By (8), we have $g \geq f$ for all $x \in \mathcal{C}$, i.e., g is a majorant of f. In addition, $F(g) = \{x \in \mathcal{C} \mid \alpha x < \beta\}$, and hence Lemma 2 follows from Lemma 1. $\square$

We are now ready to show inequality (4) and finish the proof of Theorem 2. Given a non-empty set $\mathcal{X} \subseteq \mathcal{F}_{A,b,c}$, consider the monotone discrete function $f : \mathcal{C} \to \{0,1\}$ defined by the condition $\min[f] = \mathcal{X}$. Since (1) is monotone, any true vector of f also satisfies (1):

$$x \in T(f) \quad \Rightarrow a_{k1}x_1 + \ldots + a_{kn}x_n \geq b_k$$

for all $k = 1, \ldots, r$. In addition, $f \not\equiv 0$ because $\mathcal{X} \neq \emptyset$. Thus, by Lemma 2 we have the inequalities

$$|\{x \mid a_{k1}x_1 + \ldots + a_{kn}x_n < b_k\} \cap \max[f]| \leq \sum_{x \in \mathcal{X}} p(x) \tag{9}$$

for each $k = 1, ..., r$. Now, from $\max[f] = \mathcal{I}(\mathcal{X})$ we deduce that

$$\mathcal{I}(\mathcal{F}_{A,b,c}) \cap \mathcal{I}(\mathcal{X}) \subseteq \bigcup_{k=1}^{r} \{x \mid a_{k1}x_1 + \ldots + a_{kn}x_n < b_k\} \cap \max[f],$$

and thus (4) and the theorem follows by (9). $\square$

3 Bounding the number of minimal feasible solutions

To prove inequality (5) of Theorem 5, let us consider an arbitrary non-empty antichain $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_{A,b,c})$. For any $y \in \mathcal{I}(\mathcal{F}_{A,b,c})$ we can find an index $i = \rho(y) \in [r] \stackrel{\text{def}}{=} \{1, \ldots, r\}$ such that y violates the i^{th} inequality of the system, i.e., $a_{(i)}y < b_i$, where $a_{(i)}$ and b_i denote the i^{th} row and component of A and b, respectively.

Consider a vector $x \in \mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}$ and let x_l be a positive component of x. Then there exists a vector $y_{(l)} \in \mathcal{Y}$ such that $y_{(l)} \geq x - e_l$. Let $i = \rho(y_{(l)})$ and assume without loss of generality that

$$(a_{(i)})_1 \geq (a_{(i)})_2 \geq \ldots \geq (a_{(i)})_n. \tag{10}$$

We claim that $x^{(l]} = z_{(l)}^{(l]}$, where

$$z_{(l)} = y^{(l]} + e_l. \tag{11}$$

It follows from $y_{(l)} \geq x - e_l$ that $z_{(l)}^{(l]} \geq x^{(l]}$. If $(z_{(l)})_l > x_l$, then $(y_{(l)})_l \geq x_l$, which implies $y_{(l)} \geq x$, a contradiction. Thus $(z_{(l)})_l = x_l$ holds. Moreover, if $(z_{(l)})_j > x_j$ for some $j < l$, then we have $(y_{(l)})_j \geq x_j + 1$. By (10), $a_{(i)}(y_{(l)} - e_j + e_l) < b_i$, i.e., $y_{(l)} - e_j + e_l$ is infeasible for (1). However, $y_{(l)} - e_j + e_l \geq x$ by $y_{(l)} \geq x - e_l$ and hence $y_{(l)} - e_j + e_l$ must be feasible. This shows that $x^{(l]} = z_{(l)}^{(l]}$ and consequently leads to the representation

$$x = \bigvee_{l \in [n]:\, x_l > 0} z_{(l)}, \tag{12}$$

where for vectors $v, u \in \mathcal{C}$ we let $v \vee u$ denote the component-wise maximum of v and u.

Not all of the vectors $z_{(l)}$ are necessary for this representation. Suppose that $\rho(y_{(l)}) = \rho(y_{(l')}) = i$ for some positive components x_l and $x_{l'}$ of x, and $l' < l$. Then (12) remains valid if we drop $z_{(l')}$, the vector with the smaller index l'. In other words, by sorting the i^{th} row of A and then selecting among the vectors $y_{(l)} \in \rho^{-1}(i)$ the one with the highest l, we obtain at most r vectors $z_{(i)} = z_{(l_i)}$ such that

$$x = \bigvee_{i \in [r]} z_{(i)}. \tag{13}$$

The latter representation readily implies (5). $\square$

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