完全グラフの均衡的 (C_4, C_4, C_8)-Trefoil 分解アルゴリズム

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アブストラクト

グラフ理論において、グラフの分解問題は主要な研究テーマである。C_4、C_8 をそれぞれ 4 点、8 点を通るサイクルとする。1 点を共有する辺素な 3 個のサイクル C_4、C_4、C_8 からなるグラフを (C_4, C_4, C_8)-trefoil という。本研究では、完全グラフ K_n を (C_4, C_4, C_8)-trefoil 部分グラフに均衡的に分解する分解アルゴリズムを与える。

キーワード: 均衡的 (C_4, C_4, C_8)-trefoil 分解; 完全グラフ; グラフ理論

Balanced (C_4, C_4, C_8)-Trefoil Decomposition Algorithm of Complete Graphs

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Abstract

In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a balanced (C_4, C_4, C_8)-trefoil decomposition algorithm of the complete graph K_n.

Keywords: Balanced (C_4, C_4, C_8)-trefoil decomposition; Complete graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_8 be *the* 4-*cycle* and *the* 8-*cycle*, respectively. *The* (C_4, C_4, C_8)-*trefoil* is a graph of 3 edge-disjoint cycles C_4, C_4 and C_8 with a common vertex and the common vertex is called *the center of the* (C_4, C_4, C_8)-*trefoil*.
When K_n is decomposed into edge-disjoint sum of (C_4, C_4, C_8)-trefoils, we say that K_n *has a* (C_4, C_4, C_8)-*trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_4, C_8)-trefoils, we say that K_n *has a balanced* (C_4, C_4, C_8)-*trefoil decomposition* and this

number is called *the replication number.*

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[1] and Wallis[5, Chapter 12 : Triple Systems]. Horák and Rosa[2] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this sense, our balanced (C_4, C_4, C_8)-trefoil decomposition of K_n is to be known as *a balanced (C_4, C_4, C_8)-trefoil system*.

2. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_n

We use the following notation for a (C_4, C_4, C_8)-trefoil.

Notation. We denote a (C_4, C_4, C_8)-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_1$, $v_1 - v_5 - v_6 - v_7 - v_1$, $v_1 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_{13} - v_{14} - v_1$ by $\{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7), (v_1, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14})\}$.

We have the following theorem.

Theorem. K_n has a balanced (C_4, C_4, C_8)-trefoil decomposition if and only if $n \equiv 1$ (mod 32).

Proof. (Necessity) Suppose that K_n has a balanced (C_4, C_4, C_8)-trefoil decomposition. Let b be the number of (C_4, C_4, C_8)-trefoils and r be the replication number. Then $b = n(n-1)/32$ and $r = 14(n-1)/32$. Among r (C_4, C_4, C_8)-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_4, C_8)-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/32$ and $r_2 = 13(n-1)/32$. Therefore, $n \equiv 1$ (mod 32) is necessary.

(Sufficiency) Put $n = 32t + 1$. Construct tn (C_4, C_4, C_8)-trefoils as follows:
$B_i^{(1)} = \{(i, i+4t+1, i+26t+2, i+6t+1), (i, i+4t+2, i+26t+4, i+6t+2), (i, i+2t+1, i+17t+2, i+17t+3, i+30t+4, i+29t+3, i+15t+2, i+3t+1)\}$
$B_i^{(2)} = \{(i, i+4t+3, i+26t+6, i+6t+3), (i, i+4t+4, i+26t+8, i+6t+4), (i, i+2t+2, i+17t+4, i+17t+6, i+30t+8, i+29t+6, i+15t+4, i+3t+2)\}$
$B_i^{(3)} = \{(i, i+4t+5, i+26t+10, i+6t+5), (i, i+4t+6, i+26t+12, i+6t+6), (i, i+2t+3, i+17t+6, i+17t+9, i+30t+12, i+29t+9, i+15t+6, i+3t+3)\}$
...
$B_i^{(t)} = \{(i, i+6t-1, i+30t-2, i+8t-1), (i, i+6t, i+30t, i+8t), (i, i+3t, i+19t, i+20t, i+34t, i+32t, i+17t, i+4t)\}$
$(i = 1, 2, ..., n)$,
where the additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.
Then they comprise a balanced (C_4, C_4, C_8)-trefoil decomposition of K_n.

Note. We consider the vertex set V of K_n as $V = \{1, 2, ..., n\}$.
The additions $i + x$ are taken modulo n with residues $1, 2, ..., n$.

Example 1. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_{33}.
$B_1 = \{(1, 6, 29, 8), (1, 7, 31, 9), (1, 4, 20, 21, 2, 33, 18, 5)\}$
$B_2 = \{(2, 7, 30, 9), (2, 8, 32, 10), (2, 5, 21, 22, 3, 1, 19, 6)\}$
$B_3 = \{(3, 8, 31, 10), (3, 9, 33, 11), (3, 6, 22, 23, 4, 2, 20, 7)\}$
$B_4 = \{(4, 9, 32, 11), (4, 10, 1, 12), (4, 7, 23, 24, 5, 3, 21, 8)\}$

$B_5 = \{(5,10,33,12),(5,11,2,13),(5,8,24,25,6,4,22,9)\}$
$B_6 = \{(6,11,1,13),(6,12,3,14),(6,9,25,26,7,5,23,10)\}$
$B_7 = \{(7,12,2,14),(7,13,4,15),(7,10,26,27,8,6,24,11)\}$
$B_8 = \{(8,13,3,15),(8,14,5,16),(8,11,27,28,9,7,25,12)\}$
$B_9 = \{(9,14,4,16),(9,15,6,17),(9,12,28,29,10,8,26,13)\}$
$B_{10} = \{(10,15,5,17),(10,16,7,18),(10,13,29,30,11,9,27,14)\}$
$B_{11} = \{(11,16,6,18),(11,17,8,19),(11,14,30,31,12,10,28,15)\}$
$B_{12} = \{(12,17,7,19),(12,18,9,20),(12,15,31,32,13,11,29,16)\}$
$B_{13} = \{(13,18,8,20),(13,19,10,21),(13,16,32,33,14,12,30,17)\}$
$B_{14} = \{(14,19,9,21),(14,20,11,22),(14,17,33,1,15,13,31,18)\}$
$B_{15} = \{(15,20,10,22),(15,21,12,23),(15,18,1,2,16,14,32,19)\}$
$B_{16} = \{(16,21,11,23),(16,22,13,24),(16,19,2,3,17,15,33,20)\}$
$B_{17} = \{(17,22,12,24),(17,23,14,25),(17,20,3,4,18,16,1,21)\}$
$B_{18} = \{(18,23,13,25),(18,24,15,26),(18,21,4,5,19,17,2,22)\}$
$B_{19} = \{(19,24,14,26),(19,25,16,27),(19,22,5,6,20,18,3,23)\}$
$B_{20} = \{(20,25,15,27),(20,26,17,28),(20,23,6,7,21,19,4,24)\}$
$B_{21} = \{(21,26,16,28),(21,27,18,29),(21,24,7,8,22,20,5,25)\}$
$B_{22} = \{(22,27,17,29),(22,28,19,30),(22,25,8,9,23,21,6,26)\}$
$B_{23} = \{(23,28,18,30),(23,29,20,31),(23,26,9,10,24,22,7,27)\}$
$B_{24} = \{(24,29,19,31),(24,30,21,32),(24,27,10,11,25,23,8,28)\}$
$B_{25} = \{(25,30,20,32),(25,31,22,33),(25,28,11,12,26,24,9,29)\}$
$B_{26} = \{(26,31,21,33),(26,32,23,1),(26,29,12,13,27,25,10,30)\}$
$B_{27} = \{(27,32,22,1),(27,33,24,2),(27,30,13,14,28,26,11,31)\}$
$B_{28} = \{(28,33,23,2),(28,1,25,3),(28,31,14,15,29,27,12,32)\}$
$B_{29} = \{(29,1,24,3),(29,2,26,4),(29,32,15,16,30,28,13,33)\}$
$B_{30} = \{(30,2,25,4),(30,3,27,5),(30,33,16,17,31,29,14,1)\}$
$B_{31} = \{(31,3,26,5),(31,4,28,6),(31,1,17,18,32,30,15,2)\}$
$B_{32} = \{(32,4,27,6),(32,5,29,7),(32,2,18,19,33,31,16,3)\}$
$B_{33} = \{(33,5,28,7),(33,6,30,8),(33,3,19,20,1,32,17,4)\}$.

This decomposition can be written as follows:
$B_i = \{(i,i+5,i+28,i+7),(i,i+6,i+30,i+8),(i,i+3,i+19,i+20,i+1,i+32,i+17,i+4)\}$
$(i = 1,2,...,33)$.

Example 2. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_{65}.
$B_i^{(1)} = \{(i,i+9,i+54,i+13),(i,i+10,i+56,i+14),(i,i+5,i+36,i+37,i+64,i+61,i+32,i+7)\}$
$B_i^{(2)} = \{(i,i+11,i+58,i+15),(i,i+12,i+60,i+16),(i,i+6,i+38,i+40,i+3,i+64,i+34,i+8)\}$
$(i = 1,2,...,65)$.

Example 3. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_{97}.
$B_i^{(1)} = \{(i,i+13,i+80,i+19),(i,i+14,i+82,i+20),(i,i+7,i+53,i+54,i+94,i+90,i+47,i+10)\}$
$B_i^{(2)} = \{(i,i+15,i+84,i+21),(i,i+16,i+86,i+22),(i,i+8,i+55,i+57,i+1,i+93,i+49,i+11)\}$
$B_i^{(3)} = \{(i,i+17,i+88,i+23),(i,i+18,i+90,i+24),(i,i+9,i+57,i+60,i+5,i+96,i+51,i+12)\}$
$(i = 1,2,...,97)$.

Example 4. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_{129}.
$B_i^{(1)} = \{(i,i+17,i+106,i+25),(i,i+18,i+108,i+26),(i,i+9,i+70,i+71,i+124,i+119,i+$
$62,i+13)\}$
$B_i^{(2)} = \{(i,i+19,i+110,i+27),(i,i+20,i+112,i+28),(i,i+10,i+72,i+74,i+128,i+122,i+$

$64, i+14)\}$
$B_i^{(3)} = \{(i, i+21, i+114, i+29), (i, i+22, i+116, i+30), (i, i+11, i+74, i+77, i+3, i+125, i+66, i+15)\}$
$B_i^{(4)} = \{(i, i+23, i+118, i+31), (i, i+24, i+120, i+32), (i, i+12, i+76, i+80, i+7, i+128, i+68, i+16)\}$
$(i = 1, 2, ..., 129)$.

Case 5. Example 5. Balanced (C_4, C_4, C_8)-trefoil decomposition of K_{161}.
$B_i^{(1)} = \{(i, i+21, i+132, i+31), (i, i+22, i+134, i+32), (i, i+11, i+87, i+88, i+154, i+148, i+77, i+16)\}$
$B_i^{(2)} = \{(i, i+23, i+136, i+33), (i, i+24, i+138, i+34), (i, i+12, i+89, i+91, i+158, i+151, i+79, i+17)\}$
$B_i^{(3)} = \{(i, i+25, i+140, i+35), (i, i+26, i+142, i+36), (i, i+13, i+91, i+94, i+1, i+154, i+81, i+18)\}$
$B_i^{(4)} = \{(i, i+27, i+144, i+37), (i, i+28, i+146, i+38), (i, i+14, i+93, i+97, i+5, i+157, i+83, i+19)\}$
$B_i^{(5)} = \{(i, i+29, i+148, i+39), (i, i+30, i+150, i+40), (i, i+15, i+95, i+100, i+9, i+160, i+85, i+20)\}$
$(i = 1, 2, ..., 161)$.

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