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グラフ及び領域空間に関する大域丸めの幾何学的性質について

浅野哲夫（北陸先端科学技術大学院大学）加藤直樹（京都大学）玉木久夫（明治大学）徳山豪 (東北大学)

概要　ハイパーグラフ $\mathcal{H}=(V,\mathcal{F})$ と $[0,1]$-値ベクトル $\mathbf{a}\in[0,1]^V$ が与えられた時、$\mathcal{H}$ に関する $\mathbf{a}$ の大域丸めとは二値ベクトル $\alpha\in\{0,1\}^V$ で、全てのハイパーエッジ F に対して $|\sum_{v\in F}(\mathbf{a}(v)-\alpha(v))|<1$ が成り立つものの事を言う。本論文では、$\mathbf{a}$ の大域丸め全体の集合の幾何学的及び組合せ的性質を考察する。　具体的には、ハイパーグラフが最短路公理を満たすとき、大域丸めの集合が単体をなすことを予想し、幾何的な領域空間や、直並列グラフの最短路ハイパーグラフに対してこの予想を示す。

On Geometric Structure of Global Roundings for Graphs and Range Spaces

Tetsuo Asano (JAIST) **t-asano@jaist.ac.jp**
Naoki Katoh (Kyoto University) **naoki@archi.kyoto-u.ac.jp**
Hisao Tamaki (Meiji University) **tamaki@cs.meiji.ac.jp**
Takeshi Tokuyama (Tohoku University) **tokuyama@dais.is.tohoku.ac.jp**

Abstract. Given a hypergraph $\mathcal{H}=(V,\mathcal{F})$ and a $[0,1]$-valued vector $\mathbf{a}\in[0,1]^V$, its global rounding is a binary (i.e.,$\{0,1\}$-valued) vector $\alpha\in\{0,1\}^V$ such that $|\sum_{v\in F}(\mathbf{a}(v)-\alpha(v))|<1$ holds for each $F\in\mathcal{F}$. We study geometric (or combinatorial) structure of the set of global roundings of $\mathbf{a}$ using the notion of compatible set with respect to the discrepancy distance. We conjecture that the set of global roundings forms a simplex if the hypergraph satisfies “shortest-path” axioms, and prove it for some special cases including some geometric range spaces and the shortest path hypergraph of a series-parallel graph.

1 Introduction

Rounding problem is a central problem in computer science and computer engineering. Given a real number a, its rounding is either its floor $\lfloor a\rfloor$ or ceiling $\lceil a\rceil$. Then, we want to consider how to round a set of n real numbers each of which is assigned to an element of a set $V=\{v_1,v_2,\dots,v_n\}$ with a given structure. We can assume that each number is in the range $[0,1]$, so that the input set can be considered as $\mathbf{a}\in[0,1]^V$ and the output rounding is $\alpha\in\{0,1\}^V$. Throughout this paper, we use a Greek (resp. bold) character for representing a binary (resp. real-valued) function on V.

We assume that the structure on V is represented by a hypergraph $\mathcal{H}=(V,\mathcal{F})$ where $\mathcal{F}\subset 2^V$ is the set of hyperedges. For simplicity, we assume without loss of generality that $\mathcal{F}$ contains all the singletons. We say α is a *global rounding* of $\mathbf{a}$ iff $w_F(\alpha)=\sum_{v\in F}\alpha(v)$ is a rounding (i.e., either floor or ceiling) of $w_F(\mathbf{a})=\sum_{v\in F}\mathbf{a}(v)$ for each $F\in\mathcal{F}$. Let $\Gamma_{\mathcal{H}}(\mathbf{a})$ be the set of all global roundings of $\mathbf{a}$.

We can rephrase the global rounding condition as $D_{\mathcal{H}}(\mathbf{a},\alpha)<1$, where $D_{\mathcal{H}}$ is the *discrepancy distance* between $\mathbf{a}$ and $\mathbf{b}$ in $[0,1]^V$ defined by

$$D_{\mathcal{H}}(\mathbf{a},\mathbf{b})=\max_{F\in\mathcal{F}}|w_F(\mathbf{a})-w_F(\mathbf{b})|.$$

Thus, $\Gamma_{\mathcal{H}}(\mathbf{a})$ is the set of integral points in the open unit ball about $\mathbf{a}$ by considering $D_{\mathcal{H}}$ as the distance. $\Gamma_{\mathcal{H}}(\mathbf{a})\neq\emptyset$ for every $\mathbf{a}$ iff $\mathcal{H}$ is unimodular [5]. However, except for the above unimodular condition for the nonemptiness and some results on its cardinality, little is known on the structure of $\Gamma_{\mathcal{H}}(\mathbf{a})$. We remark that $\sup_{\mathbf{a}\in[0,1]^V}\min_{\alpha\in\{0,1\}^V}D_{\mathcal{H}}(\mathbf{a},\alpha)$ is the *linear discrepancy* of $\mathcal{H}$, and considered as a key concept in hypergraph theory and combinatorial geometry [5, 7, 10]. In this paper, we study the geometric property of $\Gamma_{\mathcal{H}}(\mathbf{a})$.

We say that a hypergraph $\mathcal{H}$ has the *simplex property* if $\Gamma_{\mathcal{H}}(\mathbf{a})$ is (the vertex set of) a simplex (possibly a degenerate one or empty) regarding it as a set of n-dimensional points for any $\mathbf{a} \in [0,1]^V$. Our main aim is to investigate classes of hypergraphs that have the simplex property.

The global rounding condition is directly written in an integer programming formula, and thus from the viewpoint of mathematical programming, we have interesting classes of integer programming problems for which the solution space is a simplex while the corresponding LP polytope is not always a simplex.

The simplex property is motivated by recent results on $\mu(\mathcal{H}) = \max_{\mathbf{a}\in[0,1]^V} |\Gamma_{\mathcal{H}}(\mathbf{a})|$, which is the maximum number of global roundings. $\mu(\mathcal{H})$ can never become less than $n+1$ for any hypergraph since n unit vectors and the zero vector always form $\Gamma_{\mathcal{H}}(\mathbf{a})$ for a suitable $\mathbf{a}$. In general, $\mu(H)$ may become exponential in n. However, Sadakane *et al.*[12] discovered that $\mu(\mathcal{I}_n) = n+1$ where $\mathcal{I}_n$ is the hypergraph on $V = \{1, 2, .., n\}$ with the edge set $\{[i,j]; 1 \le i \le j \le n\}$ consisting of all subintervals of V. A corresponding global rounding is called a *sequence rounding*, which is a convenient tool in digitization of a sequence analogue data.

Given this discovery, it is natural to ask for which class of hypergraphs the property $\mu(\mathcal{H}) = n+1$ holds. Moreover, there should be combinatorial (or geometric) reasoning why $\mu(\mathcal{H}) = n+1$ holds for those hypergraphs. Naturally, the simplex property implies that $\mu(\mathcal{H}) = n+1$ since a d-dimensional simplex has $d+1$ vertices, and indeed $\mathcal{I}_n$ has the simplex property.

Shortest paths and range spaces

$\mathcal{I}_n$ has $n(n+1)/2$ hyperedges, and the authors do not know any hypergraph with less than $n(n+1)/2$ hyperedges (including n singletons) that has the simplex property. Thus, it is reasonable to consider some natural classes of hypergraphs with $n(n+1)/2$ hyperedges.

Consider a connected graph $G = (V, E)$ in which each edge e has a positive length $\ell(e)$. We fix a total ordering $\{v_1, v_2, \ldots, v_n\}$ on V. This ordering is inherited to any subset of V. For each pair (v_i, v_j) of vertices in V such that $i < j$, let $p(v_i, v_j)$ be the shortest path between them. If there are more than one shortest paths between them, we consider the lexicographic ordering among the paths induced from the ordering on V, and select the one with the first one in this ordering. Let $P(v_i, v_j)$ be the set of vertices on $p(v_i, v_j)$ including the terminal nodes v_i and v_j. We also define $P(v, v) = \{v\}$ for each $v \in V$. Let $\mathcal{F}(G) = \{P(v_i, v_j) : 1 \le i \le j \le n\}$, and call $\mathcal{H}(G) = (V, \mathcal{F}(G))$ the *shortest-path hypergraph* associated with G.

It is conjectured that $\mu(\mathcal{H}(G)) = n+1$ if G is a connected graph with n vertices [1]. Note that $\mathcal{H}(G) = \mathcal{I}_n$ if G is a path. The conjecture has been proved for for trees, cycles, and outerplanar graphs [1, 13]. However, those proofs are complicated and case dependent. We try to establish a more structured theory considering the following deeper conjecture.

Conjecture 1.1 *For any connected graph G, $\mathcal{H}(G)$ satisfies the simplex property.*

This conjecture was proposed in [1] by the authors where the simplex property was called "affine independence property" since vertices of a simplex are affine independent as a set of vectors. So far, the conjecture has been proved only for trees, unweighted complete graphs, and unweighted (square) meshes. We prove that the simplex property is invariant under some graph-theoretic connection operations, and as a consequence, we show that the conjecture holds for series-parallel graphs.

In addition to significantly extending the verified classes of hypergraphs for both of the weaker and stronger conjectures, our theory also simplifies the proofs of known results. For example, that the weaker conjecture holds for cycles is one of the main results of [1] and its proof therein is quite involved. In our framework, it is almost trivial that the stronger conjecture holds for cycles (see Section 3).

From a computational-geometric viewpoint, $\mathcal{I}_n$ can be considered as the 1-dimensional range space corresponding to intervals, and thus we try to extend the theory to geometric range spaces.

We generalize the argument to *axiomatic shortest-path* hypergraphs (defined later), and prove the simplex property for some geometric range spaces such as the space of isothetic right-angle triangles.

Algorithmic implication

The theory is not only combinatorially interesting but is applied to algorithm design on the rounding problems. The algorithmic question of how to obtain a low-discrepancy rounding of given $\mathbf{a}$ is important in several applications. For example, consider the problem of digital halftoning in image processing, where the gray-scale value of each pixel has to be rounded into a binary value. This problem is formulated as that of obtaining a low-discrepancy rounding, in which the hypergraph corresponds to a family of certain local sets of pixels, and several methods have been proposed[2, 3, 11]. Unfortunately, for a general hypergraph, it is NP-complete to decide whether a given input $\mathbf{a}$ has a global rounding or not, and hence it is NP-hard to compute a rounding with the minimum discrepancy. Thus, a practical approach is to consider a special hypergraph for which we can compute a low-discrepancy rounding efficiently.

It is folklore that the unimodularity condition means that the vertices of the ball (w.r.t. $D_{\mathcal{H}}$) are integral, and an LP solution automatically gives an IP solution. Thus, a global rounding always exists and can be computed in polynomial time if $\mathcal{H}$ is unimodular, and therefore in the literature [2, 3, 8] unimodular hypergraphs are mainly considered.

Here, we consider another case where an integer programming problem can be solved in polynomial time: If the number of integral points in the solution space is small (i.e. of polynomial size), and there is an enumeration algorithm that is polynomial in the output size (together with the input size), we can solve the problem in polynomial time. We show that enumeration of all global roundings can be done in polynomial time for several (non-unimodular) hypergraphs with the simplex property by applying this framework.

2 Combinatorial and linear algebraic tools

2.1 Compatible set

The set of binary functions on V can be regarded as the n-dimensional hypercube $C_n = \{0,1\}^n$, where $n = |V|$. Consider an integer-valued distance f on C_n. We call a subset A of C_n a *compatible set* with respect to f if $f(x,y) = 1$ for any pair $x \neq y$ of A. In other words, A is a compatible set if and only if it is a unit diameter set. Property of a compatible set is highly dependent on f: If f is the L_∞ distance, the hypercube itself is a compatible set, while the cardinality of a compatible set for the Hamming distance is at most two. By definition, $D_{\mathcal{H}}$ gives an integer-valued distance on the hypercube C_n.

Definition 1 *A set of binary functions on V is called $\mathcal{H}$-compatible if it is a compatible set with respect to $D_{\mathcal{H}}$. In other words, $|w_F(\alpha) - w_F(\beta)| \leq 1$ holds for every hyperedge F of $\mathcal{H}$ for any elements α and β of the set.*

$\Gamma_{\mathcal{H}}(\mathbf{a})$ is always an $\mathcal{H}$-compatible set, since the $D_{\mathcal{H}}$ distance between two global roundings must be integral and less than 2. Conversely, any maximal $\mathcal{H}$-compatible set is $\Gamma_{\mathcal{H}}(\mathbf{g})$, where $\mathbf{g}$ is the center of gravity of the compatible set. Thus, it suffices to show the simplex property for compatible sets instead of sets of global roundings.

2.2 General properties

The simplex property is monotone, that is,

Lemma 2.1 *If $\mathcal{H} = (V, \mathcal{F})$ has the simplex property and $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{H}' = (V, \mathcal{F}')$ does, too.*

Recall that a set $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ of vectors is affine dependent if and only if there are real numbers $c_1, c_2, \ldots, c_m$ satisfying (1) at least one of them is non-zero, (2) $\sum_{1 \leq i \leq m} c_i = 0$, and (3) $\sum_{1 \leq i \leq m} c_i \mathbf{a}_i = \mathbf{0}$.

A set A of binary assignments on V is called *minimal affine dependent* if it is an affine dependent set as a set of vectors in the n-dimensional real vector space ($n = |V|$) and every proper subset of it is affine independent.

For a binary assignment α on V and a subset X of V, $\alpha|_X$ denotes the restriction of α on X. Given a set A of binary assignments on V, its restriction to X is $A|_X = \{\alpha|_X : \alpha \in A\}$. Note that the set is not a multi-set, and we only keep a single copy even if $\alpha|_X = \beta|_X$ for different α and β in A.

For assignments α on X and β on Y $\alpha \oplus \beta$ is an assignment on $V = X \cup Y$ obtained by concatenating α and β: That is, $\alpha \oplus \beta(v) = \alpha(v)$ if $v \in X$, and $\alpha \oplus \beta(v) = \beta(v)$ if $v \in Y$. By definition, $\alpha \oplus \beta$ is only defined if $\alpha(v) = \beta(v)$ for each $v \in X \cap Y$. The following is our key lemma:

Lemma 2.2 *Let A be a minimal affine dependent set on V, and let $V = X \cup Y$. If $A|_X$ and $A|_Y$ are affine independent, then $A|_{X\cap Y}$ has only one assignment.*

Proof: Since A is affine dependent, there exists a constant $c(\alpha)$ for each $\alpha \in A$ such that $\sum_{\alpha \in A} c(\alpha) = 0$ and $\sum_{\alpha\in A} c(\alpha)\alpha = \mathbf{0}$, and at least one $c(\alpha)$ is nonzero. We consider projection of these formulae to X to have formulae
$\sum_{\beta \in A|_X} C(\beta) = 0$ and $\sum_{\beta\in A|_X} C(\beta)\beta = \mathbf{0}$, where $C(\beta) = \sum_{\alpha\in A, \alpha|_X = \beta} c(\alpha)$. Because of affine independence of $A|_X$, $C(\beta) = 0$ for each $\beta \in A|_X$. Let us consider $\tau \in A|_{X\cap Y}$. Let $A(\tau) = \{\alpha \in A : \alpha|_{X\cap Y} = \tau\}$, and $A_X(\tau) = \{\beta \in A|_X : \beta|_{X\cap Y} = \tau\}$. We select τ such that there exists $\alpha \in A(\tau)$ satisfying $c(\alpha) \neq 0$. Let $\eta = \sum_{\alpha\in A(\tau)} c(\alpha)\alpha$. Then, $\eta|_X = \sum_{\beta\in A_X(\tau)} C(\beta)\beta$, and it is $\mathbf{0}$ since $C(\beta) = 0$ for each β. Similarly $\eta|_Y = \mathbf{0}$. Thus, $\eta = \mathbf{0}$. Moreover, $\sum_{\alpha\in A(\tau)} c(\alpha) = \sum_{\beta\in A_X(\tau)} C(\beta)$ and hence it is 0. This means that $A(\tau)$ is affine dependent. Because of minimality of A, $A = A(\tau)$, and we have the lemma. □

Given a subset S of V, we can consider the induced hypergraph $\mathcal{H}|_S = (S, \mathcal{F} \cap 2^S)$. Naturally, if a set A of binary assignments on V is compatible for $\mathcal{H}$, $A|_S$ is compatible for $\mathcal{H}|_S$.

By definition, a subset of a compatible set is also a compatible set. Thus, the concept of minimal affine dependent compatible set (possibly an empty set) is well defined. We have the following corollary of Lemma 2.2:

Corollary 2.3 *Consider a hypergraph $\mathcal{H} = (V, \mathcal{F})$ and a minimal affine dependent compatible set A. Suppose that $V = X \cup Y$ and each of $\mathcal{H}|_X$ and $\mathcal{H}|_Y$ has the simplex property. Then, for any pair α and α' in A, $\alpha(v) = \alpha'(v)$ for each $v \in X \cap Y$.*

Definition 2 *A vertex v of a hypergraph $\mathcal{H}$ is called a double-covered vertex if there exist suitable subsets X and Y such that $V = X \cup Y$, $v \in X\cap Y$, and both of $\mathcal{H}|_X$ and $\mathcal{H}|_Y$ have the simplex property. We say $S \subset V$ is double-covered if every element of S is double-covered.*

Definition 3 *For a subset S of vertices of a hypergraph $\mathcal{H} = (V, \mathcal{F})$, a set A of assignments on V is called S-contracted if $\alpha(v) = 0$ for each pair $v \in S$ and $\alpha \in A$.*

Theorem 2.4 *Let $\mathcal{H} = (V, \mathcal{F})$ be a hypergraph, and let $S \subset V$ be a double-covered set. Then, if every S-contracted compatible set is affine independent, $\mathcal{H}$ has the simplex property.*

Proof: Assume on the contrary that $\mathcal{H}$ does not have the simplex property. Thus, we have an affine dependent compatible set, and hence have a minimal affine dependent compatible set A. From Corollary 2.3, we can assume that all assignments of A take the same value on each element of S. Thus, if we replace the value to 0 at every $v \in S$, the revised set $\tilde{A}$ is also compatible and minimal affine dependent, since we subtract the same vector from each member of A to obtain $\tilde{A}$. However, $\tilde{A}$ is S-contracted, and hence contradicts the hypothesis. □

Corollary 2.5 *If V itself is double-covered, $\mathcal{H} = (V, \mathcal{F})$ has the simplex property.*

2.3 Axiomatic shortest path hypergraph

Definition 4 *A hypergraph $\mathcal{H} = (V, \mathcal{F})$ is called an ASP (axiomatic shortest path) hypergraph if $\mathcal{F} = \{f(u, v) | u, v \in V \times V\}$ satisfies the following conditions: (1): $f(u, u) = \{u\}$.*
(2): $f(u, v) = f(u', v')$ if and only if $(u, v) = (u', v')$ as unordered pairs (one-to-one property).
(3): For any $s, t \in f(u, v)$, $f(s, t) \subset f(u, v)$ (monotonicity).

Definition 5 *Given an ASP hypergraph $\mathcal{H} = (V, \mathcal{F})$, a subset S of V is called a shortest-path-closed subset (SPC subset) if $f(u, v) \subseteq S$ for any pair u and v in S.*

Lemma 2.6 *Given an ASP hypergraph $\mathcal{H} = (V, \mathcal{F})$ and an SPC subset S of V, $\mathcal{H}|_S$ is also an ASP hypergraph.*

3 Shortest path hypergraphs

Definition 6 *A subgraph $G' = (V', E')$ of $G = (V, E)$ is called an SPC subgraph if any shortest path in G' is a shortest path in G.*

Lemma 3.1 *If $G' = (V', E')$ is an SPC subgraph of $G = (V, E)$, V' is an SPC subset of V with respect to $\mathcal{H}(G)$, and $\mathcal{H}(G)|_{V'} = \mathcal{H}(G')$.*

Lemma 3.2 *Consider $\mathcal{H} = \mathcal{H}(G)$ for $G = (V, E)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be SPC subgraphs such that $V_1 \cup V_2 = V$. Then, if both $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property, each vertex in $V_1 \cap V_2$ is double-covered.*

Proposition 3.3 *If G is a cycle, $\mathcal{H}(G)$ has the simplex property.*

Proof: We give a cyclic ordering $v_1, v_2, \ldots, v_n$ of the vertices. For the vertex v_1, let $V_1 = \{v_1, v_2, v_3, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, v_{k+2}, \ldots, v_n, v_1\}$ where k is the largest index for which the shortest path from v_1 to v_k goes through v_2. Let G_1 and G_2 are induced subgraphs associated with V_1 and V_2, respectively. Since G_1 and G_2 are paths, it is known [1] that $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property. It is clear the G_1 and G_2 are SPC subgraphs, and $V_1 \cap V_2 = \{v_1\}$, and $V_1 \cup V_2 = V$. Thus, from Lemma 3.2, v_1 is double covered. This argument holds for any cyclic ordering, and thus every vertex of V is double-covered. Thus, from Corollary 2.5, $\mathcal{H}(G)$ has the simplex property. □

A graph $G = (V, E)$ is a series connection of two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if there exists a vertex (joint vertex) v such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{v\}$, and $E_1 \cup E_2 = E$.

Theorem 3.4 ([1]) *Let G be a series connection of two connected graphs G_1 and G_2. Then, if both $\mathcal{H}(G_1)$ and $\mathcal{H}(G_2)$ have the simplex property, $\mathcal{H}(G)$ does.*

Definition 7 *A connected graph $G = (V, E)$ has a 3-parallel decomposition if there exist two vertices u and v such that G is decomposed into nonempty connected graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$ such that (1) $V = V_1 \cup V_2 \cup V_3$, (2) $V_1 \cap V_2 = V_2 \cap V_3 = V_1 \cap V_3 = \{u, v\}$, and (3) E is the disjoint union of E_1, E_2, and E_3. (see Fig. 1).*

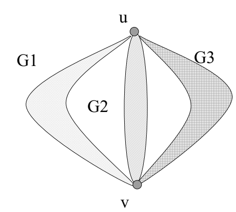

Figure 1: 3-parallel decomposition of G.

Consider a family Ψ of connected graphs, and assume that it is *closed under the subgraph operation*; that is, any connected subgraph of $G \in \Psi$ is also in Ψ. A graph $G \in \Psi$ is a *minimal counterexample* for the simplex property in Ψ if $\mathcal{H}(G)$ does not satisfy the simplex property but $\mathcal{H}(G')$ has the simplex property for every connected subgraph G' of G.

Theorem 3.5 *A minimal counterexample G for the simplex property in Ψ is 2-connected, and does not have a 3-parallel decomposition.*

Proof: 2-connectivity follows from Theorem 3.4. Thus, we assume that G has a 3-parallel decomposition at u and v, and derive a contradiction. We define the following three subgraphs of G: $G_{(1,2)}$ is the union of G_1 and G_2, $G_{(1,3)}$ is the union of G_1 and G_3, and $G_{(2,3)}$ is the union of G_2 and G_3. These graphs are connected and hence satisfy the simplex property because of the minimality of G. By symmetry, we can assume that the shortest path between u and v is in G_1. Then, both $G_{(1,2)}$ and $G_{(1,3)}$ are SPC subgraphs. Thus, $\mathcal{H}(G)|_{V(G_{(1,2)})} = \mathcal{H}(G_{(1,2)})$ and $\mathcal{H}(G)|_{V(G_{(1,3)})} = \mathcal{H}(G_{(1,3)})$, where $V(G_{(i,j)})$ is the vertex set of $G_{(i,j)}$. Thus, it is clear that each vertex of G_1 is double-covered.

A vertex x in $V(G_{(2,3)})$ is called *biased* if either the shortest path in G from x to u goes through v or the shortest path from x to v goes through u. We claim that a biased vertex is double-covered. Without loss of generality, we assume that x is a vertex of G_2 and the shortest path $\mathbf{p}$ from x to v goes through u. Then, any vertex of G_2 on $\mathbf{p}$ is also biased, and $G_{(1,3)} \cup \mathbf{p}$ is an SPC subgraph. Thus, v is in the intersection of two SPC subgraphs $G_{(1,3)} \cup \mathbf{p}$ and $G_{(1,2)}$, and hence double-covered. Thus, $S = V(G_1) \cup B$ is double-covered, where B is the set of all biased vertices.

Now, we are ready to apply Theorem 2.4. Consider an arbitrary S-contracted compatible set A of $\mathcal{H}(G)$. We claim that A is also $\mathcal{H}(G_{(2,3)})$ compatible. If this claim is true, A must be affine independent (since $\mathcal{H}(G_{(2,3)})$ has the simplex property), and we can conclude that $\mathcal{H}(G)$ has the simplex property from Theorem 2.4, so that we have contradiction.

We give a proof for the claim. Let α and β be any two members of A. Consider any shortest path $\mathbf{p}$ of $G_{(2,3)}$. Let x and y be endpoints of $\mathbf{p}$, and let P be the vertex set of the path. It suffices to show the compatibility $|\alpha(P) - \beta(P)| \leq 1$.

If all the vertices on $\mathbf{p}$ are in S, $\alpha(P) = \beta(P) = 0$, and the compatibility condition is trivial. Thus, we assume there exist vertices in $V \setminus S$ on $\mathbf{p}$. Let x_0 be the nearest vertex in $(V \setminus S) \cap P$ to x. The subpath $\mathbf{p}_0$ of $\mathbf{p}$ between x_0 and y is the shortest path in $G_{(2,3)}$ between them. Let P_0 be the set of vertices on it.

Consider the shortest path $\mathbf{q}$ with respect to G between x_0 and y. If it contains both u and v on it, either the shortest path between x_0 and u contains v or that between x_0 and v contains u. Thus, x_0 must be biased, and hence in S, contradicting our hypothesis. Therefore, without loss of generality, we can assume $\mathbf{q}$ does not contain v. This means that $\mathbf{q}$ contains no vertex of $G_1 \setminus \{u\}$, since otherwise $\mathbf{q}$ must go through u twice. Thus, $\mathbf{q}$ is in $G_{(2,3)}$, and hence $\mathbf{q} = \mathbf{p}_0$ since the shortest path between given two vertices is unique in our definition of the shortest path hypergraph. Thus, from the compatibility on a shortest path of G, $|\alpha(P_0) - \beta(P_0)| \leq 1$. Since assignment on each vertex of S is 0 for each of α and β, we have the compatibility $|\alpha(P) - \beta(P)| \leq 1$ on $\mathbf{p}$. Thus, A is $\mathcal{H}(G_{(2,3)})$ compatible, and we have the claim. □

Thus, the simplex property holds for a graph that is constructed by applying a series of 3-parallel connections and series connections from pieces (such as paths, cycles, unit edge-length complete graphs, and unit edge-length meshes) for which the simplex property is known to hold. We give a typical example in the following: A graph is *series-parallel* if it does not have a subdivision of the complete graph K_4 as its subgraph. Here, a subdivision of a graph is obtained by replacing edges of the original graph with chains. A connected graph is *outerplanar* if and only if it has a planar drawing in which every vertex lies on the outerface boundary. An edge that is not on the outerface boundary is called a *chord*. A series parallel graph is planar, and an outerplanar graph is series-parallel.

Theorem 3.6 *If G is series-parallel, $\mathcal{H}(G)$ has the simplex property.*

Proof: Clearly, the family of connected series-parallel graphs is closed under the subgraph operation, and we consider its minimal counterexample G. By Theorem 3.5, G is 2-connected. If G is not outerplanar, G has a vertex v in the interior of the outerface cycle C. Since G is 2-connected, v is connected to at least two vertices of C without using edges on C. If v is connected to three vertices of C, the union of these paths and C contains a subdivision of K_4, and we have contradiction. Thus, v is connected to exactly two vertices u_1 and u_2 of C, and we have 3-parallel decomposition at u_1 and u_2. Thus, G must be outerplanar. If G has a chord, G has 3-parallel decomposition at the end vertices of the chord. Thus, G does not have a chord. However, a 2-connected outerplanar graph without a chord must be a cycle, and we have already shown the simplex property for cycles. Thus, we have the theorem. □

As a corollary, we have $\mu(G) = n + 1$ for a series-parallel graph, extending the result for an outerplanar graph given in [13].

Moreover, it can be observed that any (non-cycle) 2-connected series parallel graph has a 3-parallel decomposition in which two of the components are paths from a classification of substructures of series parallel graphs given by Juvan *et al.* [9] (also see [14]). Using this observation and the argument given in [13] for outerplanar graphs, we have the following (we omit details in this version):

Theorem 3.7 *We can enumerate all the global roundings of an input* **a** *for the shortest path hypergraph of a series-parallel graph with* n *vertices in* $O(n^3)$ *time.*

4 Geometric problems

We consider some geometric hypergraphs that are ASP hypergraphs. Consider a set V of n points on a plane. For each pair $u = (x_u, y_u)$ and $v = (x_v, y_v)$ of points, uv is the line segment connecting them. Let $B(u,v)$ be the region below the segment uv; that is, $B(u,v) = \{(x,y)|x \in [x_u, x_v], y - y_u \leq \frac{y_v - y_u}{x_v - x_u}(x - x_u)\}$ if $x_u \neq x_v$. If $x_u = x_v$, we define $B(u,v) = uv$. Let $R(u,v)$ be the closed isothetic rectangle which has u and v in its diagonal position, and let $T(u,v) = B(u,v) \cap R(u,v)$ be the lower right-angle isothetic triangle which has uv as its longest boundary edge. We define $T(u,u) = R(u,u) = B(u,u) = uu = \{u\}$.

We consider hypergraphs $\mathcal{S} = (V, \{V \cap uv : u,v \in V\})$, $\mathcal{B} = (V, \{V \cap B(u,v) : u,v \in V\})$, $\mathcal{R} = (V, \{V \cap R(u,v) : u,v \in V\})$, and $\mathcal{T} = (V, \{V \cap T(u,v) : u,v \in V\})$. See Fig. 2 to get intuition.

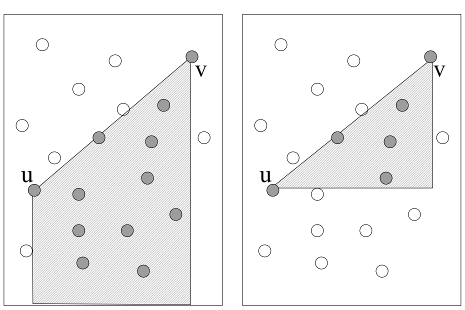

Figure 2: $B(u,v)$ (left) and $T(u,v)$ (right).

Lemma 4.1 *$\mathcal{S}$, $\mathcal{B}$, and $\mathcal{T}$ are ASP hypergraphs for any point set V. $\mathcal{R}$ is an ASP hypergraph if there are no four points of V forming corners of an isothetic rectangle.*

4.1 Simplex property of range spaces

Theorem 4.2 *Each of $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$ have the simplex property. If there are no four points of V forming corners of an isothetic rectangle, $\mathcal{R}$ has the simplex property.*

Proof: We prove the simplex property by induction on the number of horizontal lines and that of vertical lines on which V lies. We only deal with $\mathcal{T}$ here because of space limitation. If V lies on a horizontal line ℓ, the problem is reduced to the sequence rounding problem. We assume that the statement holds if the point set lies on less than M horizontal lines or less than N vertical lines.

Suppose that V lies on M horizontal lines and also lies on N vertical lines. Let $X_{\geq 2} = V \setminus X_1$ and $X_{\leq N-1} = V \setminus X_N$. They are SPC subsets of V. Let $\mathcal{T}^+ = \mathcal{T}|_{X_{\geq 2}}$ and $\mathcal{T}^- = \mathcal{T}|_{X_{\leq N-1}}$. From Lemma 2.6, they are ASP hypergraphs, and by induction hypothesis, have the simplex property. Thus, $X_{\geq 2} \cap X_{\leq N-1} = V \setminus (X_1 \cup X_N)$ is double-covered.

Similarly, we can see that $V \setminus (Y_1 \cup Y_M)$ is double-covered. Since union of two double-covered sets is also double-covered, $S = [V \setminus (X_1 \cup X_N)] \cup [V \setminus (Y_1 \cup Y_M)]$ is double-covered. Thus, we can apply Theorem 2.4, and consider the restriction of $\mathcal{T}$ to $V \setminus S$. Any point in $V \setminus S$ must be at a corner of the minimum enclosing isothetic rectangle of V, thus $V \setminus S$ has at most four points, for which we can directly show the simplex property of the restriction of $\mathcal{T}$. Thus, $\mathcal{T}$ has the simplex property. □

We remark that $\mathcal{R}$ is smaller than the range space corresponding to all isothetic rectangles. However, since $\mathcal{R}$ has the simplex property, the range space of all isothetic rectangles also has the simplex property because of Lemma 2.1. Similarly, since $\mathcal{T}$ has the simplex property, the range space of all isothetic right-angle triangles has the simplex property.

4.2 Enumeration Algorithms

We can design a polynomial-time algorithm for enumerating all the global roundings of an input real assignment **a** for each of $\mathcal{B}$, $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$. We briefly explain the algorithm for $\mathcal{T}$. Basically, we can apply a building-up (or divide-and-conquer) strategy, in which we first compute the restrictions on $X_{\geq \lceil n/2 \rceil}$ and $X_{\leq \lceil n/2 \rceil - 1}$ recursively, and check the rounding condition for $\mathcal{T}$ on each possible concatenated rounding. It takes $O(n^2)$ time for testing each concatenated rounding by using an efficient range-searching method, and hence the total time complexity becomes $O(n^4)$. This is highly contrasted to the fact that it is NP hard to decide the existence of a global rounding for the family of all 2×2 square regions in a grid [3, 4].

If we consider $\mathcal{R}$, the linear discrepancy is known to be $O(\log^3 n)$ and $\Omega(\log n)$ [6] , and hence it is expected that a given input may have no global rounding. Thus, we may consider a heuristic algorithm for computing a nice (not necessarily global) rounding by using the building-up strategy in which we select K best roundings (with respect to the discrepancy) from those obtained by concatenating pairs of assignments constructed in the previous stage to proceed to the next stage. Our theorem implies that if we set $K \geq n + 1$, we never miss a global rounding if it exists.

5 Concluding remarks

If we can replace 3-parallel decomposition with 2-parallel decomposition in Theorem 3.5, we can prove the conjecture, since any 2-connected graph is decomposed into e and $G \setminus \{e\}$ at the endpoints of any edge e. However, it is not known whether $\mu(\mathcal{H}(G))$ is polynomially bounded in general. Another interesting question is whether there is a hypergraph with the simplex property with less than $n(n+1)/2$ hyperedges (including singletons).

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