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三角格子点上の単位円グラフに対する多重彩色

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あらまし　与えられた非負整数 m と n に対して，$P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1+y\boldsymbol{e}_2) \mid x \in \{0,1,\ldots,m-1\},\ y \in \{0,1,\ldots,n-1\}\}$ と定義する．ここで $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$，$\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2,\sqrt{3}/2)$ である，つまり $P(m,n)$ は 2 次元平面上の三角格子点の部分集合を表す．$P(m,n)$ を頂点集合とし，頂点間のユークリッド距離が d 以下であるときに隣接と定義する無向グラフを $T_{m,n}(d)$ とする．本稿では $T_{m,n}(d)$ がパーフェクトであるための必要条件と十分条件を議論する．具体的には，$\forall m \in \mathrm{Z}_+$ に対して $T_{m,n}(d)$ がパーフェクトである必要十分条件は $d \geqq \sqrt{n^2-3n+3}$ であることを示す．非負整数頂点重み $\boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$ が与えられたとき，それぞれの頂点 $v \in P(m,n)$ に $w(v)$ 色が割り当てられ，隣接するどの頂点対も同じ色を共有しないような $P(m,n)$ への色の割り当てを $(T_{m,n}(d),\boldsymbol{w})$ の多重彩色という．本稿では $P(m,n)$ がパーフェクトであるときに $(T_{m,n}(d),\boldsymbol{w})$ を多重彩色する効率的な算法も示す．本稿の主な成果である $P(m,n)$ のパーフェクト性を利用すると一般の $(T_{m,n}(d),\boldsymbol{w})$ の多重彩色に対する多項式時間精度保証付き近似解法を構成できる．こうして構成した算法は高々 $\alpha(d)\omega+\mathrm{O}(d^3)$ 色を用いた多重彩色の解を算出する．ここで ω は重み付きクリーク数である．$d=1,\sqrt{3},2,\sqrt{7},3$ のとき，対応する近似率 $\alpha(d)$ はそれぞれ $(4/3),(5/3),(5/3),(7/4),(7/4)$ である．$d>1$ のとき $\alpha(d) \leq \left(1+\frac{2}{\sqrt{3}+\frac{2\sqrt{3}-3}{d}}\right)$ である．また本稿では，$(T_{m,n}(d),\boldsymbol{w})$ を $(4/3)\omega$ 色で多重彩色できるか否か判定する問題は NP-完全であることを示す．

キーワード　多重彩色，単位円グラフ，パーフェクトグラフ，近似算法

Multicoloring Unit Disk Graphs on Triangular Lattice Points

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Abstract　Given a pair of non-negative integers m and n, $P(m,n)$ denotes a subset of 2-dimensional triangular lattice points defined by $P(m,n) \stackrel{\text{def.}}{=} \{(x\boldsymbol{e}_1+y\boldsymbol{e}_2) \mid x \in \{0,1,\ldots,m-1\},\ y \in \{0,1,\ldots,n-1\}\}$ where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2,\sqrt{3}/2)$. Let $T_{m,n}(d)$ be an undirected graph defined on vertex set $P(m,n)$ satisfying that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. In this paper, we discuss a necessary and sufficient condition that $T_{m,n}(d)$ is perfect. More precisely, we show that [$\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect] if and only if $d \geqq \sqrt{n^2-3n+3}$. Given a non-negative vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$, a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ is an assignment of colors to $P(m,n)$ such that each vertex $v \in P(m,n)$ admits $w(v)$ colors and every adjacent pair of two vertices does not share a common color. We also give an efficient algorithm for muliticoloring $(T_{m,n}(d),\boldsymbol{w})$ when $P(m,n)$ is perfect. In general case, our results on the perfectness of $P(m,n)$ implies a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d),\boldsymbol{w})$. Our algorithm finds a multicoloring which uses at most $\alpha(d)\omega+\mathrm{O}(d^3)$ colors, where ω denotes the weighted clique number. When $d=1,\sqrt{3},2,\sqrt{7},3$, the approximation ratio $\alpha(d)=(4/3),(5/3),(5/3),(7/4),(7/4)$, respectively. When $d>1$, we showed that $\alpha(d) \leq \left(1+\frac{2}{\sqrt{3}+\frac{2\sqrt{3}-3}{d}}\right)$. We also showed the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ with strictly less than $(4/3)\omega$ colors.

Key words　multicoloring, unit disk graph, perfect graph, approximation algorithm

1. Introduction

Given a pair of non-negative integers m and n, $P(m,n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by

$$P(m,n) \stackrel{\text{def.}}{=} \left\{ (x\boldsymbol{e}_1 + y\boldsymbol{e}_2) \;\middle|\; \begin{array}{l} x \in \{0,1,2,\ldots,m-1\}, \\ y \in \{0,1,2,\ldots,n-1\} \end{array} \right\}$$

where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathrm{R}^2$ and a positive real d, a *unit disk graph*, denoted by (P,d), is an undirected graph with vertex set P such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. We denote the unit disk graph $(P(m,n),d)$ by $T_{m,n}(d)$.

Given an undirected graph H and a non-negative integer vertex weight $\boldsymbol{w}'$ of H, a *multicoloring* of $(H,\boldsymbol{w}')$ is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on $(H,\boldsymbol{w}')$ finds a multicoloring of $(H,\boldsymbol{w}')$ which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [4], minimum integer weighted coloring [15] or w-coloring [12].

In this paper, we study weighted unit disk graphs on triangular lattice points $(T_{m,n}(d),\boldsymbol{w})$. First, we show a necessary and sufficient condition that $T_{m,n}(d)$ is a perfect graph. If the graph is perfect, we can solve the multicoloring problem easily. Next, we propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d),\boldsymbol{w})$. Our algorithm is based on the well-solvable case that the given graph is perfect. For any $d \geqq 1$, our algorithm finds a multicoloring which uses at most

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right)\omega + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right)\lfloor d+1 \rfloor^2$$

colors, where ω denotes the weighted clique number. Table 1 shows the values of the above approximation ratio in case that d is small.

表 1
Table 1 Approximation ratio

d	$1\cdots$	$\sqrt{3}\cdots$	$\sqrt{7}\cdots$	$2\sqrt{3}\cdots$	$\sqrt{13}\cdots$	$\sqrt{19}\cdots$
ratio	4/3	5/3	7/4	2	9/5	2
d	$\sqrt{21}\cdots$	$3\sqrt{3}\cdots$	$\sqrt{31}\cdots$	$\cdots$		∞
ratio	11/6	2	13/7	$\cdots$		$1+2/\sqrt{3}$

We also show the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d),\boldsymbol{w})$ which uses strictly less than $(4/3)\omega$ colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and Reed [9] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [9], [12] independently gave (4/3)-approximation algorithms for this problem. In case that a given graph H is a square lattice graph or a hexagonal lattice graph, the graph H becomes bipartite and so we can obtain an optimal multicoloring of $(H,\boldsymbol{w}')$ in polynomial time (see [9] for example). Halldórsson and Kortsarz [5] studied planar graphs and partial k-trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [6], [8], [14]. Here we note that the approximation ratio of our algorithm is less than $1 + 2/\sqrt{3} < 2.155$ for any $d \geqq 1$.

2. Well-Solvable Cases and Perfectness

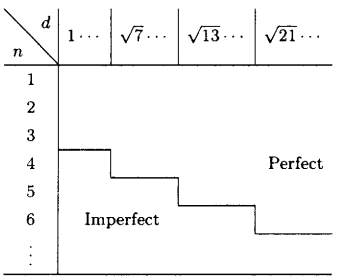
In this section, we discuss some well-solvable cases such that the multicoloring number is equivalent to the weighted clique number.

An undirected graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

[Theorem 1] When $n \geqq 1$ and $d \geqq 1$, we have the following;
[$\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect] if and only if
$d \geqq \sqrt{n^2 - 3n + 3}$.

Table 2 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small n and d.

表 2
Table 2 Perfectness and imperfectness



To show the above theorem, we introduce some definitions. We say that an undirected graph has a transitive orientation property, if each edge can be assigned a one-way direction in such a way that the resulting directed graph (V,F) satisfies

that $[(a,b) \in F$ and $(b,c) \in F$ imply $(a,c) \in F]$. An undirected graph which is transitively orientable is called *comparability graph*. The complement of a comparability graph is called *co-comparability graph*. It is well-known that every co-comparability graph is perfect.

[Lemma 1] For any integer $n \geqq 1$, if $d \geqq \sqrt{n^2-3n+3}$, then $T_{m,n}(d)$ is a co-comparability graph.

Proof: When $n \leqq 2$, the statement is trivial. It suffices to show that the complement of $T_{m,n}(d)$ is a comparability graph when $n \geqq 3$ and $d \geqq \sqrt{n^2-3n+3}$. Let $\overline{T}_{m,n}(d)$ be the complement of $T_{m,n}(d)$. We direct each edge in $\overline{T}_{m,n}(d)$ as follows. For any edge $e = \{v_1, v_2\}$ in $\overline{T}_{m,n}(d)$, we direct the edge e from v_1 to v_2 when the x-coordinate of v_1 is strictly less than that of v_2. We show that the obtained directed graph, denoted by G', satisfies the transitivity.

Clearly, G' is acyclic. Assume that G' contains a pair of directed edges (v_1, v_2) and (v_2, v_3). We denote the position of v_i by (x_i, y_i) where x_i and y_i are the x-coordinate and the y-coordinate, respectively. The definition of G' implies that $x_1 < x_2 < x_3$. In the following, we show that $x_2 - x_1 > d/2$.
(Case 1) Consider the case that $|y_2 - y_1| < (\sqrt{3}/2)(n-1)$. Then, it is clear that $|y_2 - y_1| \leqq (\sqrt{3}/2)(n-2)$. Since the distance between v_1 and v_2 is greater than d and $n \geqq 3$, we have that $x_2 - x_1 > \sqrt{d^2 - |y_2 - y_1|^2} \geqq \sqrt{d^2 - (3/4)(n-2)^2} > \sqrt{d^2 - (3/4)(n^2 - 3n + 3)} \geqq \sqrt{d^2 - (3/4)d^2} = d/2$.
(Case 2) Assume that $|y_2 - y_1| \geqq (\sqrt{3}/2)(n-1)$. Since $v_1, v_2 \in P(m,n)$, it is clear that $|y_2 - y_1| = (\sqrt{3}/2)(n-1)$. Without loss of generality, we can assume that $(x_2, y_2) = (x_1, y_1) + (x'\boldsymbol{e}_1 + (n-1)\boldsymbol{e}_2)$ for some integer x'. Since $0 < x_2 - x_1 = (n-1)/2 + x'$, we have $x' > -(n-1)/2$. If $x' \leqq -1$, then $|x_2 - x_1|^2 + |y_2 - y_1|^2 = (x' + (n-1)/2)^2 + (3/4)(n-1)^2 \leqq (-1 + (n-1)/2)^2 + (3/4)(n-1)^2 = n^2 - 3n + 3 \leqq d^2$ and it contradicts with the non-adjacency between v_1 and v_2 on $T_{m,n}(d)$. Thus the integrality of x' implies that $x' \geqq 0$ and $|x_2 - x_1| = (n-1)/2 + x' \geqq (n-1)/2 = |y_2 - y_1|/\sqrt{3}$. Then the inequalities $d^2 < |x_2 - x_1|^2 + |y_2 - y_1|^2 \leqq |x_2 - x_1|^2 + 3|x_2 - x_1|^2 = 4|x_2 - x_1|^2$ implies that $d/2 < x_2 - x_1$.

Similarly, we can show that $x_3 - x_2 > d/2$. Thus we have $x_3 - x_1 > d$ and the distance between v_1 and v_3 is greater than d. From the definition of G', the digraph G' contains the edge (v_1, v_3). ∎

The following lemma deals with the special case that $n = 3$, $d = 1$.

[Lemma 2] For any $m \in \mathrm{Z}_+$ and $1 \leqq \forall d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.

Proof: We only need to consider the case that $d = 1$, since $T_{m,n}(d) = T_{m,n}(1)$ when $1 \leqq d < \sqrt{3}$. Let H be an induced subgraph of $T_{m,3}(1)$. When $\omega(H) \leqq 2$, H has no 3-cycle. Then it is easy to show that H has no odd cycle and thus $\chi(H) = \omega(H)$, since H is bipartite. If $\omega(H) \geqq 3$, then it is clear that $\omega(H) = 3$ and $\chi(H) \leqq 3$, since $\omega(T_{m,3}(1)) = 3$ and $T_{m,3}(1)$ has a trivial 3-coloring. ∎

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.

From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph G has an *odd-hole*, if G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect. In the following, we denote a point $(x\boldsymbol{e}_1 + y\boldsymbol{e}_2) \in P(m,n)$ by $\langle x, y \rangle$.

[Lemma 3] If $1 \leqq d < \sqrt{7}$, then $\forall m \geqq 5$, $T_{m,4}(d)$ has at least one odd-hole.

Proof: If $1 \leqq d < \sqrt{3}$, then a subgraph induced by

$$\{ \langle 2,0\rangle, \langle 1,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle \}$$

is a 9-hole. If $\sqrt{3} \leqq d < 2$, then a subgraph induced by

$$\{ \langle 3,0\rangle, \langle 1,1\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,1\rangle \}$$

is a 7-hole. If $2 \leqq d < \sqrt{7}$, then a subgraph induced by

$$\{ \langle 2,0\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \langle 3,2\rangle, \langle 3,0\rangle \}$$

is a 5-hole. When $1 \leqq d < \sqrt{7}$, $T_{5,4}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

[Lemma 4] If $1 \leqq d < \sqrt{13}$, then $\forall m \geqq 6$, $T_{m,5}(d)$ has at least one odd-hole.

Proof: If $1 \leqq d < \sqrt{7}$, then odd-holes in the proof of Lemma 3 are induced subgraph of $T_{6,5}(d)$. If $\sqrt{7} \leqq d < 3$, then a subgraph induced by

$$\{ \langle 2,0\rangle, \langle 0,2\rangle, \langle 1,4\rangle, \langle 4,2\rangle, \langle 4,0\rangle \}$$

is a 5-hole. If $3 \leqq d < \sqrt{13}$, then a subgraph induced by

$$\{ \langle 3,0\rangle, \langle 0,3\rangle, \langle 2,4\rangle, \langle 5,3\rangle, \langle 5,0\rangle \}$$

is a 5-hole. When $1 \leqq d < \sqrt{13}$, $T_{6,5}(d)$ has at least one odd-hole, and hence the proof is completed. ∎

[Lemma 5] For any integer $n \geqq 4$, if $1 \leqq d < \sqrt{n^2-3n+3}$, then $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is imperfect.

Proof: In the following, we show that $\forall n \geqq 4$, if $1 \leqq d < \sqrt{n^2-3n+3}$, then $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ has at least one odd-hole, by induction on n. When $n = 4, 5$, it is clear from Lemmas 3 and 4, respectively.

Now we consider the case that $n = n' \geqq 6$ under the assumption that if $1 \leqq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3}$, then $\exists m' \in \mathrm{Z}_+$, $T_{m',n'-1}(d)$ has at least one odd-hole. If $1 \leqq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3} = \sqrt{n'^2 - 5n' + 7}$, then $T_{m',n'}(d)$ has at least one odd-hole, since $T_{m',n'-1}(d)$ is an induced subgraph of $T_{m',n'}(d)$. In the remained case that

$\sqrt{n'^2-5n'+7} \leqq d < \sqrt{n'^2-3n'+3}$, the set of points

$\{\langle n'-3,0\rangle, \langle 0,n'-2\rangle, \langle n'-4,n'-1\rangle, \langle 2n'-7,n'-2\rangle, \langle 2n'-6,0\rangle\}$

is contained in $P(m'',n')$, if $m''=2n'-5$. It is easy to see that the above five vertices induces a 5-hole of $T_{m'',n'}(d)$, when $n' \geqq 6$ and $\sqrt{n'^2-5n'+7} \leqq d < \sqrt{n'^2-3n'+3}$ ∎

Lemma 5 shows the imperfectness of every graph which violates a condition of Theorem 1. Thus, we completed a proof of Theorem 1. From the above lemmas, the following is immediate.

[Corollary 1] Let $d>1$ be a real number. Then, $T_{m,n}(d)$ is a co-comparability graph, if and only if $n \leqq \frac{3+\sqrt{4d^2-3}}{2}$.

Next, we consider the multicoloring problems. Given an undirected graph $G=(V,E)$ and vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^V$, the *multicoloring number* $\chi(G,\boldsymbol{w})$ is the least number of colors required in a multicoloring of $(G,\boldsymbol{w})$. The *weighted clique number* $\omega(G,\boldsymbol{w})$ is the weight of a maximum weight clique in $(G,\boldsymbol{w})$. It is clear that $\chi(G,\boldsymbol{w}) \geqq \omega(G,\boldsymbol{w})$. First, we describe a property which helps the multicoloring case theoretically. We say that an undirected graph G has an *odd-antihole*, if the complement of G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5.

[Lemma 6] Undirected graph $G=(V,E)$ is perfect if and only if $[\forall \boldsymbol{w} \in \mathrm{Z}_+^V,\ \chi(G,\boldsymbol{w})=\omega(G,\boldsymbol{w})\ ]$ holds.

Proof: Let $G_w=(V_w,E_w)$ be a graph obtained from G by replacing each vertex v with $w(v)$-clique. If two vertices u,v of G_w correspond to a unique vertex of G, we say that the pair $\{u,v\}$ belongs to a same class. It suffices to show that G is perfect if and only if G_w is perfect. Since G is an induced subgraph of G_w, the perfectness of G_w implies that G is perfect. Next we show that if G_w is not perfect then G is not perfect. In [1], it was shown that if a graph is not perfect, there exists an odd-hole or odd-antihole.

Consider the case that G_w contains an odd-hole C. If C contains a pair of vertices in a same class, it contradicts with the assumption that C is an induced subgraph. Otherwise, C is also an odd-hole of G and thus G is not perfect.

Consider the case that G_w contains an odd-antihole $\overline{C}$. We only need to consider the case that $\overline{C}$ contains a pair of vertices $\{u,v\}$ in a same class. Clearly, u and v are adjacent on G_w. Since the complement of $\overline{C}$ is an odd cycle, u has a pair of vertices u',u'' such that neither $\{u,u'\}$ nor $\{u,u''\}$ is adjacent pair. Since $\{u,v\}$ belongs to a same class, both $\{v,u'\}$ and $\{v,u''\}$ are non-adjacent pairs. Thus (u,u',v,u'') forms a 4-cycle in the complement of $\overline{C}$. It contradicts with the assumption that $\overline{C}$ is an odd-antihole. ∎

By combining Theorem 1 and Lemma 6 we have the following.

[Theorem 2] When $n \geqq 1$, the following property holds;

$[\forall m \in \mathrm{Z}_+$ and $\forall \boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)},\ \chi(T_{m,n}(d),\boldsymbol{w})=\omega(T_{m,n}(d),\boldsymbol{w})]$ if and only if $d \geqq \sqrt{n^2-3n+3}$.

Lastly, we discuss some algorithmic aspects. Assume that we have a co-comparability graph G and related digraph H which gives a transitive orientation of the complement of G. Then each independent set of G corresponds to a chain (directed path) of H. The multicoloring problem on G is essentially equivalent to the minimum size chain cover problem on H. Every clique of G corresponds to an anti-chain of H. Thus the equality $\omega(G)=\chi(G)$ is obtained from Dilworth's decomposition theorem [2]. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem (see [13] for example).

In the following, we describe an algorithm for multicoloring an weighted graph $(T_{m,3}(1),\boldsymbol{w})$. We denote the set of colors by $C^*=\{1,2,\ldots,\omega^*\}$ where $\omega^*=\omega(T_{m,3}(1),\boldsymbol{w})$. The following algorithm finds an assignment of colors $c: P(m,n) \to 2^{C^*}$ such that $\forall v \in P(m,n)$, $|c(v)|=w(v)$ and for every edge $\{u,v\} \in T_{m,3}(1)$, $c(u) \cap c(v)=\emptyset$.

For any $x \in \{0,1,\ldots,m-1\}$, we denote the points $x\boldsymbol{e}_1+2\boldsymbol{e}_2$, $x\boldsymbol{e}_1+1\boldsymbol{e}_1$, $x\boldsymbol{e}_1+0\boldsymbol{e}_2$ by t_{x+1}, u_{x+1}, v_x, respectively. Thus $\{t_1,t_2,\ldots,t_m\}$, $\{u_1,u_2,\ldots,u_m\}$ and $\{v_0,v_1,\ldots,v_{m-1}\}$ form a partition of $P(m,3)$. Without loss of generality, we can assume that $w(v_0)=w(t_m)=w(u_m)=0$. Our algorithm assigns colors to vertices in the following manner. Assume that we have a multicoloring $c: P' \to 2^C$ where $P'=\{t_1,t_2,\ldots,t_j\} \cup \{u_1,u_2,\ldots,u_j\} \cup \{v_0,v_1,\ldots,v_j\}$ satisfying that $\forall i \in \{1,2,\ldots,j\}$, $c(t_i) \subseteqq c(v_i)$ or $c(t_i) \supseteqq c(v_i)$. Next, we assign colors to u_{j+1}. Since $w(v_0)=w(t_m)=w(u_m)=0$, we can assume that $w(t_j) \geqq w(v_j)$ without loss of generality. Since $\{u_j,t_j,u_{j+1}\}$ is a 3-clique, $|c(u_j)|+|c(t_j)|+w(u_{j+1}) \leqq w^*$. Thus there exists a subset of colors C_1 with $|C_1|=w(u_{j+1})$ and C_1 is disjoint with $c(u_j) \cup c(t_j)$. Then we set $c(u_{j+1})$ to C_1. Next, we assign colors to t_{j+1}. there exists a set of colors $C_2 \subseteqq C^* \setminus (c(t_j) \cup c(u_{j+1}))$ with $|C_2|=w(t_{j+1})$, since $\{t_j,u_{j+1},t_{j+1}\}$ is a 3-clique. Then set $c(t_{j+1})$ to C_2. Lastly we assign colors to v_{j+1}.

(Case 1) If $w(t_{j+1}) \geqq w(v_{j+1})$, then set $c(v_{j+1})$ to a subset of $c(t_{j+1})$ whose cardinality is $w(v_{j+1})$.

(Case 2) Consider the case that $w(t_{j+1}) < w(v_{j+1})$ and $w(t_j)+w(t_{j+1}) \geqq w(v_j)+w(v_{j+1})$. Then there exists a subset of colors $C_3 \subseteqq c(t_j) \setminus c(v_j)$ whose cardinality is $w(v_{j+1})-w(t_{j+1})$. We set $c(v_{j+1})=c(t_{j+1}) \cup C_3$.

(Case 3) Consider the case that $w(t_{j+1}) < w(v_{j+1})$ and $w(t_j)+w(t_{j+1}) < w(v_j)+w(v_{j+1})$. Then we set $c(v_{j+1})=c(t_{j+1}) \cup (c(t_j) \setminus c(v_j)) \cup C_4$ where C_4 and $v(t_j) \cup c(u_{j+1}) \cup c(t_{j+1})$ are disjoint and $w(v_{j+1})=w(t_{j+1})+(w(t_j)-w(v_j))+$

$|C_4|$. Since $\{v_j, u_{j+1}, v_{j+1}\}$ is a 3-clique, it is easy to see that there exists such a subset of colors.

A naive implementation of the above procedure gives a pseudo polynomial time algorithm, since the algorithm maintains the set of colors C^* explicitly. If we represent the assigned set of colors by the union of some intervals and implement the above procedure carefully, we can obtain a polynomial time algorithm with respect to m.

3. Approximation Algorithm

In this section, we propose an approximation algorithm for multicoloring the graph $(T_{m,n}(d), \boldsymbol{w})$. When $d = 1$, McDiarmid and Reed [9] proposed an approximation algorithm for $(T_{m,n}(1), \boldsymbol{w})$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 1/3$ colors.

In the following, we propose an approximation algorithm for $(T_{m,n}(d), \boldsymbol{w})$ when $d > 1$. The basic idea of our algorithm is similar to the shifting strategy [7].

[Theorem 3] When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ such that the number of required colors is bounded by

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega(T_{m,n}(d), \boldsymbol{w}) + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \chi(T_{m,n}(d)).$$

Proof: We describe an outline of the algorithm. For simplicity, we define $K_1 = \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor$ and $K_2 = \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor + \lfloor \frac{2}{\sqrt{3}} d \rfloor$.

First, we construct K_2 vertex weights $\boldsymbol{w}'_k$ for $k \in \{0, 1, \ldots, K_2 - 1\}$ by setting

$$w'_k(x,y) = \begin{cases} 0, & y \in \{k, k+1, \ldots, k + \lfloor \frac{2}{\sqrt{3}} d \rfloor - 1\} \pmod{K_2}, \\ K_1 \left\lfloor \frac{w(x,y)}{K_1} \right\rfloor, & \text{otherwise.} \end{cases}$$

Next, we exactly solve K_2 multicoloring problems defined by K_2 pairs $(T_{m,n}(d), \boldsymbol{w}'_k)$, $k \in \{0, 1, \ldots, K_2 - 1\}$ and obtain K_2 multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), \boldsymbol{w}'_k) = \omega(T_{m,n}(d), \boldsymbol{w}'_k)$ for any $k \in \{0, 1, \ldots, K_2 - 1\}$. Put $\boldsymbol{w}'' = \boldsymbol{w} - \sum_{k=0}^{K_2-1} \boldsymbol{w}'_k$. Then each element of $\boldsymbol{w}''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), \boldsymbol{w}'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$colors. Lastly, we output the direct sum of $K_2 + 1$ multicolorings obtained above. The definition of the weight vector $\boldsymbol{w}'_k$ implies that $\forall k \in \{0, 1, \ldots, K_2 - 1\}$, $K_1\, \omega(T_{m,n}(d), \boldsymbol{w}'_k) \leqq \omega(T_{m,n}(d), \boldsymbol{w})$. Thus, the obtained multicoloring uses at most $(K_2/K_1)\omega(T_{m,n}(d), \boldsymbol{w}) + (K_1 - 1)\chi(T_{m,n}(d))$ colors. ∎

The following lemma gives the chromatic number of $T_{m,n}(d)$.

[Lemma 7] If m, n are sufficiently large, then $\chi(T_{m,n}(d)) = \widehat{d}^{\,2}$ where $\widehat{d}$ is the minimum Euclidean distance between two points in $P(m,n)$ subject to that distance being greater than d. Clearly, $d < \widehat{d} \leqq \lfloor d + 1 \rfloor$.

Proof: See McDiarmid [9] for example. ∎

When d is small, Table 1 shows the approximation ratio. The following corollary gives a simple upper bound of the approximation ratio.

[Corollary 2] For any $d \geqq 1$, we have

$$1 + \frac{\left\lfloor \frac{2}{\sqrt{3}} d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor} \leqq 1 + \frac{2}{\sqrt{3} + \frac{2\sqrt{3}-3}{d}}.$$

Here we note that if we apply our algorithm in the case that $d = 1$, then the algorithm finds a multicoloring which uses at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 6$ colors.

4. Hardness Result

In this section, we discuss the hardness of our problem. If we deal with a rational number d as an input data, then the problem for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ is NP-hard, since multicoloring of $(T_{m,n}(1), \boldsymbol{w})$ is shown to be NP-hard by McDiarmid and Reed [9]. It is clear that even if we consider the case that d is integer and we define the input size of d is also d (not $O(\log d)$), the problem remains NP-hard. Thus we only need to consider the case that d is a constant. The following lemma gives an idea of our proof of hardness.

[Lemma 8] For any $d \geqq 1$, the triangular lattice graph is an induced subgraph of $T_{m,n}(d)$.

Proof: Let $a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ be a farthest neighbor of $(0,0)$ on $T_{m,n}(d)$. Put $\boldsymbol{e}'_1 = (a, b)$, $\boldsymbol{e}'_2 = (\frac{1}{2}a - \frac{\sqrt{3}}{2}b, \frac{\sqrt{3}}{2}a + \frac{1}{2}b)$, and $P' = \{(x\boldsymbol{e}'_1 + y\boldsymbol{e}'_2) \in P(m,n) \mid x \in \mathrm{Z},\ y \in \mathrm{Z}\}$. Then the subgraph of $T_{m,n}(d)$ induced by P' is a triangular lattice graph. ∎

[Theorem 4] Let d be a constant rational number. Given a pair $(T_{m,n}(d), \boldsymbol{w})$, it is NP-complete to determine whether $(T_{m,n}(d), \boldsymbol{w})$ is multicolorable with strictly less than $(4/3)\omega(T_{m,n}(d), \boldsymbol{w})$ colors or not.

Proof: It is known to be NP-complete to determine the 3-multicolorability of a given 4-colorable weighted triangular lattice graph $(T_{m,n}(1), \boldsymbol{w})$ [9]. We can reduce a 3-multicolorability problem on $T_{m,n}(1)$ to a 3-multicolorability problem on $T_{m',n'}(d)$, by copying the vertex weights of the original problem onto a sparse triangular lattice defined in

表 3
Table 3 Unit disk graphs on square lattice points

n \ d	$1\cdots$	$\sqrt{2}\cdots$	$2\cdots$	$\sqrt{5}\cdots 2\sqrt{2}\cdots$
1				
2				Perfect
3				
4		Imperfect		

the proof of Lemma 8. The obtained instance has $O(mnd^2)$ vertices. Since d is a constant, the time complexity of the reduction procedure is bounded by a polynomial of the input size of $(T_{m,n}(1), \boldsymbol{w})$. ∎

5. Discussion

In this paper, we dealt with the triangular lattice. In the following, we discuss the square lattice. Given a pair of non-negative integers m and n, $Q(m,n) \stackrel{\text{def.}}{=} \{0,1,2,\ldots,m-1\} \times \{0,1,2,\ldots,n-1\}$ denotes the subset of 2-dimensional integer square lattice points. We denote the unit disk graph $(Q(m,n),d)$ by $S_{m,n}(d)$. In case that $d < \sqrt{2}$, it is clear that $S_{m,n}(d) = S_{m,n}(1)$ and the graph is bipartite for any m and n. If $d = \sqrt{2}$, we proposed a (4/3)-approximation algorithm for multicoloring $(S_{m,n}(\sqrt{2}), \boldsymbol{w})$ in our previous paper [11]. We also showed the NP-hardness of the problem.

Unfortunately, Theorem 1 is not extensible to the square lattice case. Table 3 shows the perfectness and imperfectness of unit disk graphs on the square lattice for small n and d. The perfectness of $T_{m,3}(\sqrt{2})$ was shown in [11]. The graph $S_{m,3}(2)$ contains a 5-hole: $\{(0,0),(2,0),(2,1),(1,2),(0.2)\}$.

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