ディジタルハーフトーニングへの応用に向けての魔方陣の一般化 (1)[1]

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要旨 本論文では魔方陣を一般化した零 $k \times k$ discrepancy 行列を導入する．この行列は $n \times n$ 行列で，その要素は 0 から n^2-1 の整数値からなり，どの $k \times k$ 部分行列の値も一致する．まず，n と k に対して零 $k \times k$ discrepancy 行列が構成できるかが問題となる．本論文では零 $k \times k$ discrepancy 行列の存在の条件を述べる．特に $n = k^m$ である時，零 $k \times k$ discrepancy 行列の構成法について詳述する．このような零 $k \times k$ discrepancy 行列はディジタル・ハーフトーニングに用いられるディザ行列として良い性質を持つと考えられる．

A Generalization of Magic Squares with Applications to Digital Halftoning (1)[1]

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Abstract A *semimagic square* of order n is an $n \times n$ matrix containing the integers $0, \ldots, n^2-1$ arranged in such a way that each row and column add up to the same value. We generalize this notion to that of a *zero $k \times k$-discrepancy matrix* by replacing the requirement that the sum of each row and each column be the same by that of requiring that the sum of the entries in each $k \times k$ square contiguous submatrix be the same. We show that such matrices exist if k and n are both even, and do not if k and n are are relatively prime. Further, the existence is also guaranteed whenever $n = k^m$, for some integers $k, m \geq 2$. We present a space-efficient algorithm for constructing such a matrix.

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1 Introduction

A Latin square of order n is an $n \times n$ matrix consists of n sets of the numbers 0 to $n-1$ arranged in such a way that no row or column contains the same two numbers. A *semimagic square* is an $n \times n$ matrix filled with the numbers $0, \ldots, n^2-1$ in such a way that the sum of the numbers in each row and each column are the same. Magic squares and related classes of integer matrices have been studied extensively (for an exhaustive bibliography, see [8] and the references therein).

This paper generalizes the notion of a semimagic square by replacing the requirement that all row and column sums be the same by the analogous requirement for all $k \times k$ contiguous square submatrices; we call such $n \times n$ matrices *zero* $k \times k$*-discrepancy matrices* of order (k,n). Let $\mathbb{N}(k,n)$ be the set of all such matrices. In this paper we prove that $\mathbb{N}(k,n)$ is non-empty if k and n are both even, and empty if they are relatively prime. Further, we show by an explicit construction that $\mathbb{N}(k,k^m) \neq \emptyset$ for any integers $k,m \geq 2$.

Another property plays an important role in the latter construction of zero $k \times k$-discrepancy matrices. A characterization of matrices with this property is also given in this paper.

Our investigation is motivated by an application described below, but intuitively we seek a matrix filled with distinct integers in an as uniform a manner as possible. The analogous geometric problem of distributing n points uniformly in a unit square has been studied extensively in the literature [6, 12]. Usually, a family of regions is introduced to evaluate the uniformity of a point distribution. If the points of an n-point set P are uniformly distributed, for any region R in the family the number of points in R should be close to $\frac{1}{n}\text{area}(R)$, where $\frac{1}{n}$ is the point density of P in the entire square. Thus, the *discrepancy of* P *in a region* R is defined as the difference between this value and the actual number of points of P in R. The *discrepancy of the point distribution* P with respect to the family of regions is defined by the maximum such difference, over all regions.

The problem of establishing discrepancy bounds for various classes of regions has been studied extensively [10]. One of the simplest families is that of axis-parallel rectangles for which $\Theta(\log n)$ bound is known [6, 12]. In the context of digital halftoning, a family of axis-parallel squares (contiguous square submatrices) over a matrix is appropriate for measuring the uniformity since human eye perception is usually modeled using weighted sum of intensity levels with Gaussian coefficients over square regions around each pixel [3]. Thus, the matrices discussed in this paper can be used as dither matrices in which integers are arranged in an apparently random manner to be used as variable thresholds. Small matrix size tends to generate visible artifacts. In this sense the dither matrix of size 8×8 designed by Floyd and Steinberg [7] may be too small. A common way to construct a larger dither matrix is to use local search under some criterion based on spatial frequency distribution of the resulting matrix. Such dither matrices are called blue-noise masks [13, 15, 16, 17]. One disadvantage of a blue-noise mask is its high space complexity. There appears to be no way to avoid storing the entire matrix. The zero $k \times k$-discrepancy matrices of order (k,k^m) we construct, on the other hand, are such that we can generate any one element by a simple integer calculation requiring only m seed matrices, each of size $k \times k$.

2 Preliminaries

Generalizing the notion of a semimagic square, we consider an $n \times n$ matrix containing all the integers $0, \ldots, n^2-1$ such that the entries contained in every contiguous $k \times k$ submatrix add up to the same value.

More formally, for integers $m,n > 1$, let $\mathbb{Z}(n,m)$ be the class of all $n \times n$ integer matrices with entries from the set $\{0, \ldots, m-1\}$ and let $\mathbb{Z}(n) \subset \mathbb{Z}(n,n^2)$ be the set of those $n \times n$ matrices which contain every value $0, \ldots, n^2-1$ exactly once.

A contiguous $k \times k$ submatrix (or *region*, hereafter) $R_{i,j} = R_{i,j}^{(k)}$ with its upper left corner at (i,j) is defined by

$$R_{i,j}^{(k)} = \{(i',j') \mid i' = i, \ldots, i+k-1 \text{ and } j' = j, \ldots, j+k-1\},$$

where indices are calculated modulo n.[4] Given a matrix P and a region $R_{i,j}$ of size k, $P(R_{i,j})$ denotes the sum of the elements of P in locations given by $R_{i,j}$. Analogously, define a $C_{i,j} = C_{i,j}^{(k)}$ to be the $k \times 1$ region of a matrix starting at (i,j) and $P(C_{i,j})$ to be the sum of elements of P in the locations given by $C_{i,j}$. We are interested in *all* $k \times k$ regions in an $n \times n$ matrix:

$$\mathscr{F}_{k,n} = \{R_{i,j}^{(k)} \mid i,j = 0,1,\ldots,n-1\}.$$

The $k \times k$*-discrepancy* $\mathscr{D}_{k,n}(P)$ of an $n \times n$ matrix P for the family $\mathscr{F}_{k,n}$ is defined as

$$\mathscr{D}_{k,n}(P) = \max_{R \in \mathscr{F}_{k,n}} P(R) - \min_{R' \in \mathscr{F}_{k,n}} P(R').$$

In this paper we focus on the existence of matrices $P \in \mathbb{Z}(n)$ with $k \times k$-discrepancy $\mathscr{D}_{k,n}(P) = 0$. In other words, we are interested in the existence and construction of matrices in $\mathbb{Z}(n)$ all of whose contiguous $k \times k$ submatrices have equal sums. Let $\mathbb{N}(k,n)$ be the set of all such zero-$k \times k$-discrepancy matrices of order (k,n).

Theorem 1 *The set* $\mathbb{N}(k,n)$ *of zero-*$k \times k$*-discrepancy matrices of order* (k,n) *has the following properties:*

(a) $\mathbb{N}(k,n)$ *is non-empty if* k *and* n *are both even.*

(b) $\mathbb{N}(k,n)$ *is empty if* k *and* n *are relatively prime.*

(c) $\mathbb{N}(k,n)$ *is empty if* k *is odd and* n *is even.*

(d) $\mathbb{N}(k,k^m)$ *is non-empty for any integers* k *and* m, $k \geq 2, m \geq 2$.

In addition, using the above results, we can show that there is no $n \times n$ matrix P that achieves zero-discrepancy simultaneously for the families $\mathscr{F}_{2,n}$ and $\mathscr{F}_{3,n}$, i.e., $\mathbb{N}(2,n) \cap \mathbb{N}(3,n) = \emptyset$,

Proof: [Theorem 1, parts (a)–(c)] To prove part (a), it suffices to show $\mathbb{N}(2,n) \neq \emptyset$ if n is even since any $k \times k$ region can be partitioned into 2×2 regions if k is even. (More generally, if k' divides k, $\mathbb{N}(k',n) \subset \mathbb{N}(k,n)$.)

Let $P = (p_{i,j}) \in \mathbb{Z}(n)$ be the matrix in which the numbers are arranged in the row-major order, that is, $p_{i,j} = in + j, i,j = 0,1,\ldots,n-1$. We classify matrix elements by their parity and rotate all the elements of odd parity by 180 degrees, i.e., for every (i,j) with $i+j$ odd, we swap $p_{i,j}$ and $p_{n-1-i,n-1-j}$. It is easily checked that the sum of elements in any 2×2 region is always $2n^2 - 2$. An example for $n = 8$ is shown in Fig. 1.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 \\ 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 \\ 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \\ 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 62 & 2 & 60 & 4 & 58 & 6 & 56 \\ 55 & 9 & 53 & 11 & 51 & 13 & 49 & 15 \\ 16 & 46 & 18 & 44 & 20 & 42 & 22 & 40 \\ 39 & 25 & 37 & 27 & 35 & 29 & 33 & 31 \\ 32 & 30 & 34 & 28 & 36 & 26 & 38 & 24 \\ 23 & 41 & 21 & 43 & 19 & 45 & 17 & 47 \\ 48 & 14 & 50 & 12 & 52 & 10 & 54 & 8 \\ 7 & 57 & 5 & 59 & 3 & 61 & 1 & 63 \end{bmatrix}$$

Figure 1: Parity rotation used in the proof of Theorem 1(a).

Turning to part (b), for a contradiction, assume that there exists a matrix $P \in \mathbb{Z}(n)$ in which the sum $P(R_{i,j})$ of elements of P over a $k \times k$ region $R_{i,j}$ is independent of i,j. In particular, $P(R_{i,j}) = P(R_{i,j+1}) = c$ for some constant c and therefore $P(C_{i,j}) = P(C_{i,j+k}) = c/k$, for all i,j.

Since k and n are relatively prime, the last relation implies that in fact $P(C_{i,j})$ is independent of j. Similar reasoning leads to the conclusion that it is independent of i as well. In particular, $P(C_{0,0}) =$

[4] Throughout this paper, index arithmetic is performed modulo matrix dimensions unless otherwise noted.

$P(C_{1,0})$, and therefore, by definition of $C_{0,0}$ and $C_{1,0}$, we must have $p_{i,0} = p_{i+k,0}$, contradicting our assumption that all the elements of P are distinct.

Finally, we consider part (c) of Theorem 1. Let $P \in \mathbb{Z}(n)$ and let k be odd and n be even. For a contradiction, assume that the values in any $k \times k$ region add up to the same number, say S, which must clearly be an integer. Summing $P(R_{i,j})$ over all i and j and observing that every entry in P appears precisely k^2 times in these sums, we conclude that

$$n^2 S = k^2(0+1+\cdots+n^2-1) = k^2\frac{n^2}{2}(n^2-1),$$

and therefore $S = k^2(n^2-1)/2$, which cannot be an integer if n is even and k is odd. This contradiction concludes the proof of Theorem 1(a)–(c). □

3 Construction of a $k^m \times k^m$-Matrix of Zero $k \times k$-Discrepancy

In this section we finish the proof of Theorem 1 by designing a $k^m \times k^m$ matrix from $\mathbb{Z}(k^m)$ for any positive integer m such that its $k \times k$-discrepancy is zero; in fact we present a proof of a stronger statement, see Theorem 5. We first show that there exists a $k^2 \times k^2$ matrix in $\mathbb{Z}(k^2)$ whose $k \times k$ discrepancy is zero, and then extend the result to $k^m \times k^m$ matrices.

Definition 1 *The* simple expansion $\tilde{P}$ *of a* $k \times k$ *matrix* P *is the matrix formed by repeating* P $k \times k$ *times, as follows:*

$$\tilde{P} = \begin{bmatrix} P & P & \cdots & P \\ P & P & \cdots & P \\ & & \vdots & \\ P & P & \cdots & P \end{bmatrix}.$$

Note that the $k \times k$*-discrepancy of* $\tilde{P}$ *is zero, as every* $k \times k$ *region contains the same set of numbers.*

Definition 2 *A* cyclic column shift *of a matrix* P *is the matrix obtained by shifting each column of* P *to the right (i.e., shifting the* j*th column to the* $(j+1)$*st column) and moving the last column to the first column. A* cyclic row shift *is similarly defined: It means shifting each row of* P *down to the next lower row (i.e., shifting* i*th row to the* $(i+1)$*st row) and moving the bottom row to the top row.*

We denote the matrix obtained by applying cyclic column shift c times and cyclic row shift r times to a $k \times k$ matrix P by $P^{(c,r)}$. That is, element (i,j) in P moves to position $((i+r) \bmod k, (j+c) \bmod k)$ in $P^{(c,r)}$. The *cyclic expansion* $\hat{P} = (\hat{p}_{i,j})$ of a $k \times k$ matrix P is a $k^2 \times k^2$ matrix defined by

$$\hat{P} = \begin{bmatrix} P^{(0,0)} & P^{(0,1)} & \cdots & P^{(0,k-1)} \\ P^{(1,0)} & P^{(1,1)} & \cdots & P^{(1,k-1)} \\ & & \vdots & \\ P^{(k-1,0)} & P^{(k-1,1)} & \cdots & P^{(k-1,k-1)} \end{bmatrix}.$$

An easy calculation shows that, for all i,j, $\hat{p}_{i,j} = p_{i',j'}$, with

$$i' = i - \lfloor j/k \rfloor \text{ and } j' = j - \lfloor i/k \rfloor \pmod{k}. \tag{1}$$

Definition 3 *A* constant-gap matrix $P = (p_{i,j})$ *is one for which*

$$p_{i,j} - p_{i,j'} = p_{i',j} - p_{i',j'} \tag{2}$$

holds for all choices of i, i', j, *and* j'.

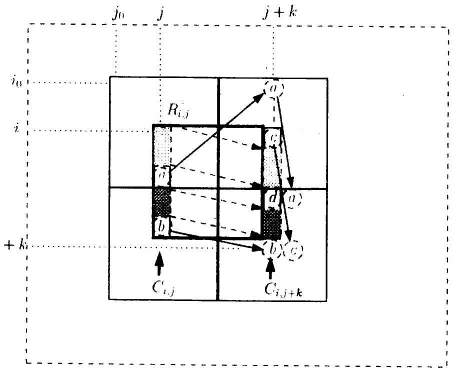

Figure 2: Illustration to the proof of Lemma 3.

Intuitively, this means that for any two columns j and j' the gap between elements in the same row is independent of the row, hence the "constant gap" name. Since (2) can be rewritten as

$$p_{i,j} - p_{i',j} = p_{i,j'} - p_{i',j'} \text{ or } p_{i,j} + p_{i',j'} = p_{i,j'} + p_{i',j},$$

rows and columns play symmetric roles in the definition. Moreover, a constant-gap matrix has the strong Monge property [1] since the sum of the main diagonal elements is equal to that of the off diagonal elements in any 2×2 submatrix.

Lemma 2 *The constant-gap property is preserved (1) under exchange of any two rows, (2) under exchange of any two columns, and, for square matrices, (3) under mirror reflection across the main diagonal.*

Proof: Immediate from the definition. □

The following lemma is a key to our construction of zero discrepancy matrices.

Lemma 3 *If P is a $k \times k$ constant-gap matrix, the $k \times k$-discrepancy of its cyclic expansion $\hat{P}$ is zero.*

Proof: Recall that $R_{i,j}$ and $C_{i,j}$ denote $k \times k$ and $k \times 1$ contiguous submatrices of $\hat{P}$, and $\hat{P}(R_{i,j})$ and $\hat{P}(C_{i,j})$ the sums of the corresponding elements in $\hat{P}$, respectively. We aim to prove $\hat{P}(R_{i,j}) = \hat{P}(R_{i,j+1})$, for all i, j. Together with $\hat{P}(R_{i,j}) = \hat{P}(R_{i+1,j})$, which is proven by a symmetric argument, this implies the statement of the theorem. By definition, $\hat{P}(R_{i,j+1}) - \hat{P}(R_{i,j}) = \hat{P}(C_{i,j+k}) - \hat{P}(C_{i,j})$; recall that all indices in $\hat{P}$ are calculated modulo k^2.

Put $i_0 = k\lfloor i/k \rfloor$ and $j_0 = k\lfloor j/k \rfloor$. To prove $\hat{P}(C_{i,j}) = \hat{P}(C_{i,j+k})$ we compare the two columns. As illustrated in Fig. 2, the part above the element (i_0+k-1, j) and the one above the element $(i+k-1, j)$ in $C_{i,j}$ both appear in $C_{i,j+k}$. Differences between $C_{i,j}$ and $C_{i,j+k}$ comprise only four elements: $a = \hat{p}_{i_0+k-1,j}, b = \hat{p}_{i+k-1,j}, c = \hat{p}_{i,j+k}, d = \hat{p}_{i_0+k,j+k}$. By cyclic row and column shifts, the four elements move in $\hat{P}$ as follows:

$$\begin{aligned}
a &= \hat{p}_{i_0+k-1,j} \rightarrow_r \hat{p}_{i_0,j+k} \rightarrow_c \hat{p}_{i_0+k,j+k+1},\\
b &= \hat{p}_{i+k-1,j} \rightarrow_r \hat{p}_{i+k,j+k},\\
c &= \hat{p}_{i,j+k} \rightarrow_c \hat{p}_{i+k,j+k+1},\\
d &= \hat{p}_{i_0+k,j+k},
\end{aligned}$$

where $\rightarrow_x$ represents the cyclic x shift and indices are calculated modulo k^2.

When $j \neq k-1 \pmod{k}$ and $i \neq k-1 \pmod{k}$, all four elements d: $\hat{p}_{i_0+k,j+k}$, a: $\hat{p}_{i_0+k,j+k+1}$, b: $\hat{p}_{i+k,j+k}$, and c: $\hat{p}_{i+k,j+k+1}$ belong to the same submatrix, namely, to $P^{(\lfloor i_0/k \rfloor+1, \lfloor j_0/k \rfloor+1)}$. Since the constant gap property is preserved by cyclic row and column shifts, we have $d-b=a-c$, and thus $a+b=c+d$. It may happen that $j+k$ and $j+k+1$ belong to different contiguous submatrices. In fact, it happens when $j=k-1 \pmod{k}$ and $i=k-1 \pmod{k}$. If $j=k-1 \pmod{k}$, we extend the sequence as follows:

$$\begin{aligned}
a &= \hat{p}_{i_0+k-1,j} \to_r \hat{p}_{i_0,j+k} \to_c \hat{p}_{i_0+k,j_0+k} \to_r \hat{p}_{i_0+k+1,j_0+2k} = \hat{p}_{i_0+k+1,j+k+1},\\
b &= \hat{p}_{i+k-1,j} \to_r \hat{p}_{i+k,j+k}, \to_r \hat{p}_{i+k+1,j+2k},\\
c &= \hat{p}_{i,j+k} \to_r \hat{p}_{i+1,j+2k}, \to_c \hat{p}_{i+k+1,j+k+1},\\
d &= \hat{p}_{i_0+k,j+k} \to_r \hat{p}_{i_0+k+1,j+2k}.
\end{aligned}$$

Then, the four elements lie in the same contiguous submatrix as we required. The case for $i=k-1 \pmod{k}$ is similar. This completes the proof of $\hat{P}(C_{i,j}) = \hat{P}(C_{i,j+k})$ and of the lemma. □

Lemma 4 *Let $P=(p_{ij})$ and $Q=(q_{ij})$ be matrices in $\mathbb{Z}(k)$. Combine $\hat{P}$ and $\tilde{Q}$ into a single matrix in two different ways, namely, put $C^{(1)} = C^{(1)}(P,Q) = (c_{ij}) = \tilde{Q} + k^2\hat{P}$ and $C^{(2)} = C^{(2)}(P,Q) = (c'_{ij}) = \hat{P} + k^2\tilde{Q}$. In other words, $c_{i,j} = \tilde{q}_{i,j} + k^2\hat{p}_{i,j}$ or $c'_{i,j} = \hat{p}_{i,j} + k^2\tilde{q}_{i,j}$, for all i,j. If P has the constant gap property, then*

(a) $C^{(1)}$ and $C^{(2)}$ are in $\mathbb{Z}(k^2)$, and

(b) their $k \times k$-discrepancy is zero.

In addition, $C^{(1)}$ and $C^{(2)}$ are distinct if $P \neq Q$. Thus $|\mathbb{N}(k,k^2)| \geq 2$.

Proof: The resulting matrices obviously belong to $\mathbb{Z}(k^2,k^4)$ and have zero discrepancy, as linear combinations of matrices of zero discrepancy. It is easy to check that $C^{(1)} \neq C^{(2)}$ if $P \neq Q$.

Thus to prove (b), it suffices to show that the elements of the matrices are all distinct. We focus on $C^{(1)}$, the argument for $C^{(2)}$ is analogous. Since $\hat{P}, \tilde{Q} \in \mathbb{Z}(k^2,k^2)$, $c_{ij} = c_{i'j'}$ implies $\hat{p}_{ij} = \hat{p}_{i'j'}$ and $\tilde{q}_{ij} = \tilde{q}_{i'j'}$. In other words, for a repeated value to occur in $C^{(1)}$, there must exist two positions (i,j) and (i',j') so that in $\hat{P}$ the same number occurs at (i,j) and (i',j'), and this also happens in $\tilde{Q}$. We argues that this is impossible. Indeed, since $\tilde{Q}$ is defined by just repeating the same matrix (with all entries distinct) k^2 times, each element stays in the same relative position in each submatrix. On the other hand, no element in a submatrix $P^{(c,r)}$ of $\hat{P}$ occurs in the same position in any other submatrix. □

We now prove a stronger version of Theorem 1d.

Theorem 5 $\mathbb{N}(k,k^m) \neq \emptyset$ *for any integers $k,m \geq 2$. Moreover, a zero-discrepancy matrix in $\mathbb{N}(k,k^m)$ can be explicitly computed in time linear in its size using $O(mk^2)$ space.*

Proof: We generalize the construction presented in Lemma 4. A matrix $M \in \mathbb{Z}(k^m)$ with zero discrepancy is defined using $m-1$ constant-gap matrices $P_0, P_1, \ldots, P_{m-2}$ of size k and one arbitrary matrix P_{m-1} of the same size (all in $\mathbb{Z}(k)$) as follows:

$$\begin{aligned}
M(i,j) = {} & k^{2(m-1)} P_0^{(\lfloor i/k^{m-1} \rfloor, \lfloor j/k^{m-1} \rfloor)}(i \bmod k, j \bmod k)\\
& + k^{2(m-2)} P_1^{(\lfloor i/k^{m-2} \rfloor \bmod k, \lfloor j/k^{m-2} \rfloor \bmod k)}(i \bmod k, j \bmod k)\\
& \ldots\\
& + k^2 P_{m-2}^{(\lfloor i/k \rfloor \bmod k, \lfloor j/k \rfloor \bmod k)}(i \bmod k, j \bmod k)\\
& + P_{m-1}(i \bmod k, j \bmod k).
\end{aligned}$$

For example, when $k=3$ and $m=3$, M is constructed as follows, where we have used R,Q,P for P_0,P_1,P_2, respectively, to avoid cumbersome notation.

$$
M=\begin{bmatrix}
P^{(0,0)} & P^{(0,0)} & P^{(0,0)} & P^{(0,1)} & P^{(0,1)} & P^{(0,1)} & P^{(0,2)} & P^{(0,2)} & P^{(0,2)}\\
P^{(0,0)} & P^{(0,0)} & P^{(0,0)} & P^{(0,1)} & P^{(0,1)} & P^{(0,1)} & P^{(0,2)} & P^{(0,2)} & P^{(0,2)}\\
P^{(0,0)} & P^{(0,0)} & P^{(0,0)} & P^{(0,1)} & P^{(0,1)} & P^{(0,1)} & P^{(0,2)} & P^{(0,2)} & P^{(0,2)}\\
P^{(1,0)} & P^{(1,0)} & P^{(1,0)} & P^{(1,1)} & P^{(1,1)} & P^{(1,1)} & P^{(1,2)} & P^{(1,2)} & P^{(1,2)}\\
P^{(1,0)} & P^{(1,0)} & P^{(1,0)} & P^{(1,1)} & P^{(1,1)} & P^{(1,1)} & P^{(1,2)} & P^{(1,2)} & P^{(1,2)}\\
P^{(1,0)} & P^{(1,0)} & P^{(1,0)} & P^{(1,1)} & P^{(1,1)} & P^{(1,1)} & P^{(1,2)} & P^{(1,2)} & P^{(1,2)}\\
P^{(2,0)} & P^{(2,0)} & P^{(2,0)} & P^{(2,1)} & P^{(2,1)} & P^{(2,1)} & P^{(2,2)} & P^{(2,2)} & P^{(2,2)}\\
P^{(2,0)} & P^{(2,0)} & P^{(2,0)} & P^{(2,1)} & P^{(2,1)} & P^{(2,1)} & P^{(2,2)} & P^{(2,2)} & P^{(2,2)}\\
P^{(2,0)} & P^{(2,0)} & P^{(2,0)} & P^{(2,1)} & P^{(2,1)} & P^{(2,1)} & P^{(2,2)} & P^{(2,2)} & P^{(2,2)}
\end{bmatrix}
$$

$$
+k^2\begin{bmatrix}
Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)}\\
Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)}\\
Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)}\\
Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)}\\
Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)}\\
Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)}\\
Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)} & Q^{(0,0)} & Q^{(0,1)} & Q^{(0,2)}\\
Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)} & Q^{(1,0)} & Q^{(1,1)} & Q^{(1,2)}\\
Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)} & Q^{(2,0)} & Q^{(2,1)} & Q^{(2,2)}
\end{bmatrix}
$$

$$
+k^4\begin{bmatrix}
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R\\
R & R & R & R & R & R & R & R & R
\end{bmatrix}.
$$

The remainder of the proof proceeds just as in that of Lemma 4; we omit the details. Recall that $P_m^{(a,b)}(i,j)=P_m((i+b) \bmod k,(j+a) \bmod k)$. Thus we can generate every entry of such a matrix without explicitly storing any information besides the m $k\times k$ matrices $P_0,\ldots,P_{m-1}$; the computation requires at most $O(m)$ additional working space. □

4 Concluding Remarks

We have introduced a discrepancy-based measure of uniformity of an $n\times n$ square matrix containing $0,1,\ldots,n^2-1$ as a generalization of a semimagic square. We have succeeded in obtaining matrices of even dimension with zero discrepancy for families of $2k\times 2k$ contiguous submatrices. For arbitrary k, we can construct a $k^m\times k^m$ matrix of $k\times k$-discrepancy zero. Moreover, such a matrix can be explicitly computed in time linear in its size using only $O(mk^2)$ space, which is a great advantage over the heuristic algorithms used for designing blue-noise masks in digital halftoning. This paper serves as a starting point of this type of investigation. A number of issues are still left open. One of the most interesting and attractive problems is to find low-discrepancy matrices, when n (dimension of the matrix) and k (the dimension of submatrix) are relatively prime to each other.

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