On the Three-Dimensional Orthogonal Drawing of Outerplanar Graphs (Extended Abstract)

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Abstract. It has been known that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing, while it has been open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing. We show in this paper that every outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.

Keywords: 3-D orthogonal drawing, bend, face, k-graph, outerplanar graph

1 Introduction

We consider the problem of generating orthogonal drawings of graphs in the space. The problem has obvious applications in the design of 3-D VLSI circuits and optoelectronic integrated systems [3, 5].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of a vertex of G. G is called a k-graph if $\Delta(G) \leq k$. The connectivity of a graph is the minimum number of vertices whose removal results in a disconnected graph or a single vertex graph. A graph is said to be k-connected if the connectivity of the graph is at least k.

It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph G is called a *3-D drawing* of G. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *2-D drawing* of G.

A *3-D orthogonal drawing* of a graph G is a 3-D drawing such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph G has a 3-D orthogonal drawing only if $\Delta(G) \leq 6$. A 3-D orthogonal drawing with no more than b bends per edge is called a *b-bend 3-D orthogonal drawing*.

Eades, Symvonis, and Whitesides [2], and Papakostas and Tollis [6] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Eades, Symvonis, and Whitesides [2] also posed an interesting open question of whether every 6-graph has a 2-bend 3-D orthogonal drawing. Wood [8] showed that every 5-graph has a 2-bend 3-D orthogonal drawing. Tayu, Nomura, and Ueno [7] showed that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing. Moreover, Nomura, Tayu, and Ueno [4] showed that every outerplanar 6-graph has a 0-bend 3-D orthogonal drawing if and only if it contains no triangle as a subgraph, while Eades, Stirk, and Whitesides [1] proved that it is NP-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing. Tayu, Nomura, and Ueno [7] also posed an interesting open question of whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

We shown in this paper the following theorem.

Theorem 1 *Every outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.*

The proof of Theorem 1 is constructive and provides a polynomial time algorithm to generate such a drawing for an outerplanar 5-graph. It is still open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

2 Preliminaries

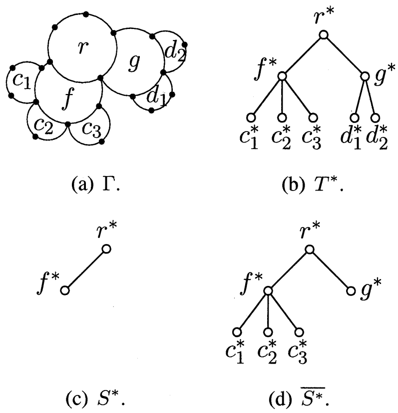
A 2-D drawing of a planar graph G is regarded as a graph isomorphic to G, and referred to as a *plane* *graph*. A plane graph partitions the rest of the plane into connected regions. A *face* is a closure of such a region. The unbounded region is referred to as the *external face*. We denote the boundary of a face f of a plane graph Γ by $b(f)$. If Γ is 2-connected then $b(f)$ is a cycle of Γ.

Given a plane graph Γ, we can define another graph Γ^* as follows: corresponding to each face f of Γ there is a vertex f^* of Γ^*, and corresponding to each edge e of Γ there is an edge e^* of Γ^*; two vertices f^* and g^* are joined by the edge e^* in Γ^* if and only if the edge e in Γ lies on the common boundary of faces f and g of Γ. Γ^* is called the *(geometric-)dual* of Γ.

A graph is said to be *outerplanar* if it has a 2-D drawing such that every vertex lies on the boundary of the external face. Such a drawing of an outerplanar graph is said to be *outerplane*. It is well-known that an outerplanar graph is a series-parallel graph. Let Γ be an outerplane graph with the external face f_o, and $\Gamma^* - f_o^*$ be a graph obtained from Γ^* by deleting vertex f_o^* together with the edges incident to f_o^*. It is easy to see that if Γ is an outerplane graph then $\Gamma^* - f_o^*$ is a forest. In particular, an outerplane graph Γ is 2-connected if and only if $\Gamma^* - f_o^*$ is a tree.

3 2-Connected Outerplanar Graphs

We first consider the case when G is 2-connected. Let G be a 2-connected outerplanar 5-graph and Γ be an outerplane graph isomorphic to G. Since Γ is 2-connected, $T^* = \Gamma^* - f_o^*$ is a tree. A vertex r^* of T^* is designated as a root, and T^* is considered as a rooted tree. If l^* is a leaf of T^* then l is called a *leaf face* of Γ. If g^* is a child of f^* in T^* then f is called the *parent face* of g, and g is called a *child face* of f in Γ. The unique edge in $b(f) \cap b(g)$ is called the *base* of g. We choose r^* so that $b(r) \cap b(f_o) \neq \emptyset$, and any edge in $b(r) \cap b(f_o)$ is defined as the base of r. Let S^* be a rooted subtree of T^* with root r^*. If S^* is consisting of just r^* then S^* is denoted by r^*. $\Gamma[S^*]$ is a subgraph of Γ induced by the vertices on boundaries of faces of Γ corresponding to the vertices of S^*. It should be noted that $\Gamma[S^*]$ is a 2-connected outerplane graph. Let f^* be a vertex in $V(T^*) - V(S^*)$ which is a child of a vertex $p^* \in V(S^*)$. $S^* + f^*$ is a subtree of T^* obtained from S^* by adding f^* and edge (f^*, p^*). Let $\overline{S^*}$ be a rooted subtree of T^* with root



(a) Γ. (b) T^*. (c) S^*. (d) $\overline{S^*}$.

Figure 1: Example of an outerplanae graph Γ, rooted tree T^*, subtrees S^* and $\overline{S^*}$ of T^*.

r^* induced by the vertices of S^* and the children of the vertices of S^*. Fig. 1 shows an example of an outerplane graph Γ, rooted tree T^*, and rooted subtrees S^* and $\overline{S^*}$.

For any face f of Γ, $b(f)$ is a cycle since Γ is 2-connected. Let

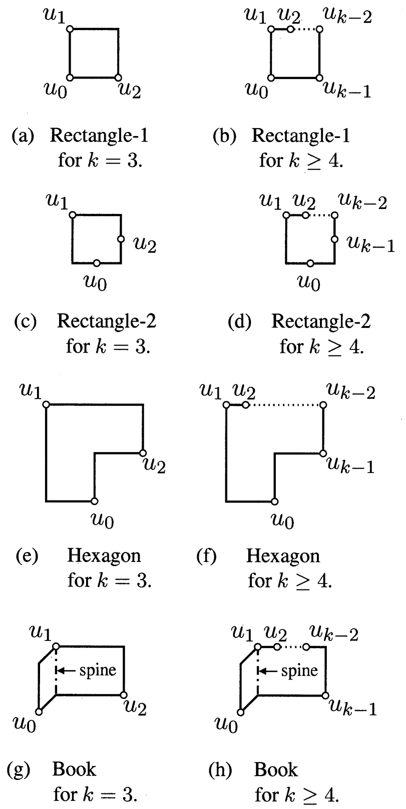
$$\begin{aligned} V(b(f)) &= \{u_i \mid 0 \leq i \leq k-1\}, \text{and} \\ E(b(f)) &= \{(u_0, u_{k-1})\} \cup \\ &\quad \{(u_i, u_{i+1}) \mid 0 \leq i \leq k-2\}, \end{aligned}$$

where (u_i, u_{k-1}) is the base of f. A 1-bend 3-D orthogonal drawing of $b(f)$ is said to be *canonical* if $b(f)$ is drawn as one of the following four configurations.

Configuration 1 (Rectangle-1) : If $k = 3$ then only (u_1, u_2) has a bend as shown in Fig. 2(a). If $k \geq 4$ then every edge has no bend, and $u_1, u_2, \ldots, u_{k-2}$ are drawn on a side of a rectangle as shown in Fig. 2(b).

Configuration 2 (Rectangle-2) : If $k = 3$ then every edge has a bend, and u_1 is at a corner of a rectangle as shown in Fig. 2(c). If $k \geq 4$ then only (u_0, u_{k-1}) and (u_0, u_1) have a bend, u_1, $u_2, \ldots, u_{k-2}$ are drawn on a side of a rectangle, and u_0 and u_{k-1} are on another different sides of the rectangle as shown in Fig. 2(d).

Configuration 3 (Hexagon) : If $k = 3$ then every edge has a bend as shown in Fig. 2(e). If $k \geq 4$ then only (u_0, u_{k-1}) and (u_0, u_1) have a bend, and $u_1, u_2, \ldots, u_{k-2}$ are on a side of a hexagon as shown in Fig. 2(f).



(a) Rectangle-1 for $k = 3$. (b) Rectangle-1 for $k \geq 4$. (c) Rectangle-2 for $k = 3$. (d) Rectangle-2 for $k \geq 4$. (e) Hexagon for $k = 3$. (f) Hexagon for $k \geq 4$. (g) Book for $k = 3$. (h) Book for $k \geq 4$.

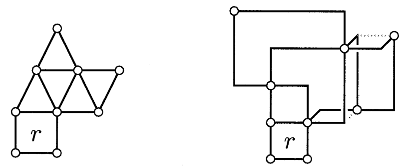
Figure 2: Rectangle-1 and -2, Hexagon, and Book.

Configuration 4 (Book) : A *book* is obtained from a rectangle by bending at a line segment, called the *spine*, parallel to a side of the rectangle. If $k = 3$ then every edge has a bend as shown in Fig. 2(g). If $k \geq 4$ then only (u_0, u_{k-1}), (u_0, u_1), and (u_{k-2}, u_{k-1}) have a bend, and u_1, $u_2, \ldots, u_{k-2}$ are on a side of a book as shown in Fig. 2(h).

A drawing of Γ is said to be *canonical* if every face is drawn canonically. Fig. 3 shows an example of an outerplane graph Γ, and a 1-bend 3-D orthogonal canonical drawing of Γ.

Roughly speaking, we will show that if $\Gamma[\overline{S^*}]$ has a 1-bend 3-D orthogonal canonical drawing then $\Gamma[\overline{S^* + f^*}]$ also has a 1-bend 3-D orthogonal canonical drawing, where f^* is a leaf of $\overline{S^*}$. The following theorem immediately follows by induction.

Theorem 2 *A 2-connected outerplanar 5-graph has a 1-bend 3-D orthogonal drawing.*



(a) Γ. (b) 1-bend 3-D orthogonal canonical drawing of Γ.

Figure 3: Example of Γ and 1-bend 3-D orthogonal canonical drawing of Γ.

3.1 Proof of Theorem 2

For a grid point $p = (p_x, p_y, p_z)$ and a vector $\boldsymbol{v} = (v_x, v_y, v_z)$, let $p+\boldsymbol{v}$ be the grid point $(p_x+v_x, p_y+v_y, p_z+v_z)$. For a unit vector $\boldsymbol{d}$, we denote $-\boldsymbol{d} = \overline{\boldsymbol{d}}$. Define that $\boldsymbol{e}_x = (1,0,0)$, $\boldsymbol{e}_y = (0,1,0)$, $\boldsymbol{e}_z = (0,0,1)$, and $D = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, \overline{\boldsymbol{e}_x}, \overline{\boldsymbol{e}_y}, \overline{\boldsymbol{e}_z}\}$. Every vector in D is called a *direction*.

A 3-D orthogonal drawing of a plane graph Γ can be regarded as a pair $\langle \phi, \rho \rangle$ of one-to-one mappings $\phi : V(\Gamma) \to \mathbb{Z}^3$ and ρ which maps edges (u, v) to internally disjoint paths on the 3-D grid $\mathcal{G}$ connecting $\phi(u)$ and $\phi(v)$. For a direction $\boldsymbol{d} \in D$ and a vertex $v \in V(\Gamma)$, $\langle \phi, \rho \rangle$ is said to be ***d*-free** at $\phi(v)$ if $\rho(e)$ does not contain the edge of $\mathcal{G}$ connecting $\phi(v)$ and $\phi(v) + \boldsymbol{d}$.

Let Γ be a 2-connected outerplane graph, and $\langle \phi, \rho \rangle$ be a 3-D orthogonal canonical drawing of Γ. Let f be a leaf face of Γ, and

$$\begin{aligned} V(b(f)) &= \{u_i \mid 0 \leq i \leq k-1\}, \\ E(b(f)) &= \{(u_0, u_{k-1})\} \cup \\ &\quad \{(u_i, u_{i+1}) \mid 0 \leq i \leq k-2\}, \end{aligned}$$

where (u_0, u_{k-1}) is the base of f. We define three unit vectors $\boldsymbol{d}_0(f, u_0)$, $\boldsymbol{d}_1(f, u_0)$, and $\boldsymbol{d}_2(f, u_0)$ as follows:

- If f is drawn as a rectangle-1, we define that $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from $\phi(u_{k-1})$ to $\phi(u_0)$, $\boldsymbol{d}_1(f, u_0) = \overline{\boldsymbol{d}_0(f, u_0)}$, and $\boldsymbol{d}_2(f, u_0)$ is a unit vector orthogonal to the rectangle.
- If f is drawn as a rectangle-2, let p be the bend of base (u_0, u_{k-1}). We define that $\boldsymbol{d}_1(f, u_0)$ is a unit vector orthogonal to the rectangle, and $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from $\phi(u_0)$ to p.
- If f is drawn as a hexagon, let p be the bend of base (u_0, u_{k-1}). We define that $\boldsymbol{d}_0(f, u_0)$ is the unit vector directed from p to $\phi(u_0)$, $\boldsymbol{d}_1(f, u_0)$

is the unit vector directed from p to $\phi(u_{k-1})$, and $d_2(f, u_0)$ is a unit vector orthogonal to the hexagon.

- If f is drawn as a book, let p be the bend of base (u_0, u_{k-1}). We define that $d_0(f, u_0)$ is the unit vector directed from $\phi(u_{k-1})$ to p, $d_1(f, u_0)$ is the unit vector directed from $\phi(u_0)$ to p, and $d_2(f, u_0)$ is the unit vector directed from the bend q of edge (u_{k-2}, u_{k-1}) to $\phi(u_{k-1})$.

A 1-bend 3-D orthogonal canonical drawing $\langle \phi, \rho \rangle$ of Γ is called a *1-bend 3-D orthogonal τ-drawing* of Γ if $\langle \phi, \rho \rangle$ satisfies one of the following conditions for every leaf face f. Let (u_0, u_{k-1}) be the base of a leaf face f.

Condition 1 : f is drawn as a rectangle-1 or hexagon, and

- if $d_\Gamma(u_0) \leq 4$ then $\langle \phi, \rho \rangle$ is $d_0(f, u_0)$-free or $d_2(f, u_0)$-free at $\phi(u_0)$,
- if $d_\Gamma(u_{k-1}) \leq 4$ then $\langle \phi, \rho \rangle$ is $d_1(f, u_0)$-free or $\overline{d_2(f, u_0)}$-free at $\phi(u_{k-1})$; (See Fig. 4(a) and (c).)

Condition 2 : f is drawn as a rectangle-2, and

- $d_\Gamma(u_0) = 5$,
- $\langle \phi, \rho \rangle$ is $d_0(f, u_0)$-free at $\phi(u_{k-1})$,
- if $d_\Gamma(u_{k-1}) \leq 3$ then $\langle \phi, \rho \rangle$ is $d_1(f, u_0)$-free at $\phi(u_{k-1})$. (See Fig. 4(b).)

Condition 3 : f is drawn as a book, and

- if $d_\Gamma(u_0) \leq 4$ then $\langle \phi, \rho \rangle$ is $d_0(f, u_0)$-free or $d_1(f, u_0)$-free at $\phi(u_0)$,
- if $d_\Gamma(u_{k-1}) \leq 4$ then $\langle \phi, \rho \rangle$ is $d_1(f, u_0)$-free or $\overline{d_0(f, u_0)}$-free at $\phi(u_{k-1})$,
- if $d_\Gamma(u_0) \leq 4$, $d_\Gamma(u_{k-1}) \leq 4$, $\langle \phi, \rho \rangle$ is not $d_0(f, u_0)$-free at $\phi(u_0)$, and $\langle \phi, \rho \rangle$ is not $d_1(f, u_0)$-free at $\phi(u_{k-1})$ then $\langle \phi, \rho \rangle$ is $d_2(f, u_0)$-free at $\phi(u_{k-1})$, and $d_\Gamma(u_{k-1}) = 4$,
- spine except for their ends is not used in the drawing; (See Fig. 4(d).)

Fig. 5 shows an example of an outerplane graph Γ, and a 1-bend 3-D orthogonal τ-drawing of Γ. In order to prove Theorem 2, it suffices to prove the following.

Theorem 3 *A 2-connected outerplanar 5-graph has a 1-bend 3-D orthogonal τ-drawing.*

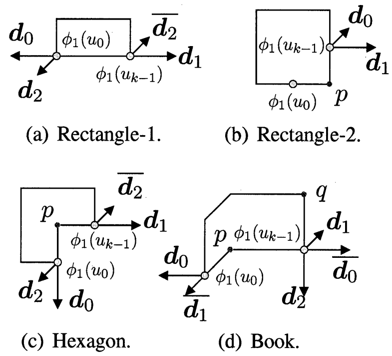


Figure 4: Directions for draiwng of face f.

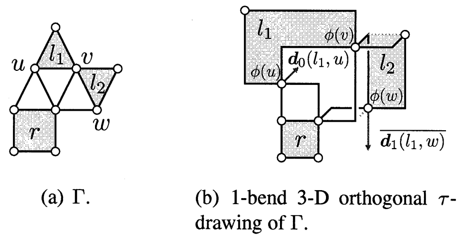


Figure 5: Example of Γ and 1-bend 3-D orthogonal τ-drawing of Γ.

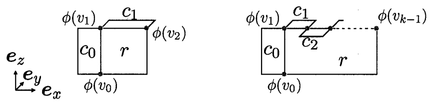
Proof (scketch). Let G be a 2-connected outerplanar 5-graph, Γ be an outerplane graph isomorphic to G, and $T^* = \Gamma^* - f_o^*$ be a tree rooted at r^*. We prove the theorem by induction. The basis of the induction is stated as follows.

Lemma 1 $\Gamma[\overline{r^*}]$ *has a 1-bend 3-D orthogonal τ-drawing.* □

Proof of Lemma 1. Let

$$\begin{aligned} V(b(r)) &= \{v_i \mid 0 \leq i \leq k-1\}, \\ E(b(r)) &= \{(v_0, v_{k-1})\} \cup \\ & \quad \{(v_i, v_{i+1}) \mid 0 \leq i \leq k-2\}, \end{aligned}$$

where (v_0, v_{k-1}) is the base of r. Let c_i be a child face of r with base (v_i, v_{i+1}) for $0 \leq i \leq k-2$, if any. Let $\langle \phi, \rho \rangle$ be a 1-bend 3-D orthogonal canonical drawing of $\Gamma[\overline{r^*}]$ as shown in Fig. 6, where c_i is drawn as rectangle-1, if any. Since $\langle \phi, \rho \rangle$ is $d_0(c_0, v_0)$-free at $\phi(v_0)$ and $d_1(c_0, v_0)$-free at $\phi(v_1)$, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_0, if any. If $k = 3$, by taking $d_2(c_1, v_1) = e_z$, $\langle \phi, \rho \rangle$ is $d_2(c_1, v_1)$-free at $\phi(v_1)$ and $d_1(c_1, v_1)$-



(a) $k = 3$. (b) $k \geq 4$.

Figure 6: Drawing of initial case.

free at $\phi(v_2)$. Therefore, $\langle \phi, \rho \rangle$ also satisfies Condition 1 for c_1, if any. Thus, we conclude that $\langle \phi, \rho \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{r^*}]$. If $k \geq 4$, by taking $\boldsymbol{d}_2(c_i, v_i) = \boldsymbol{e}_z$ for $1 \leq i \leq k-3$, $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(c_i, v_i)$-free at $\phi(v_i)$ and $\overline{\boldsymbol{d}_2(c_i, v_i)}$-free at $\phi(v_{i+1})$. Thus, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_i $(1 \leq i \leq k-3)$. Similarly, by taking $\boldsymbol{d}_2(c_{k-2}, v_{k-2}) = \overline{\boldsymbol{e}_z}$, $\langle \phi, \rho \rangle$ is $\boldsymbol{d}_2(c_{k-2}, v_{k-2})$-free at $\phi(v_{k-2})$ and $\overline{\boldsymbol{d}_2(c_{k-2}, v_{k-2})}$-free at $\phi(v_{k-1})$. Thus, $\langle \phi, \rho \rangle$ satisfies Condition 1 for c_{k-2}. So, we conclude that $\langle \phi, \rho \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{r^*}]$. □

Let S^* be a rooted subtree of T^* with root r^*. The inductive step is stated as follows.

Lemma 2 *If $\Gamma[\overline{S^*}]$ has a 1-bend 3-D orthogonal τ-drawing then $\Gamma[\overline{S^* + f^*}]$ also has a 1-bend 3-D orthogonal τ-drawing, where f^* is a leaf of $\overline{S^*}$.* □

Proof of Lemma 2 (scketch). Let $\Lambda_1 = \Gamma[\overline{S^*}]$ and $\Lambda_2 = \Gamma[\overline{S^* + f^*}]$, and let $\langle \phi_1, \rho_1 \rangle$ be a 1-bend 3-D orthogonal τ-drawing of Λ_1. We will construct a 1-bend 3-D orthogonal τ-drawing $\langle \phi_2, \rho_2 \rangle$ of Λ_2. Let

$$\begin{aligned} V(b(f)) &= \{v_i \mid 0 \leq i \leq k-1\}, \\ E(b(f)) &= \{(v_0, v_{k-1})\} \cup \\ &\quad \{(v_i, v_{i+1}) \mid 0 \leq i \leq k-2\}, \end{aligned}$$

where (v_0, v_{k-1}) is the base of f. We distinguish four cases depending on the configuration of f by $\langle \phi_1, \rho_1 \rangle$.

Case 1. f is drawn as a rectangle-1:
Without loss of generality, we assume that $\boldsymbol{d}_0(f, v_0) = \overline{\boldsymbol{e}_x}$, $\boldsymbol{d}_2(f, v_0) = \overline{\boldsymbol{e}_y}$, and z-coordinate of $\phi_1(v_1)$ is larger than that of $\phi_1(v_0)$. Let c_i be a child face of f with base (u_i, u_{i+1}) for $0 \leq i \leq k-2$, if any. We further distinguish three cases.

Case 1-1. $k = 3$:
Since $\langle \phi_1, \rho_1 \rangle$ is a 1-bend 3-D orthogonal τ-drawing, we distinguish four cases depending on free directions.

Case 1-1-1. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
Since $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$, canonical drawings of c_0 and c_1 can be added to $\langle \phi_1, \rho_1 \rangle$ as shown in Fig. 7(a), if any. Let $\langle \phi_2, \rho_2 \rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. If c_0 exists and $d_{\Lambda_2}(v_0) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_0)$ (see Fig. 7(a)). Since $\boldsymbol{d}_0(c_0, v_0) = \overline{\boldsymbol{e}_z}$ by definition, by taking $\boldsymbol{d}_2(c_0, v_0) = \overline{\boldsymbol{e}_y}$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_0(c_0, v_0)$-free or $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_0)$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{d}_2(c_0, v_0)}$-free at $\phi_2(v_1)$, since $\overline{\boldsymbol{d}_2(c_0, v_0)} = \boldsymbol{e}_y$. Thus, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0 if any. If c_1 exists and $d_{\Lambda_2}(v_2) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_2)$. Since $\boldsymbol{d}_1(c_1, v_1) = \overline{\boldsymbol{e}_y}$ by definition, by taking $\boldsymbol{d}_2(c_1, v_1) = \overline{\boldsymbol{e}_y}$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_1(c_1, v_1)$-free or $\overline{\boldsymbol{d}_2(c_1, v_1)}$-free at $\phi_2(v_2)$. So, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_1, if any, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_1, v_1)$-free at $\phi_2(v_1)$. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for the child faces of f.

We now show that $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face g of $\Gamma[S^* + f^*]$. Let u and u' be endvertices of the base of g. If $\{u, u'\} \cap \{v_0, v_{k-1}\} = \emptyset$ then $\langle \phi_2, \rho_2 \rangle$ is canonical for g, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}$-free at $\phi_2(w)$ if and only if $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}$-free at $\phi_1(w)$ for $w \in \{u, u'\}$. Otherwise, we assume without loss of generality that $u = v_0$. Then, $u' \neq v_{k-1}$. Since g is not a child face of f, none of the two edges of g incident with v_0 is contained in $E(b(f)) \cup E(b(c_0))$, and so $d_{\Lambda_2}(v_0) = 5$. Therefore, $\langle \phi_2, \rho_2 \rangle$ satisfies one of Conditions 1–3 for g. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-

(a) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.

(b) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.

(c) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

(d) $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 7: Drawing of child faces of f in Case 1-1.

D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-1-2. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
In this case, if $d_{\Gamma_1}(v_2) \leq 4$ then $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_2)$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. So, canonical drawings of c_0 and c_1 can be added to $\langle \phi_1, \rho_1 \rangle$ as shown in Fig. 7(b), if any. Let $\langle \phi_2, \rho_2 \rangle$ be the resultant 1-bend 3-D orthogonal canonical drawing. By the similar arguments to Case 1-1-1, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, if any. If c_1 exists and $d_{\Lambda_2}(v_2) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_2)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{e}_x$-free at $\phi_1(v_2)$ and $d_{\Lambda_2}(v_2) \leq 4$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_z$-free at $\phi_2(v_1)$. Since $\boldsymbol{d}_0(c_1, v_1) = \boldsymbol{e}_z$ and $\boldsymbol{d}_1(c_1, v_1) = \boldsymbol{e}_x$ by definition, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 3 for c_1. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

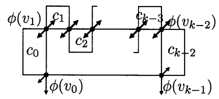
Case 1-1-3. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
In this case, if $d_{\Gamma_1}(v_0) \leq 4$ then $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 7(c), if any. If $d_{\Lambda_2}(v_0) \leq 4$ and c_0 exists then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\overline{\boldsymbol{e}_x}$-free at $\phi_1(v_0)$ and $d_{\Lambda_2}(v_0) \leq 4$. Thus by taking $\boldsymbol{d}_2(c_0, v_0) = \boldsymbol{e}_x$, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{d}_2(c_0, v_0)$-free at $\phi_2(v_1)$. Also, by the similar arguments to Case 1-1-1, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_1, if any. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-1-4. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ nor $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
In this case, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ if $d_{\Gamma_1}(v_0) \leq 4$ and $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_2)$ if $d_{\Gamma_1}(v_2) \leq 4$, since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_0 and c_1 as shown in Fig. 7(d), if any. Then by similar arguments to Case 1-1-3 and Case 1-1-2, we can see that $\langle \phi_2, \rho_2 \rangle$ satisfies Conditions 1 and 3 for c_0 and c_1, respectively. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

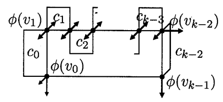
Case 1-2. $k \geq 4$, k is even.Similarly to Case 1-1, we distinguish four cases depending on free directions.

Case 1-2-1. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i ($0 \leq i \leq k-1$) as shown in Fig. 8(a), if any. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0. For each v_i with $1 \leq i \leq k-2$, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_y$- and $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_i)$. So, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i, if any. If $d_{\Lambda_2}(v_{k-1}) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free or $\boldsymbol{e}_y$-free at $\phi_2(v_{k-1})$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_y}$-free at $\phi_2(v_{k-2})$. So, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_{k-2}. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

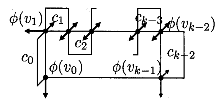
Case 1-2-2. $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
Since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_2)$. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i ($0 \leq i \leq k-1$) as shown in Fig. 8(b), if any. Then by similar arguments to Case 1-2-1, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_i with $0 \leq i \leq k-3$. Since $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_y$-free at $\phi_2(v_{k-3})$ and $\overline{\boldsymbol{e}_y}$-free



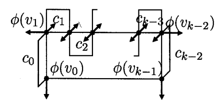
(a) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.



(b) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.



(c) $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.



(d) $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 8: Drawing of child faces of f in Case 1-2.

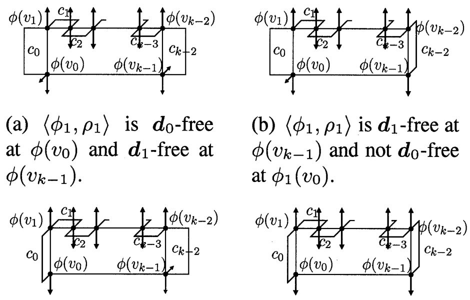
at $\phi_2(v_{k-2})$, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_{k-3}. If $d_{\Lambda_2}(v_{k-1}) \leq 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{e_z}$-free at $\phi_2(v_{k-1})$, since $\langle\phi_1,\rho_1\rangle$ is not e_x-free at $\phi_2(v_{k-1})$. So, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_{k-3}, since $\langle\phi_2,\rho_2\rangle$ is $\overline{e_y}$-free at $\phi_2(v_{k-1})$. By the similar arguments to Case 1-1-1, $\langle\phi_2,\rho_2\rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-2-3. $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$:
Since $\langle\phi_1,\rho_1\rangle$ satisfies Condition 1 for f, $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_2(f,v_0)$-free at $\phi_1(v_0)$. Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown in Fig. 8(c), if any. Then by similar arguments to Case 1-1-3, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_0, if any. Also, by the similar arguments to Case 1-1-1, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i for $2 \leq i \leq k-2$, if any. Since $\langle\phi_2,\rho_2\rangle$ is e_y-free at $\phi_2(v_1)$ and $\overline{e_y}$-free at $\phi_2(v_2)$, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_1, if any. By the similar arguments to Case 1-1-1, $\langle\phi_2,\rho_2\rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-2-4. $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$:
Since $\langle\phi_1,\rho_1\rangle$ satisfies Condition 1 for f, $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_2(f,v_0)$-free at $\phi_1(v_0)$ and $\overline{\boldsymbol{d}_2(f,v_0)}$-free at $\phi_1(v_2)$, Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown in Fig. 8(d), if any. Then by similar arguments to Case 1-2-3, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_0 and c_1, if any, by similar arguments to Case 1-2-2, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_{k-3} and c_{k-2}, if any, and by similar arguments to Case 1-2-1, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i for $2 \leq i \leq k-4$, if any. By the similar arguments to Case 1-1-1, $\langle\phi_2,\rho_2\rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-3. $k \geq 5$, k is odd. Similarly, we distinguish four cases depending on free directions.

Case 1-3-1. $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$:
Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown



(a) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi(v_0)$ and $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$.

(b) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_1$-free at $\phi(v_{k-1})$ and not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$.

(c) $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

(d) $\langle\phi_1,\rho_1\rangle$ is not $\boldsymbol{d}_0$-free at $\phi_1(v_0)$ or $\boldsymbol{d}_1$-free at $\phi_1(v_{k-1})$.

Figure 9: Drawing of child faces of f in Case 1-3.

in Fig. 9(a), if any. If c_0 exists and $d_{\Lambda_2}(v_0) \leq 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{e_z}$-free or $\overline{e_y}$-free at $\phi_2(v_0)$, since $d_{\Lambda_2}(v_0) \leq 4$ (see Fig. 9(a)). Since $\boldsymbol{d}_0(c_0,v_0) = \overline{e_z}$ by definition, by taking $\boldsymbol{d}_2(c_0,v_0) = \overline{e_y}$, $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{d}_0(c_0,v_0)$-free or $\boldsymbol{d}_2(c_0,v_0)$-free at $\phi_2(v_0)$. Also, $\langle\phi_2,\rho_2\rangle$ is $\overline{\boldsymbol{d}_0(c_0,v_0)}$-free at $\phi_2(v_1)$ by definition. Thus, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_0. For v_i with $2 \leq i \leq k-3$, $\langle\phi_2,\rho_2\rangle$ is e_y- and $\overline{e_y}$-free at $\phi_2(v_i)$. Also, $\langle\phi_2,\rho_2\rangle$ is e_y-free at $\phi_2(v_1)$ and at $\phi_2(v_{k-2})$. Thus, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i with $1 \leq i \leq k-3$ since $k-2 > 1$. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \leq 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{e_z}$-free or e_y-free at $\phi_1(v_{k-1})$. Also, $\langle\phi_2,\rho_2\rangle$ is e_z-free at $\phi_2(v_{k-2})$. So, by taking $\boldsymbol{d}_2(c_{k-2},v_{k-2}) = e_y$, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_{k-2}. By the similar arguments to Case 1-1-1, $\langle\phi_2,\rho_2\rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle\phi_2,\rho_2\rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[S^* + f^*]$.

Case 1-3-2. $\langle\phi_1,\rho_1\rangle$ is $\boldsymbol{d}_0(f,v_0)$-free at $\phi_1(v_0)$ and not $\boldsymbol{d}_1(f,v_0)$-free at $\phi_1(v_2)$:
Since $\langle\phi_1,\rho_1\rangle$ satisfies Condition 1 for f, $\langle\phi_1,\rho_1\rangle$ is $\overline{\boldsymbol{d}_2(f,v_0)}$-free at $\phi_1(v_2)$. Let $\langle\phi_2,\rho_2\rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle\phi_1,\rho_1\rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown in Fig. 9(b), if any. By the similar arguments to Case 1-3-2, $\langle\phi_2,\rho_2\rangle$ is canonical for c_i with $0 \leq i \leq k-3$, if any. If c_{k-2} exists and $d_{\Lambda_2}(v_{k-1}) \leq 4$ then $\langle\phi_2,\rho_2\rangle$ is $\overline{e_z}$-free, since $\langle\phi_1,\rho_1\rangle$ is not e_x-free at $\phi_1(v_{k-1})$. Also, $\langle\phi_2,\rho_2\rangle$ is e_z-free at $\phi_2(v_{k-2})$. So, $\langle\phi_2,\rho_2\rangle$ is $\boldsymbol{d}_0(c_{k-2},v_{k-2})$-free at $\phi_2(v_{k-2})$ and $\overline{\boldsymbol{d}_0(c_{k-2},v_{k-2})}$-free at $\phi_2(v_{k-1})$. Thus, $\langle\phi_2,\rho_2\rangle$ satisfies Condition 1 for c_i with

$1 \leq i \leq k-3$, if any, since $k-2 > 1$. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

Case 1-3-3. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
Since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition 1 for f, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$. Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown in Fig. 9(c), if any. If c_0 exists and $d_{\Lambda_2}(v_0) \leq 4$ then $\langle \phi_2, \rho_2 \rangle$ is $\overline{\boldsymbol{e}_z}$-free at $\phi_2(v_0)$, since $\langle \phi_1, \rho_1 \rangle$ is not $\overline{\boldsymbol{e}_x}$-free at $\phi_1(v_0)$. Also, $\langle \phi_2, \rho_2 \rangle$ is $\boldsymbol{e}_z$-free at $\phi_2(v_1)$. Thus, $\langle \phi_2, \rho_2 \rangle$ satisfies Condition 1 for c_0, if any. By similar arguments to Case 1-3-2, $\langle \phi_2, \rho_2 \rangle$ is canonical for c_i with $1 \leq i \leq k-2$, if any. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face.

Case 1-3-4. $\langle \phi_1, \rho_1 \rangle$ is not $\boldsymbol{d}_0(f, v_0)$-free at $\phi_1(v_0)$ and $\boldsymbol{d}_1(f, v_0)$-free at $\phi_1(v_2)$:
Since $\langle \phi_1, \rho_1 \rangle$ satisfies Condition $\overline{1 \text{ for } f}$, $\langle \phi_1, \rho_1 \rangle$ is $\boldsymbol{d}_2(f, v_0)$-free at $\phi_1(v_0)$ and $\overline{\boldsymbol{d}_2(f, v_0)}$-free at $\phi_1(v_2)$, Let $\langle \phi_2, \rho_2 \rangle$ be a 1-bend 3-D orthogonal canonical drawing obtained from $\langle \phi_1, \rho_1 \rangle$ by adding canonical drawings of c_i $(0 \leq i \leq k-1)$ as shown in Fig. 9(d), if any. By the similar arguments to Cases 1-3-2 and 1-3-3, $\langle \phi_2, \rho_2 \rangle$ is canonical for c_i with $0 \leq i \leq k-2$. By the similar arguments to Case 1-1-1, $\langle \phi_2, \rho_2 \rangle$ also satisfies one of Conditions 1–3 for any other leaf face. Thus, we conclude that $\langle \phi_2, \rho_2 \rangle$ is a 1-bend 3-D orthogonal τ-drawing of $\Gamma[\overline{S^* + f^*}]$.

The following remaining cases can be proved similarly. The proof is omitted in the extended abstract due to space limitation.

Case 2. f is drawn as a rectangle-2:

Case 3. f is drawn as a hexagon:

Case 4. f is drawn as a book: □

4 General Outerplanar Graphs

In this section, we shall complete the proof of Theorem 1. We assume without loss of generality that G is a connected outerplanar 5-graph. Let $G_1, G_2, \ldots, G_m$ be 2-connected components of G. It is well-known that $E(G)$ can be partitioned into $E(G_1), E(G_2), \ldots,$ and $E(G_m)$. An *adjacent graph* A_G of G is defined as follows: $V(A_G) = \{G_1, G_2, \ldots, G_m\}$, and $(G_i, G_j) \in E(A_G)$ if and only if $V(G_i) \cap V(G_j) \neq \emptyset$. It is easy to see that A_G is connected. Suppose $(G_1, G_2, \ldots, G_m)$ is a preorder of $V(A_G)$ obtained by applying DFS on A_G. Then a subgraph H_i of G induced by the vertices in $\bigcup_{k=1}^{i} V(G_k)$ is connected for $1 \leq i \leq m$. We prove Theorem 1 by induction on i. Since $H_1 = G_1$ is a 2-connected outerplanar 5-graph, we know by Theorem 2 that H_1 has a 1-bend 3-D orthogonal drawing. The inductive step is stated as follows.

Lemma 3 *For $1 \leq i \leq m-1$, if H_i has a 1-bend 3-D orthogonal drawing then H_{i+1} has a 1-bend 3-D orthogonal drawing.* □

This proves Theorem 1 since $H_m = G$. The proof of Lemma 3 is omitted in the extended abstract due to space limitation.

References

[1] P. Eades, C. Strik, and S. Whitesides, "The techniques of Komolgorov and Bardzin for three-dimensional orthogonal graph drawings," Information Processing Letters, vol.60, pp.97–103, 1996.

[2] P. Eades, A. Symvonis, and S. Whitesides, "Three-dimensional orthogonal graph drawing algorithms," Discrete Applied Mathematics, vol.103, pp.55–87, 2000.

[3] F.T. Leighton and A.L. Rosenberg, "Three-dimensional circuit layouts," SIAM J. Comput., vol.15, no.3, pp.793–813, 1986.

[4] K. Nomura, S. Tayu, and S. Ueno, "On the orthogonal drawing of outerplanar graphs," IEICE Trans. Fundamentals, vol.E88-A, no.6, pp.1583–1588, 2005.

[5] S. Obenaus and T. Szymanski, "Embedding of star graphs into optical meshes without bends," J. of Parallel and Distributed Computing, vol.44, no.2, pp.97–106, 1997.

[6] A. Papakostas and I. Tollis, "Algorithm for incremental orthogonal graph drawing in three dimensions," J. Graph Algorithms and Applications, vol.3, no.4, pp.81–115, 1999.

[7] S. Tayu, K. Nomura, and S. Ueno, "On the three-dimensional orthogonal drawing of series-parallel graphs," Proceedings of 2008 IEEE International Symposium on Circuits and Systems, 2008. to appear.

[8] D. Wood, "Optimal three-dimensional orthogonal graph drawing in the general position model," Theoretical Computer Science, vol.299, no.1–3, pp.151–178, 2003.