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Group Strategy-proof Mechanisms for Shuttle Facility Games

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Abstract: We study the game-theoretical structure of a scenario where a decision maker has to determine locations of stations in a transportation system. We introduce a new model on facility games, called the "shuttle facility game." A facility F is defined to be an interval with two stations over a transportation line. Then, the decision maker wishes to design a mechanism that given as input a set of intervals reported by each player, where I_i represents the commuting route of player i, determines a location for F. The profit of a facility location is defined based on the "convenience" to each player, such as the distance to the facility. A player i may try to manipulate the output of the mechanism by strategically misreporting I_i to get a higher profit. We formulate two shuttle facility games: the fixed-length and the flexible-length shuttle facility game; and prove that each admits a group strategy-proof mechanism. We prove that the social profit is also maximized by a location of F determined by our group strategy-proof mechanism, that is, a decision maker can find a location of F so that the social profit is maximized and group strategy-proofness is attained at the same time.

Keywords: mechanism design, transportation network, strategic agents, benefit ratio

1. Introduction

We investigate an extended model on facility games that applies to a transportation system. First we overview some existing facility games. Next we introduce new models in order to investigate the transportation system.

1.1 Facility Games

In the *facility game*, a decision maker wishes to design a *mechanism* that determines a location of a facility based on the votes of players. That is, for a set of candidate locations of a facility and a set of players with various profits, the decision maker designs a mechanism as a collective decision making system. The most preferable candidate for a player is a candidate that achieves the largest profit for her. Players are strategic, in the sense that a player can vote for some candidates other than her privately known most preferable candidate, in order to get a more preferable result for her. Voting which aims to manipulate the decision of a mechanism is called *strategic voting*.

To prevent strategic voting, the decision maker is interested in a mechanism that satisfies the following properties: strategy-proofness and group strategy-proofness. A mechanism is called a *strategy-proof mechanism* if no player can benefit by strategic voting. Moreover, a mechanism is called a *group strategy-proof mechanism* if there is no coalition of players such that each member of the coalition can simultaneously benefit by strategic voting. It immediately follows that if a mechanism is group strategy-proof, then it is also strategy-proof.

The decision maker wishes to design a mechanism which outputs a candidate such that the *social benefit*, i.e., the sum of individual profits is as large as possible. However, a strategy-proof mechanism does not necessarily output a candidate which maximizes the social benefit. The *benefit ratio* of a mechanism is defined to be the ratio between the largest social benefit and the social benefit obtainable by the mechanism.

In the *desirable facility game*, the most preferable location of the facility for any player is the location of the player, and her profit decreases with the distance between herself and the facility. The *obnoxious facility game* is a facility game where a mechanism determines a location of a facility that is undesirable for the players, for example, a waste treatment plant. For any player the least preferable location of the facility is the location of the player, and her profit increases with the distance between herself and the facility.

It is common in the literature that each player has a "benefit" or "utility" as her profit, their difference being that utility functions depend only on the decided facility location, whereas benefit functions also depend on the players' reports. Moulin [14] studied the desirable facility game in the line space under the condition that all players' utilities are single-peaked functions, gave necessary and sufficient conditions of strategy-proofness under such conditions, and designed a group strategy-proof mechanism with benefit ratio one. Subsequent studies gave characterizations of the desirable facility game in the multi-dimensional Euclidean space [3] and tree metrics [18]. Hakimi [7] showed that when the candidate space is defined to be the set of all points on a simple undirected graph $G = (V, E)$, including points on edges, the op-

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timal solution of this problem is a point on a vertex of the graph G, and an optimal solution can be found in $O(|V|)$ time. Several studies [1], [10], [11], [17] have been made on mechanisms for the desirable facility game, and Alon et al. [1] gave an analysis on the benefit ratios of group strategy-proof mechanisms on arbitrary undirected graphs.

Cheng et al. [5] first introduced and studied the obnoxious facility game, and designed a group strategy-proof mechanism for the obnoxious facility game in a line segment such that candidates are the two endpoints, with benefit ratio 3. Ibara and Nagamochi [8] showed that there exists no group strategy-proof mechanism in the line metric that has three or more distinct candidates. Ibara and Nagamochi [8] also characterized 2-candidate strategy-proof mechanisms in the line metric. Oomine and Nagamochi [15] showed that for the obnoxious facility game in the tree metric, there exists group strategy-proof mechanism if and only if there exists a special point such that each candidate is at the same distance from this point. It is known that finding an optimal location that maximizes the sum of all players' benefits is NP-hard in the space defined by all points on a simple graph [19].

There are several studies on the *two-facility game* where the locations for two desirable facilities are to be determined, and the profit decreases with respect to the distance between a player's location and the nearest facility location. Procaccia and Tennenholtz [17] designed a group strategy-proof mechanism for the two-facility game in the line space. Lu et al. [10] studied the two-facility game in the line, the circle, and general spaces.

Dekel et al. [6] showed that there exists a group strategy-proof mechanism with benefit ratio 3 for the desirable facility game where player i reports a given number w_i of locations. Mei et al. [13] investigated the obnoxious facility game with multiple reports per player in the line space.

Several studies propose possible relaxations of strategy-proofness [2], [4], [9], [16]. Oomine et al. [16] studied the following relaxation of strategy-proofness by introducing a parameter $\lambda \geq 1$. A mechanism is called *λ-strategy-proof mechanism* if no player can gain more than λ times her primary benefit by strategically misreporting her location. Moreover, a mechanism is called a *λ-group strategy-proof mechanism* if there is no coalition of players such that each member of the coalition can simultaneously gain more than λ times her primary benefit by strategically misreporting their locations. A 1-group strategy-proof mechanism is equivalent to the previously defined group strategy-proof mechanism. Oomine et al. [16] characterized λ-strategy-proof mechanisms in the line metric with two candidates, and they investigated the trade-off between λ-group strategy-proofness and the benefit ratio. They designed a λ-group strategy-proof mechanism whose benefit ratio is $1 + 2/\lambda$, where the benefit ratio approaches 1 as the parameter λ tends to infinity. Further, they showed lower bounds on the benefit ratio obtainable by any λ-strategy-proof mechanisms. These lower bounds are almost tight, in the sense that for odd number of players, as the number of players gets larger, the value approaches the benefit ratio of $1 + 2/\lambda$ obtainable by the proposed λ-group strategy-proof mechanism, and exactly match when the number of players is even.

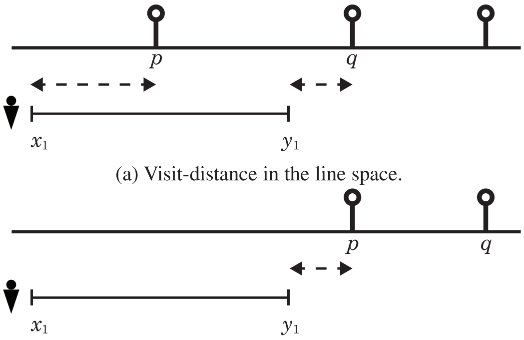


Fig. 1 The visit-distance and the intersection distance in the line space. Interval $[x_i, y_i]$ is the report of player i. (a) The points p, q are nearest stations from x_i, y_i, respectively. The dashed arrows illustrate the visit-distance. (b) Interval $[p, q]$ is the interval of stations. The dashed arrow illustrates the intersection distance.

1.2 The Shuttle Facility Game

We investigate a scenario where an extended modeling on facility games is necessary. Suppose that a policy planner has to determine the stations in a transportation system, for example the distribution of bus stops or railway stations that serve a community of individuals living in a given area. A certain number of individuals use the transportation system daily for commuting between their home and their destinations such as work place or school. Each individual uses the stations that are nearest to her home and destination. If the locations of the stations are not efficient for her, then she walks from her home to her destination directly. Each individual wishes that there are stations as near as possible to her home and her destination. We define

- the *visit-distance* for an individual to be the sum of distances from her home and her destination to the nearest respective stations,
- the *walking distance* for an individual to be the minimum of the visit-distance and the direct distance between her home and her destination, and,
- the *intersection distance* for an individual to be the minimum distance between a point in the interval of stations and a point in her interval.

To locate some number of stations, the planner asks all individuals to report two locations, namely, the end points of their commuting route. The planner determines the locations of stations based on the reports from individuals, considering the walking distances over all individuals.

We model the above scenario as a new model of a facility game. Let Ω be a set of locations. Let N be a set of players which represents all the individuals residing in the area Ω, and each player $i \in N$ has two major points of interest $x_i, y_i \in \Omega$, which represent her home location and her destination. Each player i wishes to minimize her walking distance. The decision maker wishes to design a mechanism that given as input a set of pairs of locations $\{x_i, y_i\}$ reported by each player $i \in N$, determines the locations of the stations. A group $S \subseteq N$ of players may collude and collectively report locations different from their privately known points of interest in order to decrease the walking distance for each player in the group S. The main objective is to propose a mechanism that will counteract such strategic behavior (group strategy-proofness), while respecting the collective votes of players (with a low benefit ratio).

We introduce two "shuttle facility games." First, we introduce the facility game where Ω is the one-dimensional space, the number of stations to be selected is two, the distance between the two stations is flexible, and the profit of player i is measured by certain functions including the walking distance and the intersection distance. A mechanism determines two locations of stations, based on pairs of locations $\{x_i, y_i\}$ reported by each player $i \in N$. We call the game the *flexible-length shuttle facility game*. This game is described as follows.

Designing a Mechanism for the Flexible-length Shuttle Facility game

Input: A set $N = \{1, 2, \dots, n\}$ of players,
a set of candidate intervals with flexible length for the facility location,
a set of intervals that are candidates for reports of players, and
a set of profit functions p_i for each player $i \in N$.

Output: A mechanism that outputs a facility location based on the reports of players.

Second, we introduce the facility game where Ω is the one-dimensional space, the number of stations to be selected is two, the distance between the two stations is a given real $k \geq 0$, and the profit of a player i is measured by certain functions including the walking distance and the intersection distance. A mechanism determines two locations of stations, based on pairs of locations $\{x_i, y_i\}$ reported by each player $i \in N$. We call the game the *fixed-length shuttle facility game* and the special case with $k = 0$, the *pit-stop facility game*. These games are described as follows.

Designing a Mechanism for the Fixed-length Shuttle Facility Game

Input: A set $N = \{1, 2, \dots, n\}$ of players,
a set of candidate intervals with fixed length $k \geq 0$ for the facility location,
a set of intervals that are candidates for reports of players, and
a set of profit functions p_i for each player $i \in N$.

Output: A mechanism that outputs a facility location based on the reports of players.

Designing a Mechanism for the Pit-stop Facility Game

Input: A set $N = \{1, 2, \dots, n\}$ of players,
a set of candidate points for the facility location,
a set of intervals that are candidates for reports of players, and
a set of profit functions p_i for each player $i \in N$.

Output: A mechanism that outputs a facility location based on the reports of players.

We investigate two types of profit functions for players that are common in the literature of facility games, namely, benefit functions and utility functions, and define strategy-proofness in terms of both benefit functions and utility functions in Section 2.2. We define interval-peaked benefit functions and interval-peaked utility functions as a generalization of the walking distance and intersection distance.

The fixed-length shuttle facility game with profits based on intersection-distance also applies to different scenarios, for instance, scheduling. For example, a planner asks to determine the opening time for her shop based on time intervals reported by customers, and each customer prefers that the opening time intersects the time interval reported by her, as illustrated in **Fig. 2**.

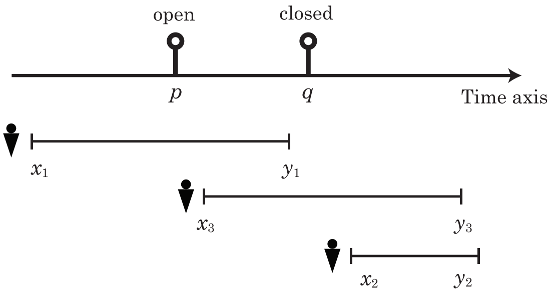


Fig. 2 A scheduling scenario. The interval $[p, q]$ is the opening time for a shop, and interval $[x_i, y_i]$ is the report of customer i of her preferred time to visit the shop. Customers 1 and 3 are satisfied since their reports intersect with the opening interval $[p, q]$, but customer 2 is dissatisfied since her reported interval does not intersect the opening interval $[p, q]$.

1.3 Our Results

In this research, we prove that

- there exists a group strategy-proof mechanism for the pit-stop facility game with interval-peaked profit functions in the line space, and we design such a mechanism,
- this mechanism is optimal, in the sense that it has benefit ratio one for the pit-stop facility game in the line space with benefit functions based on visit-distances,
- the mechanism can be adopted to obtain a group strategy-proof mechanism for the fixed-length shuttle facility game with interval-peaked profit functions, and,
- there exists a group strategy-proof mechanism for the flexible-length shuttle facility game in the line space with profit functions based on walking distances.

In Section 2, we define the notions of strategy-proofness and group strategy-proofness both with a benefit measure and a utility measure. In Section 3, we consider the pit-stop facility game, the fixed-length shuttle facility game, and the flexible-length shuttle facility game with a benefit measure. In Section 4, we consider the pit-stop facility game, the fixed-length shuttle facility game, and the flexible-length shuttle facility game with a utility measure. In Section 5, we summarize our results and propose some directions for future work.

2. Preliminaries

In this section, we define some notation pertaining to models of facility games.

2.1 Facility Games

Let $\mathbb{R}$ denote the set of real numbers. For two real numbers $x, y \in \mathbb{R}$, let $[x, y]$ denote the closed interval $\{z \in \mathbb{R} \mid x \leq z \leq y\}$. Let $\mathcal{R}$ denote the family of closed intervals $[x, y]$ such that $x, y \in \mathbb{R}$. For convenience, for functions whose arguments are from the family of closed intervals $\mathcal{R}$, such as $g : \mathcal{R} \times \mathbb{R} \to \mathbb{R}$, we explicitly write the arguments, for example $g(x, y, t)$ denotes $g([x, y], t)$.

For a set A, let $|A|$ denote the number of elements of A, and for a positive integer m, let $A^{\langle m \rangle}$ denote the family of all mul-

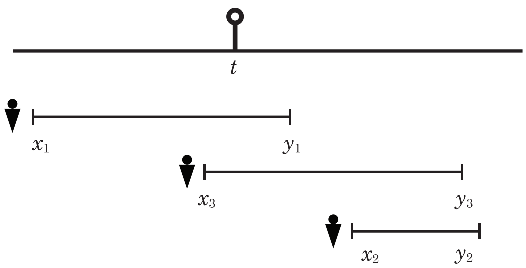


Fig. 3 The pit-stop facility game in the line space. Point t is the location of the facility. Interval $[x_i, y_i]$ is the report of player i.

tisets $\{a_1, a_2, \ldots, a_m\}$ such that $a_i \in A$, $i = 1, 2, \ldots, m$. Let $N = \{1, 2, \ldots, n\}$ be a set of players. For a subset $S \subseteq N$, let $\overline{S}$ denote $N \setminus S$. Similarly, for a player $i \in N$, let $\bar{i}$ denote $N \setminus \{i\}$. For a fixed non-negative real number k, define $\mathcal{K}_k \triangleq \{[x, x+k] \mid x \in \mathbb{R}\}$ to be the set of all closed intervals of length k.

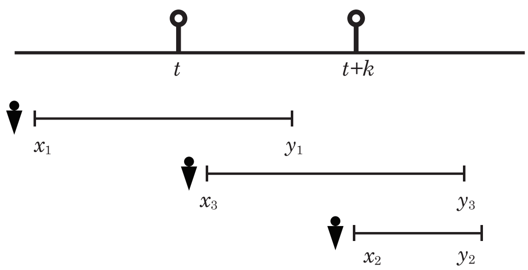
We introduce mechanisms for facility games with one facility. Let Ω_{facility} be a set of candidates of facility locations. Let Ω_{player} be a set of candidates for reports of players. To decide a location of the desirable facility in the set Ω_{facility}, each player reports one element of Ω_{player}. Let $\chi_i \in \Omega_{\text{player}}$ be the report of player i. We call the multiset $\chi = \{\chi_1, \chi_2, \ldots, \chi_n\}$ a *profile*. For a profile χ and a subset $S \subseteq N$, let χ_S denote the multiset $\{\chi_i \mid i \in S\}$. A *mechanism* is defined to be a function $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$ that outputs a location of the facility based on a given profile.

Next, we formalize the concept of a facility game. We consider two types of profit functions for facility locations in order to evaluate mechanisms. A higher value of an evaluation function implies that the output of the mechanism is more preferable.

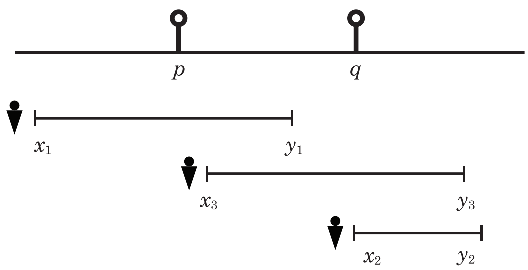
First, we introduce the concept of *benefit* as an evaluation function. The benefit $\beta_i : \Omega_{\text{player}} \times \Omega_{\text{facility}} \to \mathbb{R}$ of player i is defined as an evaluation function over the set of facility locations and reports of players, and its value increases with the preference of the facility's location for player i. A player can strategically report a candidate that is not most preferable for her in order to get a more preferable output from a mechanism. Our aim is to design a mechanism that outputs a more preferable location if each player reports her most preferable candidate. We formulate the following problem of designing a mechanism for the facility game with a benefit measure, which we call FG-BM.

FG-BM$(N, \Omega_{\text{facility}}, \Omega_{\text{player}}, \mathcal{B})$
Input: A set $N = \{1, 2, \ldots, n\}$ of players,
a set Ω_{facility} of candidates for the facility location,
a set Ω_{player} of candidates for reports of players, and
a set $\mathcal{B} = \left\{\beta_i : \Omega_{\text{player}} \times \Omega_{\text{facility}} \to \mathbb{R} \mid i \in N\right\}$ of benefit functions for each player $i \in N$.
Output: A mechanism $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$.

When the facility location is a point in the line and reports of players are intervals in the line, we call the special case FG-BM$(N, \mathbb{R}, \mathcal{R}, \mathcal{B})$ of the FG-BM *the pit-stop facility game with benefit measure*, PFG-BM. **Figure 3** illustrates the pit-stop facility game in the line space. When the facility location is an interval with a fixed length $k \geq 0$ in the line and reports of players are intervals in the line, we call the special case FG-BM$(N, \mathcal{K}_k, \mathcal{R}, \mathcal{B})$ of the FG-BM *the fixed-length shuttle facility game with benefit measure*, FL-SFG-BM. **Figure 4** (a) illustrates the fixed-length



(a) The fixed-length shuttle facility game in the line space.



(b) The flexible-length shuttle facility game in the line space.

Fig. 4 Shuttle facility games. An interval $[x_i, y_i]$ is the report of player i. (a) The fixed-length shuttle facility game; Interval $[t, t+k]$ is the location of the facility. (b) The flexible-length shuttle facility game; Interval $[p, q]$ is the location of the facility.

shuttle facility game in the line space. When the facility location is an interval with flexible length in the line and reports of players are intervals in the line, we call the special case FG-BM$(N, \mathcal{R}, \mathcal{R}, \mathcal{B})$ of the FG-BM *the flexible-length shuttle facility game with benefit measure*, SFG-BM. Figure 4 (b) illustrates the flexible-length shuttle facility game in the line space.

Next, we introduce the concept of *utility* as an evaluation function. The utility $u_i : \Omega_{\text{facility}} \to \mathbb{R}$ of player i is defined to be an evaluation function over the set of facility locations, and its value increases with the preference of the facility's location for player i. We formulate the following problem of designing a mechanism for the facility game with a utility measure, which we call FG-UM.

FG-UM$(N, \Omega_{\text{facility}}, \Omega_{\text{player}}, \mathcal{U})$
Input: A set $N = \{1, 2, \ldots, n\}$ of players,
a set Ω_{facility} of candidates for the facility location,
a set Ω_{player} of candidates for the reports of players, and
a set $\mathcal{U} = \left\{u_i : \Omega_{\text{ facility}} \to \mathbb{R} \mid i \in N\right\}$ of utility functions for each player $i \in N$.
Output: A mechanism $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$.

When the facility location is a point in the line and reports of players are intervals in the line, we call the special case FG-UM$(N, \mathbb{R}, \mathcal{R}, \mathcal{U})$ of the FG-UM *the pit-stop facility game with utility measure*, PFG-UM. When the facility location is an interval with fixed length $k \geq 0$ in the line and reports of players are intervals in the line, we call the special case FG-UM$(N, \mathcal{K}_k, \mathcal{R}, \mathcal{U})$ of the FG-UM *the fixed-length shuttle facility game with utility measure*, FL-SFG-UM. When the facility location is an interval with flexible length in the line and reports of players are intervals in the line, we call the special case FG-UM$(N, \mathcal{R}, \mathcal{R}, \mathcal{U})$ of the FG-UM *the flexible-length shuttle facility game with utility measure*, SFG-UM.

2.2 Strategy-proofness and Group Strategy-proofness

Strategy-proof and group strategy-proof mechanisms do not provide any incentive for players to report any candidates other than their truly most preferable ones. In this section, we formally define the notions of strategy-proof and group strategy-proof mechanisms.

Given a set N of players, a set Ω_{facility} of candidates for facility locations, a set Ω_{player} of candidates for reports of players and a set $\mathcal{B}$ of benefit functions β_i, for the FG-BM$(N, \Omega_{\text{facility}}, \Omega_{\text{player}}, \mathcal{B})$, a mechanism $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$ is said to be *group strategy-proof* (GSP for short) if for any non-empty subset $S \subseteq N$ of players and two profiles $\chi, \chi' \in \Omega_{\text{player}}^{\langle n \rangle}$ such that $\chi_{\overline{S}} = \chi'_{\overline{S}}$, there exists a player $i \in S$ satisfying

$$\beta_i(\chi_i, f(\chi)) \geq \beta_i(\chi_i, f(\chi')). \tag{1}$$

In essence, assuming a profile χ to be a truthful one and profile χ' to be strategically reported one where a set S of players collude and report strategically, Eq. (1) states that there exists at least one player in the set S such that the benefit obtained using the truthful report and the output $f(\chi)$ of the mechanism using the truthful report, is at least as high as the benefit hypothetically obtained by using the truthful report of the player and the output $f(\chi')$ of the mechanism resulting from the strategic report.

With weaker constraints, a mechanism $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$ is said to be *strategy-proof* (SP for short) if Eq. (1) holds for all singleton sets $S \subseteq N$ i.e., with $|S| = 1$.

For a profile χ and a point $t \in \Omega_{\text{facility}}$, the *social benefit* is defined to be the total benefit $\sum_{i=1}^{n} \beta_i(\chi_i, t)$ of all players.

Given a mechanism $f : \Omega_{\text{player}}^{\langle n \rangle} \to \Omega_{\text{facility}}$, the *benefit ratio* of the mechanism f is defined to be

$$\sup \left\{ \frac{\sup\left\{\sum_{i=1}^{n} \beta_i(\chi_i, t) \mid t \in \Omega_{\text{facility}}\right\}}{\sum_{i=1}^{n} \beta_i(\chi_i, f(\chi))} \mid \chi \in \Omega_{\text{player}}^{\langle n \rangle} \right\}.$$

Strategy-proofness and group strategy-proofness are similarly defined for the facility game with a utility measure, except that utility measures by definition do not depend on the players' profile, but only on the mechanism's outcome, and hence we omit stating them explicitly here.

2.3 Function Peakedness

For a function $g : \Omega_{\text{facility}} \to \mathbb{R}$, an inclusion-wise maximal set $\Omega(g) \in \Omega_{\text{facility}}$ is called the *peak* of g if for any $y \in \Omega(g), z \in \Omega_{\text{facility}} \setminus \Omega(g)$ it holds that $g(y) > g(z)$.

We follow the definition of Moulin [14] for single-peakedness. A function $g : \mathbb{R} \to \mathbb{R}$ is called *single-peaked* if there exists a real number $a \in \mathbb{R}$ that for any $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with $x_1 < x_2 < a < x_3 < x_4$, satisfies

$$g(x_1) \leq g(x_2) < g(a) > g(x_3) \geq g(x_4).$$

Further, a function $g : \mathbb{R} \to \mathbb{R}$ is called *interval-peaked* if there exist two numbers $a, b \in \mathbb{R}, a \leq b$, that for any $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ with $x_1 < x_2 < a \leq x_3 \leq b < x_4 < x_5$, satisfy

$$g(x_1) \leq g(x_2) < g(a) = g(x_3) = g(b) > g(x_4) \geq g(x_5).$$

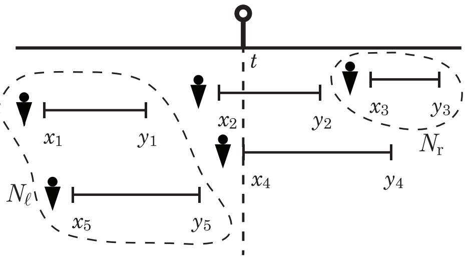


Fig. 5 The left set $N_\ell(\chi, t)$ and the right set $N_{\mathrm{r}}(\chi, t)$ of a point $t \in \mathbb{R}$, enclosed by dashed lines.

2.4 The Lowest Balanced Mechanism and the Respective Median Mechanism

Before introducing our mechanism, we define some notation. For a profile $\chi = \{[x_1, y_1], [x_2, y_2], \ldots, [x_n, y_n]\}$, we define the *left set* $N_\ell \subseteq N$ and the *right set* $N_{\mathrm{r}} \subseteq N$ of a point $t \in \mathbb{R}$ to be

$$N_\ell(\chi, t) \triangleq \{i \in N \mid t > y_i\} \quad \text{and} \quad N_{\mathrm{r}}(\chi, t) \triangleq \{i \in N \mid t < x_i\},$$

as illustrated in **Fig. 5**.

We call the infimum $p^* \in \mathbb{R}$ of the set $\{p \in \mathbb{R} \mid |N_{\mathrm{r}}(\chi, p)| \leq |N_\ell(\chi, p)|\}$ the *lowest balanced point* of χ.

A mechanism $f(\chi)$ is called the *lowest balanced mechanism* if for a given profile χ the mechanism outputs the lowest balanced point of χ.

For a profile χ and a mechanism f, let $[f_x(\chi), f_y(\chi)]$ denote the output of mechanism f. First, we define a median function. For a finite set H of real numbers, $\mathrm{med}(H) \triangleq z \in H$ such that

$$|\{w \in H \mid w \leq z\}| \geq |H|/2 \text{ and } |\{w \in H \mid w \geq z\}| > |H|/2. \tag{2}$$

A mechanism $f : \mathcal{R}^{\langle n \rangle} \to \mathcal{R}$ is called the *respective median mechanism* if for a profile $\chi = \{[x_1, y_1], \ldots, [x_n, y_n]\}$, it holds that

$$f(\chi) = [\mathrm{med}(\{x_i \mid i = 1, \ldots, n\}), \mathrm{med}(\{y_i \mid i = 1, \ldots, n\})].$$

3. Mechanisms for the Shuttle Facility Game with a Benefit Measure

In this section, we consider three facility games, the flexible-length shuttle facility game with a benefit measure, i.e., SFG-BM, the fixed-length shuttle facility game with a benefit measure, i.e., FL-SFG-BM, and as its special case, the pit-stop facility game with a benefit measure, i.e., PFG-BM. We start with the PFG-BM first, before considering the more general FL-SFG-BM.

3.1 The Pit-stop Facility Game (PFG-BM)

First, we design a GSP mechanism for the PFG-BM with interval-peaked benefit functions.

We define the concept of *interval-peakedness*. A function $g : \mathcal{R} \times \mathbb{R} \to \mathbb{R}$ is *interval-peaked* if and only if for any $[x, y] \in \mathcal{R}$ and $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$ such that $t_1 < t_2 < x \leq t_3 \leq y < t_4 < t_5$, it holds that

$$\begin{aligned} g(x, y, t_1) \leq g(x, y, t_2) < g(x, y, x) &= g(x, y, t_3) \\ &= g(x, y, y) > g(x, y, t_4) \geq g(x, y, t_5). \end{aligned}$$

Theorem 1 The lowest balanced mechanism is GSP for the

PFG-BM with interval-peaked benefit functions.

Proof. To derive a contradiction, assume that the mechanism f is not GSP.

Then, there exist a non-empty subset $S \subseteq N$ and two profiles $\chi = \{\chi_1 = [x_1, y_1], \ldots, \chi_n = [x_n, y_n]\}, \chi' \in \mathcal{R}^{\langle n \rangle}$ such that

$$\chi_{\overline{S}} = \chi'_{\overline{S}}, \tag{3}$$

$$\beta_i(\chi_i, f(\chi)) < \beta_i(\chi_i, f(\chi')), \quad \forall i \in S. \tag{4}$$

For any $i \in S$, since the benefit function β_i is interval-peaked, from Eq. (4) it follows that exactly one of Eqs. (5) and (6) is satisfied

$$f(\chi) < f(\chi') \text{ and } f(\chi) < x_i, \tag{5}$$

$$f(\chi) > f(\chi') \text{ and } y_i < f(\chi). \tag{6}$$

Case 1. Assume that Eq. (5) holds. Let $t \in \mathbb{R}$ be a real number such that $f(\chi) < t < f(\chi')$ and $t < x_i, \forall i \in S$. From the definition of N_{r}, we get that

$$S \subseteq N_{\mathrm{r}}(\chi, t).$$

From Eq. (3), only players in S $(\subseteq N_{\mathrm{r}}(\chi, t))$ change their reports between χ and χ'. The cardinality of the right set $N_{\mathrm{r}}(\chi, t)$ does not increase if players in S change their reports from χ to χ', and therefore $N_{\mathrm{r}}(\chi, t) \supseteq N_{\mathrm{r}}(\chi', t)$. Hence, it holds that

$$|N_{\mathrm{r}}(\chi, t)| \geq \left|N_{\mathrm{r}}(\chi', t)\right|. \tag{7}$$

Similarly, the cardinality of the left set $N_\ell(\chi, t)$ does not decrease if players in S change their reports from χ to χ'. Therefore, $N_\ell(\chi, t) \subseteq N_\ell(\chi', t)$. Hence, it holds that

$$\left|N_\ell(\chi', t)\right| \geq |N_\ell(\chi, t)|. \tag{8}$$

Since $f(\chi')$ is the lowest balanced point of χ' and $t < f(\chi')$, we get that

$$\left|N_{\mathrm{r}}(\chi', t)\right| > \left|N_\ell(\chi', t)\right|. \tag{9}$$

From Eqs. (7)–(9), it follows that

$$|N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)|.$$

Since the size $|N_{\mathrm{r}}(\chi, z)|$ of the set $N_{\mathrm{r}}(\chi, z)$ (resp., $|N_\ell(\chi, z)|$ of $N_\ell(\chi, z)$) is monotonically nonincreasing (resp., nondecreasing) with respect to z, from $f(\chi) < t$, it follows that

$$|N_{\mathrm{r}}(\chi, f(\chi))| \geq |N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)| \geq |N_\ell(\chi, f(\chi))|.$$

However, this contradicts the fact that $f(\chi)$ is the lowest balanced point of χ.

Case 2. Assume that Eq. (6) holds true. The proof is similar to that of Case 1. First, let $t \in \mathbb{R}$ be a real number such that $f(\chi') < t < f(\chi)$ and $y_i < t, \forall i \in S$. From the definition of N_ℓ, it holds that $S \subseteq N_\ell(\chi, t)$. We have $|N_{\mathrm{r}}(\chi', t)| \geq |N_{\mathrm{r}}(\chi, t)|$ and $|N_\ell(\chi, t)| \geq |N_\ell(\chi', t)|$. Since $f(\chi)$ is the lowest balanced point of χ and $t < f(\chi)$, we get that $|N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)|$. Therefore, it holds that $|N_{\mathrm{r}}(\chi', t)| > |N_\ell(\chi', t)|$. From the monotonicity of $|N_{\mathrm{r}}(\chi', z)|$ and $f(\chi') < t$, we have $|N_{\mathrm{r}}(\chi', f(\chi'))| \geq |N_{\mathrm{r}}(\chi', t)| > |N_\ell(\chi', t)| \geq |N_\ell(\chi', f(\chi'))|$. This contradicts the fact that $f(\chi')$ is the lowest balanced point of χ'. □

Theorem 2 The lowest balanced mechanism maximizes the social benefit for the PFG-BM with benefit functions $\beta_i(x_i, y_i, t) = -(|t - x_i| + |t - y_i|)$, $i \in N$.

Proof. Let $\mathrm{sb}(\chi, t)$ denote the social benefit $\sum_{i \in N} \beta_i(x_i, y_i, t)$. We show that the social benefit takes the maximum at the lowest balanced point.

Since the benefit functions β_i, $i \in N$, are continuous, the social benefit is also continuous. The benefit function $\beta_i(x_i, y_i, t)$ for player i is piecewise linear and is not differentiable at $t = x_i, y_i$. Since the social benefit is the sum of the benefit functions, the social benefit $\mathrm{sb}(\chi, t)$ is also piecewise linear and is not differentiable at $t \in \{x_i, y_i \mid (x_i, y_i) \in \chi\}$. We find the gradients of the social benefit at $t \in \mathbb{R} \setminus \{x_i, y_i \mid (x_i, y_i) \in \chi\}$. For $t \in \mathbb{R} \setminus \{x_i, y_i \mid (x_i, y_i) \in \chi\}$, let $\mathrm{gr}(\chi, t)$ be the gradient of $\mathrm{sb}(\chi, t)$ with respect to t.

For facility location t, the player set N is partitioned into three sets: $N_{\mathrm{r}}(\chi, t)$, $N_\ell(\chi, t)$, and $\bar{N}(\chi, t) \triangleq N \setminus (N_{\mathrm{r}}(\chi, t) \cup N_\ell(\chi, t))$.

For $i \in N_{\mathrm{r}}(\chi, t)$, by $t < x_i \leq y_i$ it holds that $\beta_i(x_i, y_i, t) = -(x_i - t + y_i - t) = 2t - x_i - y_i$. For $i \in N_\ell(\chi, t)$, by $x_i \leq y_i < t$ it holds that $\beta_i(x_i, y_i, t) = -(t - x_i + t - y_i) = -2t + x_i + y_i$. For $i \in \bar{N}(\chi, t)$, by $x_i \leq t \leq y_i$ it holds that $\beta_i(x_i, y_i, t) = -(t - x_i + y_i - t) = x_i - y_i$. Therefore

$$\begin{aligned}
\mathrm{sb}(\chi, t) &= \sum_{i \in N} \beta_i(x_i, y_i, t) \\
&= \sum_{i \in N_{\mathrm{r}}(\chi, t)} \beta_i(x_i, y_i, t) + \sum_{i \in N_\ell(\chi, t)} \beta_i(x_i, y_i, t) + \sum_{i \in \bar{N}(\chi, t)} \beta_i(x_i, y_i, t) \\
&= \sum_{i \in N_{\mathrm{r}}(\chi, t)} (2t - x_i - y_i) + \sum_{i \in N_\ell(\chi, t)} (-2t + x_i + y_i) + \sum_{i \in \bar{N}(\chi, t)} (x_i - y_i).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{gr}(\chi, t) &= \sum_{i \in N_{\mathrm{r}}(\chi, t)} 2 + \sum_{i \in N_\ell(\chi, t)} (-2) + \sum_{i \in \bar{N}(\chi, t)} 0 \\
&= 2\,|N_{\mathrm{r}}(\chi, t)| - 2\,|N_\ell(\chi, t)|.
\end{aligned}$$

From the monotonicity of $|N_{\mathrm{r}}(\chi, t)|$ and $|N_\ell(\chi, t)|$, and the definition of the lowest balanced mechanism, for the lowest balanced point p^*, it holds that

$$\begin{aligned}
&\mathrm{gr}(\chi, t) > 0 \quad \text{if } t < p^*, \\
&\mathrm{gr}(\chi, t) \leq 0 \quad \text{if } t \geq p^*.
\end{aligned}$$

Therefore the social benefit $\mathrm{sb}(\chi, t)$ is a function with a global maximum at its lowest balanced point. □

3.2 The Fixed-length Shuttle Facility Game (FL-SFG-BM)

We design a GSP mechanism for the fixed-length shuttle facility game where the benefit function of each player is interval-peaked with respect to the leftmost of the candidate facility locations.

We illustrate two examples. First, for a facility location $[t, t + k] \in \mathcal{R}$ and a player's report $[x, y] \in \mathcal{R}$, the benefit function $\beta(x, y, t, t + k) = -(|t - x| + |t + k - y|)$ is interval-peaked with peak at the interval $[x, y - k]$, and decreases with respect to the walking distance.

Next, for a facility location $[t, t + k] \in \mathcal{R}$ and a player's report

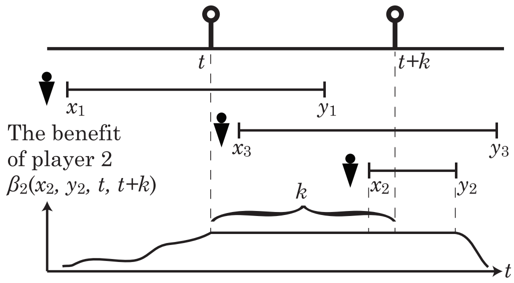


Fig. 6 The fixed-length shuttle facility game with interval-peaked benefit functions that decrease with respect to the intersection distance.

$[x, y] \in \mathcal{R}$, we focus on interval-peaked functions that take a maximum if the intersection of the two intervals $[t, t+k]$ and $[x, y]$ is not empty. Such functions, for example

$$\beta(x, y, t, t+k) = -\min\{|p - z| \mid p \in [t, t+k], z \in [x, y]\},$$

are interval-peaked with peak at the interval $[x - k, y]$, and decrease with respect to the intersection distance, as illustrated in **Fig. 6**.

In both cases, the peak of the benefit function is easily determined from the facility length k and a player's report $[x, y]$.

Theorem 3 Let $I_k = (N, \mathcal{K}_k, \mathcal{R}, \mathcal{B})$ be an instance of the FL-SFG-BM. Assume that there exist interval-peaked benefit functions $\widetilde{\beta}_i : \mathcal{R} \times \mathbb{R} \to \mathbb{R}$, $i \in N$, and a mapping $h : \mathcal{R} \to \mathcal{R}$ such that $\widetilde{\beta}_i(h([x, y]), t) = \beta_i(x, y, t, t+k)$. For a profile $\chi = \{[x_1, y_1], \ldots, [x_n, y_n]\}$, let $h(\chi)$ denote the profile $\{h([x_1, y_1]), \ldots, h([x_n, y_n])\}$. Let $\widetilde{I} = (N, \mathbb{R}, \mathcal{R}, \widetilde{\mathcal{B}} = \{\widetilde{\beta}_i \mid i \in N\})$ be an instance of the PFG-BM, and let $\widetilde{f}$ be any GSP mechanism for $\widetilde{I}$ with benefit ratio $\rho_{\widetilde{f}}$. Then, the mechanism $f(\chi) \triangleq \left[\widetilde{f}(h(\chi)), \widetilde{f}(h(\chi)) + k\right]$ is a GSP mechanism for I_k with benefit ratio at most $\rho_{\widetilde{f}}$.

Proof. We show that for any non-empty subset $S \subseteq N$ and two profiles $\chi, \chi' \in \mathcal{R}^{\langle n \rangle}$ of I_k satisfying $\chi_{\overline{S}} = \chi'_{\overline{S}}$, there exists a player i such that $\beta_i(\chi_i, f(\chi)) \geq \beta_i(\chi_i, f(\chi'))$.

From $\chi_{\overline{S}} = \chi'_{\overline{S}}$ for the two profiles $h(\chi)$ and $h(\chi')$ it holds that $h(\chi)_{\overline{S}} = h(\chi')_{\overline{S}}$. Since the mechanism $\widetilde{f}$ is GSP, we get that

$$\widetilde{\beta}_i(h([x_i, y_i]), \widetilde{f}(h(\chi))) \geq \widetilde{\beta}_i(h([x_i, y_i]), \widetilde{f}(h(\chi'))).$$

From the definition of the benefit function $\widetilde{\beta}_i$, we get that

$$\begin{aligned} &\beta_i(x_i, y_i, \widetilde{f}(h(\chi)), \widetilde{f}(h(\chi)) + k) \\ &\quad \geq \beta_i(x_i, y_i, \widetilde{f}(h(\chi')), \widetilde{f}(h(\chi')) + k). \end{aligned}$$

From the definition of the mechanism f, it follows that

$$\beta_i(x_i, y_i, f(\chi)) \geq \beta_i(x_i, y_i, f(\chi')),$$

and therefore, the mechanism f is GSP.

We show that the benefit ratio of f is at most $\rho_{\widetilde{f}}$. Note that the image of the mapping h is included in $\mathcal{R}$. From the definition, the benefit ratio of f is

$$\sup\left\{\frac{\sup\left\{\sum_{i=1}^n \beta_i(\chi_i, t, t+k) \mid t \in \mathbb{R}\right\}}{\sum_{i=1}^n \beta_i(\chi_i, f(\chi))} \mid \chi \in \mathcal{R}^{\langle n \rangle}\right\}.$$

From the definition of $\widetilde{\beta}_i$ and $\widetilde{f}$, we get that

$$\begin{aligned} &\sup\left\{\frac{\sup\left\{\sum_{i=1}^n \beta_i(\chi_i, t, t+k) \mid t \in \mathbb{R}\right\}}{\sum_{i=1}^n \beta_i(\chi_i, f(\chi))} \mid \chi \in \mathcal{R}^{\langle n \rangle}\right\} \\ &= \sup\left\{\frac{\sup\left\{\sum_{i=1}^n \beta_i(\chi_i, t, t+k) \mid t \in \mathbb{R}\right\}}{\sum_{i=1}^n \beta_i(\chi_i, \widetilde{f}(h(\chi)), \widetilde{f}(h(\chi)) + k)} \mid \chi \in \mathcal{R}^{\langle n \rangle}\right\} \\ &= \sup\left\{\frac{\sup\left\{\sum_{i=1}^n \widetilde{\beta}_i(h(\chi_i), t) \mid t \in \mathbb{R}\right\}}{\sum_{i=1}^n \widetilde{\beta}_i(h(\chi_i), \widetilde{f}(h(\chi)))} \mid \chi \in \mathcal{R}^{\langle n \rangle}\right\} \\ &\leq \sup\left\{\frac{\sup\left\{\sum_{i=1}^n \widetilde{\beta}_i(\chi_i, t) \mid t \in \mathbb{R}\right\}}{\sum_{i=1}^n \widetilde{\beta}_i(\chi_i, \widetilde{f}(\chi))} \mid \chi \in \mathcal{R}^{\langle n \rangle}\right\} \\ &= \rho_{\widetilde{f}}, \end{aligned}$$

and thus proving the theorem. □

Observe that due to Theorem 2 it is possible to define a benefit function for the FL-SFG-BM such that the mechanism defined in Theorem 3 achieves benefit ratio 1.

3.3 The Flexible-length Shuttle Facility Game (SFG-BM)

We show that there exists a GSP mechanism for the flexible-length shuttle facility game FG-BM$(N, \mathcal{R}, \mathcal{R}, \mathcal{B})$ with benefit functions based on walking distances.

Theorem 4 The respective median mechanism for the SFG-BM$(N, \mathcal{R}, \mathcal{R}, \mathcal{B})$, with benefit functions $\beta_i([x_i, y_i], [p, q]) = -\min(y_i - x_i, |x_i - p| + |y_i - q|)$, $i \in N$, is GSP.

Proof. To derive a contradiction, assume that the respective median mechanism f is not GSP. Then, there exist a non-empty subset $S \subseteq N$ and two profiles $\chi = \{\chi_1 = [x_1, y_1], \ldots, \chi_n = [x_n, y_n]\}, \chi' = \{\chi'_1 = [x'_1, y'_1], \ldots, \chi'_n = [x'_n, y'_n]\} \in \mathcal{R}^{\langle n \rangle}$ such that

$$\chi_{\overline{S}} = \chi'_{\overline{S}}, \tag{10}$$

$$\beta_i(\chi_i, f(\chi)) < \beta_i(\chi_i, f(\chi')), \quad \forall i \in S. \tag{11}$$

For any $i \in S$, since the benefit function $\beta_i([x_i, y_i], [p, q])$ is $-\min(y_i - x_i, |x_i - p| + |y_i - q|)$, from Eq. (11) it holds that $y_i - x_i > |x_i - f_x(\chi')| + \left|y_i - f_y(\chi')\right|$, and therefore

$$\left|x_i - f_x(\chi')\right| + \left|y_i - f_y(\chi')\right| < |x_i - f_x(\chi)| + \left|y_i - f_y(\chi)\right|. \tag{12}$$

There are four cases:

$$f_x(\chi) < f_x(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| \geq \left|f_y(\chi) - f_y(\chi')\right|, \tag{13}$$

$$f_y(\chi) < f_y(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| < \left|f_y(\chi) - f_y(\chi')\right|, \tag{14}$$

$$f_x(\chi) > f_x(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| \geq \left|f_y(\chi) - f_y(\chi')\right|, \tag{15}$$

$$f_y(\chi) > f_y(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| < \left|f_y(\chi) - f_y(\chi')\right|. \tag{16}$$

Case 1. Assume that Eq. (13) is satisfied.

To show that for any $i \in S$, $x_i > f_x(\chi)$, assume that for some $i \in S$, it holds that $x_i \leq f_x(\chi) < f_x(\chi')$. From Eq. (13) and the triangle inequality, it holds that

$$\begin{aligned} &\left|x_i - f_x(\chi')\right| + \left|y_i - f_y(\chi')\right| \\ &\quad = f_x(\chi) - x_i + f_x(\chi') - f_x(\chi) + \left|y_i - f_y(\chi')\right| \\ &\quad = |f_x(\chi) - x_i| + \left|f_x(\chi') - f_x(\chi)\right| + \left|y_i - f_y(\chi')\right| \\ &\quad \geq |f_x(\chi) - x_i| + \left|f_y(\chi') - f_y(\chi)\right| + \left|y_i - f_y(\chi')\right| \\ &\quad \geq |f_x(\chi) - x_i| + \left|y_i - f_y(\chi)\right|. \end{aligned}$$

This contradicts Eq. (12). Hence, for any $i \in S$ it holds that

$x_i > f_x(\chi)$.

From Eq. (10) only players in S change their reports between χ and χ'. The cardinality of the set $\{i \in N \mid x_i > f_x(\chi)\}$ does not increase if players in S change their reports from χ to χ', that is,

$$|\{i \in N \mid x_i > f_x(\chi)\}| \geq \left|\{i \in N \mid x'_i > f_x(\chi)\}\right|.$$

From Eq. (13) we get

$$\left|\{i \in N \mid x'_i > f_x(\chi)\}\right| \geq \left|\{i \in N \mid x'_i \geq f_x(\chi')\}\right|.$$

From the definition of the mechanism f and Eq. (2), we get

$$|\{i \in N \mid x_i > f_x(\chi)\}| \leq n/2.$$

Therefore it holds that

$$\left|\{i \in N \mid x'_i \geq f_x(\chi')\}\right| \leq n/2,$$

which contradicts the definition of the mechanism f.

Case 2. Assume that Eq. (14) is satisfied. Similarly, for any $i \in S$, it holds that $y_i > f_y(\chi)$. From Eq. (10) only players in S change their reports between χ and χ'. The cardinality of the set $\left\{i \in N \mid y_i > f_y(\chi)\right\}$ does not increase if players in S change their reports from χ to χ', that is,

$$\left|\left\{i \in N \mid y_i > f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i > f_y(\chi)\right\}\right|.$$

From Eq. (14) we get $\left|\left\{i \in N \mid y'_i > f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i \geq f_y(\chi')\right\}\right|$. From the definition of the mechanism f, we get $\left|\left\{i \in N \mid y_i > f_y(\chi)\right\}\right| \leq n/2$. Therefore it holds that $\left|\left\{i \in N \mid y'_i \geq f_y(\chi')\right\}\right| \leq n/2$, which contradicts the definition of the mechanism f.

Case 3. Assume that Eq. (15) is satisfied. Similarly, for any $i \in S$, it holds that $x_i < f_x(\chi)$. From Eq. (10) only players in S change their reports between χ and χ'. The cardinality of the set $\{i \in N \mid x_i < f_x(\chi)\}$ does not increase if players in S change their reports from χ to χ', that is,

$$|\{i \in N \mid x_i < f_x(\chi)\}| \geq \left|\{i \in N \mid x'_i < f_x(\chi)\}\right|.$$

From Eq. (15) we get $\left|\left\{i \in N \mid x'_i < f_x(\chi)\right\}\right| \geq \left|\left\{i \in N \mid x'_i \leq f_x(\chi')\right\}\right|$. From the definition of the mechanism f, we get $|\{i \in N \mid x_i < f_x(\chi)\}| < n/2$. Therefore it holds that $\left|\left\{i \in N \mid x'_i \leq f_x(\chi')\right\}\right| < n/2$, which contradicts the definition of the mechanism f.

Case 4. Assume that Eq. (16) is satisfied. Similarly, for any $i \in S$, it holds that $y_i < f_y(\chi)$. From Eq. (10) only players in S change their reports between χ and χ'. The cardinality of the set $\left\{i \in N \mid y_i < f_y(\chi)\right\}$ does not increase if players in S change their reports from χ to χ', that is,

$$\left|\left\{i \in N \mid y_i < f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i < f_y(\chi)\right\}\right|.$$

From Eq. (16) we get $\left|\left\{i \in N \mid y'_i < f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i \leq f_y(\chi')\right\}\right|$. From the definition of the mechanism f, we get $\left|\left\{i \in N \mid y_i < f_y(\chi)\right\}\right| < n/2$. Therefore it holds that $\left|\left\{i \in N \mid y'_i \leq f_y(\chi')\right\}\right| < n/2$, which contradicts the definition of the mechanism f. □

4. Mechanisms for the Shuttle Facility Game with a Utility Measure

In this section, we consider three facility games, the flexible-length shuttle facility game with a utility measure, i.e. SFG-UM, the fixed-length shuttle facility game with a utility measure, i.e. FL-SFG-UM, and as its special case, the pit-stop facility game with a utility measure, i.e. PFG-UM. We start with the PFG-UM first, before considering the more general FL-SFG-UM.

4.1 The Pit-stop Facility Game (PFG-UM)

First, we design a GSP mechanism for the PFG-UM with interval-peaked utility functions.

Recall that the lowest balanced mechanism is a mechanism that given a profile χ, outputs the lowest balanced point of χ, namely, the infimum of the set $\{p \in \mathbb{R} \mid |N_r(\chi, p)| \leq |N_\ell(\chi, p)|\}$ (see Fig. 5).

Theorem 5 The lowest balanced mechanism is GSP for the PFG-UM with interval-peaked utility functions.

Proof. Let $\Omega(u_i) = [a_i, b_i]$ be the peak of the utility function u_i of player $i \in N$. Let A be the profile $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\} \in \mathcal{R}^{\langle n \rangle}$.

To derive a contradiction, we assume that the lowest balanced mechanism f is not GSP. Then, there exist a non-empty subset $S \subseteq N$ and two profiles $\chi = \{\chi_1 = [x_1, y_1], \ldots, \chi_n = [x_n, y_n]\}, \chi' \in \mathcal{R}^{\langle n \rangle}$ such that

$$\chi_{\overline{S}} = \chi'_{\overline{S}}, \tag{17}$$
$$\chi_S = A_S, \text{ namely, } \chi_i = [x_i, y_i] = [a_i, b_i], \quad \forall i \in S \tag{18}$$
$$u_i(f(\chi)) < u_i(f(\chi')), \quad \forall i \in S. \tag{19}$$

Since the interval $[a_i, b_i]$ is the peak of the interval-peaked utility u_i, for any $i \in S$, from Eq. (19) it follows that exactly one of Eqs. (20) and (21) is satisfied.

$$f(\chi) < f(\chi') \text{ and } f(\chi) < a_i = x_i, \quad \forall i \in S, \tag{20}$$
$$f(\chi') < f(\chi) \text{ and } y_i = b_i < f(\chi), \quad \forall i \in S. \tag{21}$$

Case 1. Assume that Eq. (20) holds true. Let $t \in \mathbb{R}$ be a real number such that $f(\chi) < t < f(\chi')$ and $t < x_i$, $\forall i \in S$. From the definition of N_r, we get that $S \subseteq N_r(\chi, t)$. From Eq. (17), only players in S ($\subseteq N_r(\chi, t)$) change their reports between χ and χ'. The cardinality of the right set $N_r(\chi, t)$ does not increase by changing their reports from χ to χ'. Therefore it holds that $N_r(\chi, t) \supseteq N_r(\chi', t)$. Hence, we get that

$$|N_r(\chi, t)| \geq \left|N_r(\chi', t)\right|. \tag{22}$$

Similarly, the cardinality of the left set $N_\ell(\chi, t)$ does not decrease by changing reports of players in S from χ to χ', and therefore it holds that $N_\ell(\chi, t) \subseteq N_\ell(\chi', t)$. Hence, we get that

$$\left|N_\ell(\chi', t)\right| \geq |N_\ell(\chi, t)|. \tag{23}$$

Since $f(\chi')$ is the lowest balanced point of χ' and $t < f(\chi')$, it holds that

$$\left|N_r(\chi', t)\right| > \left|N_\ell(\chi', t)\right|. \tag{24}$$

From Eqs. (22)-(24), it follows that

$$|N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)|\,.$$

Since the size $|N_{\mathrm{r}}(\chi, z)|$ of the set $N_{\mathrm{r}}(\chi, t)$ (resp,. $|N_\ell(\chi, z)|$) is monotonically nonincreasing (resp,. nondecreasing) with respect to z, from $f(\chi) < t$, it follows that

$$|N_{\mathrm{r}}(\chi, f(\chi))| \geq |N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)| \geq |N_\ell(\chi, f(\chi))|\,.$$

However, this contradicts the fact that $f(\chi)$ is the lowest balanced point of χ.

Case 2. We assume that Eq. (21) holds true. The proof is similar to that of Case 1. First, let $t \in \mathbb{R}$ be a real number such that $f(\chi') < t < f(\chi)$ and $y_i < t$, $\forall i \in S$. From the definition of N_ℓ, it holds that $S \subseteq N_\ell(\chi, t)$. We have $|N_{\mathrm{r}}(\chi', t)| \geq |N_{\mathrm{r}}(\chi, t)|$, $|N_\ell(\chi, t)| \geq |N_\ell(\chi', t)|$. Since $f(\chi)$ is the lowest balanced point of χ and $t < f(\chi)$, we get that $|N_{\mathrm{r}}(\chi, t)| > |N_\ell(\chi, t)|$. Therefore, it holds that $|N_{\mathrm{r}}(\chi', t)| > |N_\ell(\chi', t)|$. From the monotonicity of $|N_{\mathrm{r}}(\chi', z)|$ and $f(\chi') < t$, we have $|N_{\mathrm{r}}(\chi', f(\chi'))| \geq |N_{\mathrm{r}}(\chi', t)| > |N_\ell(\chi', t)| \geq |N_\ell(\chi', f(\chi'))|$. This contradicts the fact that $f(\chi')$ is the lowest balanced point of χ', from where the claim follows. □

4.2 The Fixed-length Shuttle Facility Game (FL-SFG-UM)

Next, we design a GSP mechanism for the FL-SFG-UM with interval-peaked utility functions.

Theorem 6 Let $I_k = (N, \mathcal{K}_k, \mathcal{R}, \mathcal{U})$ be an instance of the FL-SFG-UM such that $\widetilde{u}_i(t) \triangleq u_i(t, t+k)$ is an interval-peaked utility function $\widetilde{u}_i : \mathbb{R} \to \mathbb{R}$, $i \in N$. Let $\widetilde{I} = (N, \mathcal{R}, \mathcal{R}, \widetilde{\mathcal{U}} = \{\widetilde{u}_i \mid i \in N\})$ be an instance of the PFG-UM. For any GSP mechanism $\widetilde{f}$ for $\widetilde{I}$, the mechanism $f(\chi) \triangleq \left[\widetilde{f}(\chi), \widetilde{f}(\chi) + k\right]$ is a GSP mechanism for I_k.

Proof. Let $[a_i, b_i]$ be the peak of utility function u_i. Let A be the profile

$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. We show that for any non-empty subset $S \subseteq N$ and two profiles $\chi, \chi' \in \mathcal{R}^{\langle n \rangle}$ of I_k such that $\chi_{\overline{S}} = \chi'_{\overline{S}}$ and $\chi_S = A_S$, there exists a player i such that $u_i(f(\chi)) \geq u_i(f(\chi'))$. Since the mechanism $\widetilde{f}$ is GSP, we get that $\widetilde{u}_i(\widetilde{f}(\chi)) \geq \widetilde{u}_i(\widetilde{f}(\chi'))$. From the definition of the utility function $\widetilde{u}_i$, we get that

$$u_i(\widetilde{f}(\chi), \widetilde{f}(\chi) + k) \geq u_i(\widetilde{f}(\chi'), \widetilde{f}(\chi') + k).$$

From the definition of the mechanism f, it follows that

$$u_i(f(\chi)) \geq u_i(f(\chi')),$$

and therefore, the mechanism f is GSP. □

4.3 The Flexible-length Shuttle Facility Game (SFG-UM)

We show that there exists a GSP mechanism for the flexible-length shuttle facility game FG-UM$(N, \mathcal{R}, \mathcal{R}, \mathcal{U})$ with utility functions based on walking distances.

Theorem 7 The respective median mechanism is GSP for the SFG-UM$(N, \mathcal{R}, \mathcal{R}, \mathcal{U})$ with utility functions defined to be $u_i([p, q]) = -\min(b_i - a_i, |a_i - p| + |b_i - q|)$, $a_i, b_i \in \mathbb{R}$, for each $i \in N$.

Proof. Let $\Omega(u_i) = [a_i, b_i]$ be the peak of the utility function u_i of player $i \in N$. Let A be the profile $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\} \in \mathcal{R}^{\langle n \rangle}$.

To derive a contradiction, we assume that the respective median mechanism f is not GSP. Then, there exist a non-empty subset $S \subseteq N$ and two profiles $\chi = \{\chi_1 = [x_1, y_1], \ldots, \chi_n = [x_n, y_n]\}, \chi' = \{\chi'_1 = [x'_1, y'_1], \ldots, \chi'_n = [x'_n, y'_n]\} \in \mathcal{R}^{\langle n \rangle}$ such that

$$\chi_{\overline{S}} = \chi'_{\overline{S}}, \tag{25}$$

$$\chi_S = A_S,\ \text{namely}, \chi_i = [x_i, y_i] = [a_i, b_i], \quad \forall i \in S \tag{26}$$

$$u_i(f(\chi)) < u_i(f(\chi')), \quad \forall i \in S. \tag{27}$$

For any $i \in S$, since the utility function $u_i([p, q])$ is $-\min(b_i - a_i, |a_i - p| + |b_i - q|)$, from Eq. (27) it holds that $y_i - x_i > |x_i - f_x(\chi')| + |y_i - f_y(\chi')|$, and therefore

$$\left|x_i - f_x(\chi')\right| + \left|y_i - f_y(\chi')\right| < |x_i - f_x(\chi)| + \left|y_i - f_y(\chi)\right|. \tag{28}$$

There are four cases:

$$f_x(\chi) < f_x(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| \geq \left|f_y(\chi) - f_y(\chi')\right|, \tag{29}$$

$$f_y(\chi) < f_y(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| < \left|f_y(\chi) - f_y(\chi')\right|, \tag{30}$$

$$f_x(\chi) > f_x(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| \geq \left|f_y(\chi) - f_y(\chi')\right|, \tag{31}$$

$$f_y(\chi) > f_y(\chi') \text{ and } \left|f_x(\chi) - f_x(\chi')\right| < \left|f_y(\chi) - f_y(\chi')\right|. \tag{32}$$

Case 1. Assume that Eq. (29) is satisfied.

To show that for any $i \in S$, $x_i > f_x(\chi)$, assume that for some $i \in S$, it holds that $x_i \leq f_x(\chi) < f_x(\chi')$. From Eq. (29) and the triangle inequality, it holds that

$$\begin{aligned}
&\left|x_i - f_x(\chi')\right| + \left|y_i - f_y(\chi')\right| \\
&\quad = f_x(\chi) - x_i + f_x(\chi') - f_x(\chi) + \left|y_i - f_y(\chi')\right| \\
&\quad = |f_x(\chi) - x_i| + \left|f_x(\chi') - f_x(\chi)\right| + \left|y_i - f_y(\chi')\right| \\
&\quad \geq |f_x(\chi) - x_i| + \left|f_y(\chi') - f_y(\chi)\right| + \left|y_i - f_y(\chi')\right| \\
&\quad \geq |f_x(\chi) - x_i| + \left|y_i - f_y(\chi)\right|.
\end{aligned}$$

This contradicts Eq. (28). Hence, for any $i \in S$ it holds that $x_i > f_x(\chi)$.

From Eq. (25) only players in S change their reports between χ and χ'. The cardinality of the set $\{i \in N \mid x_i > f_x(\chi)\}$ does not increase if players in S change their reports from χ to χ', that is,

$$|\{i \in N \mid x_i > f_x(\chi)\}| \geq \left|\{i \in N \mid x'_i > f_x(\chi)\}\right|.$$

From Eq. (29) we get

$$\left|\{i \in N \mid x'_i > f_x(\chi)\}\right| \geq \left|\{i \in N \mid x'_i \geq f_x(\chi')\}\right|.$$

From the definition of the mechanism f, we get

$$|\{i \in N \mid x_i > f_x(\chi)\}| \leq n/2.$$

Therefore it holds that

$$\left|\{i \in N \mid x'_i \geq f_x(\chi')\}\right| \leq n/2,$$

which contradicts the definition of the mechanism f.

Case 2. Assume that Eq. (30) is satisfied. Similarly, for any $i \in S$, it holds that $y_i > f_y(\chi)$. From Eq. (25) only players in S change their reports between χ and χ'. The cardinality of the set $\left\{i \in N \mid y_i > f_y(\chi)\right\}$ does not increase if players in S change their reports from χ to χ', that is,

$$\left|\left\{i \in N \mid y_i > f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i > f_y(\chi)\right\}\right|.$$

From Eq. (30) we get $\left|\left\{i \in N \mid y'_i > f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i \geq f_y(\chi')\right\}\right|$. From the definition of the mechanism f, we get $\left|\left\{i \in N \mid y_i > f_y(\chi)\right\}\right| \leq n/2$. Therefore it holds that $\left|\left\{i \in N \mid y'_i \geq f_y(\chi')\right\}\right| \leq n/2$. This contradicts the definition of the mechanism f.

Case 3. Assume that Eq. (31) is satisfied. Similarly, for any $i \in S$, it holds that $x_i < f_x(\chi)$. From Eq. (25) only players in S change their reports between χ and χ'. The cardinality of the set $\{i \in N \mid x_i < f_x(\chi)\}$ does not increase if players in S change their reports from χ to χ', that is,

$$|\{i \in N \mid x_i < f_x(\chi)\}| \geq \left|\{i \in N \mid x'_i < f_x(\chi)\}\right|.$$

From Eq. (31) we get $\left|\left\{i \in N \mid x'_i < f_x(\chi)\right\}\right| \geq \left|\left\{i \in N \mid x'_i \leq f_x(\chi')\right\}\right|$. From the definition of the mechanism f, we get $|\{i \in N \mid x_i < f_x(\chi)\}| < n/2$. Therefore it holds that $\left|\left\{i \in N \mid x'_i \leq f_x(\chi')\right\}\right| < n/2$. This contradicts the definition of the mechanism f.

Case 4. Assume that Eq. (32) is satisfied. Similarly, for any $i \in S$, it holds that $y_i < f_y(\chi)$. From Eq. (25) only players in S change their reports between χ and χ'. The cardinality of the set $\left\{i \in N \mid y_i < f_y(\chi)\right\}$ does not increase if players in S change their reports from χ to χ', that is,

$$\left|\left\{i \in N \mid y_i < f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i < f_y(\chi)\right\}\right|.$$

From Eq. (31) we get

$$\left|\left\{i \in N \mid y'_i < f_y(\chi)\right\}\right| \geq \left|\left\{i \in N \mid y'_i \leq f_y(\chi')\right\}\right|.$$

From the definition of the mechanism f, we get $\left|\left\{i \in N \mid y_i < f_y(\chi)\right\}\right| < n/2$. Therefore it holds that $\left|\left\{i \in N \mid y'_i \leq f_y(\chi')\right\}\right| < n/2$, which contradicts the definition of the mechanism f. □

5. Conclusion

We introduced two shuttle facility games, the fixed-length shuttle facility game and the flexible-length shuttle facility game, as well as a special case of the above problems, the pit-stop facility game. We investigated two types of players' profits, benefit functions that take a facility location and a player's report as arguments, and utility functions that only take a facility location as an argument. We proved that each of the game problems admits a group strategy-proof mechanism:

- the lowest balanced mechanism for the pit-stop facility game in the line space with interval-peaked profit functions,
- the lowest balanced mechanism for the fixed-length shuttle facility game in the line space with interval-peaked profit functions, and,
- the respective median mechanism for the flexible-length shuttle facility game in the line space with profit functions based on the walking distances.

We also investigated the benefit ratio and proved that for the pit-stop facility game in the line space with benefit functions based on the visit-distance, the benefit ratio of the lowest balanced mechanism is one. In our models, we showed that there exists a group strategy-proof mechanism for the pit-stop facility game in the line space and a group strategy-proof mechanism for the fixed-length shuttle facility game with interval-peaked profit functions and that the lowest balanced mechanism is group strategy-proof. We also showed that when the players' benefit functions are defined to be the negative of the visit-distance, the benefit ratio of the lowest balanced mechanism for the pit-stop facility game is one. Finally, we prove that when the players' benefit functions decrease with respect to the walking distance, there exists a group strategy-proof mechanism for the flexible-length shuttle facility game.

For future work, it remains to investigate whether the respective median mechanism is group strategy-proof or not for the flexible-length shuttle facility game with other benefit functions. Furthermore it remains to investigate the benefit ratio for the flexible-length shuttle facility game, that is, whether the respective median mechanism is optimal or not and if not, what is the lower bound on benefit ratio of strategy-proof mechanisms for the flexible-length shuttle facility game in the line space. Next, it remains to investigate whether there exists a group strategy-proof mechanism for the shuttle facility game in the different spaces, such as a tree or a circle.

Also, we are interested in the facility game related to the design of transportation systems. For example, it is interesting to investigate the time for commuting as a player's profit, instead of the distance. The time for commuting includes two types of the time, the walking time between player's locations and stations and time in the transportation between the stations. Since there are many stations in a real transportation system, it also remains to investigate the facility game to determine more than two stations in the line, a tree or a simple graph space.

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