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Variant of Wythoff's Game-Corner Two Rooks

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Abstract: This article presents an impartial game, "Corner Two Rooks." This game is a variant of "Corner the Queen" that is mathematically equivalent to Wythoff's game. In "Corner the Queen," a single chess queen is placed on a large grid of squares. Each player can move the queen any number of steps toward the upper-left corner of the grid, vertically, horizontally, or diagonally. The player who moves the queen into the upper-left corner is the winner. In this work, the authors use two rooks of chess instead of the queen, and a rook can jump over another rook but not onto another. There is a restriction on the distance that a rook can travel in each turn. This game can be considered as a misère game of the traditional Nim game with four piles and a restriction of the number of stones to be removed in each turn. The authors present the set of $\mathscr{P}$-positions of the game using a theorem for misère games. When there is no restriction on the distance that a rook can travel in each turn, we obtain a similar result in which the set of $\mathscr{P}$-positions is simpler.

Keywords: Combinatorial game theory, Nim, Impartial games, Misère games

1. Introduction

In this paper, we discuss a game that is a variant of Nim, Wythoff's game, and Welter's game. Nim has been studied by numerous mathematicians. In classic Nim, two players take turns and remove stones from one of several piles. The player who removes the last stone or stones is the winner. Bouton [2] presented a winning strategy for this game using the Nim-sum (exclusive OR).

Definition 1. *We define* $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P))$ *as the ruleset of two-player impartial games in normal (resp. misère) play, as follows: Let* $\mathcal{H}$ *denote all positions of the game and* $\mathcal{E} \subset \mathcal{H}$ *be the set of end positions. Let* $\mathrm{move}(P) : \mathcal{H} \to 2^{\mathcal{H}}$ *be the set of all positions reached from position* $P \in \mathcal{H}$. $\mathrm{move}(P) = \emptyset$ *if and only if* $P \in \mathcal{E}$. *In the game, from game position* $P \in \mathcal{H}$*, the current player alternately selects one of the elements in* $\mathrm{move}(P)$*, and the player who reaches one of the end positions is the winner (resp. loser), i.e., the player who selects a position* $P' \in move(P) \cap \mathcal{E}$ *is the winner (resp. loser). In this paper, we assume that every ruleset is loopfree, that is, we have no sequence of positions* $P_1, P_2, \ldots, P_k$ *such that* $P_i \in \mathrm{move}(P_{i-1})$ *for every* $1 < i \leq k$ *and* $P_1 = P_k$.

Clearly, an impartial game without draws has two outcome classes. These are described as follows:

Definition 2.
(a) *$\mathscr{N}$-positions are positions from which the next player can force a win as long as they play correctly at every stage.*
(b) *$\mathscr{P}$-positions are positions from which the previous player (the player who will play after the next player) can force a win as long as they play correctly at every stage.*

From Definition 2, we obtain the following theorem.

Theorem 1.
(a) *For a ruleset* $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P)), \mathcal{E}$ *is a subset of $\mathscr{P}$-positions (resp. $\mathscr{N}$-positions) in normal (resp. misère) play.*
(b) *If every* $P' \in \mathrm{move}(P)$ *is an $\mathscr{N}$-position, then P is a $\mathscr{P}$-position.*
(c) *If there exists* $P' \in \mathrm{move}(P)$ *which is a $\mathscr{P}$-position, then P is an $\mathscr{N}$-position.*

For a game position $(a_1, a_2, \ldots, a_u)$, we abbreviate $\mathrm{move}((a_1, a_2, \ldots, a_u))$ as $\mathrm{move}(a_1, a_2, \ldots, a_u)$ when there will be no confusion.

Definition 3 (Nim)**.** *Let* $\mathbb{N}_0$ *be the set of all nonnegative integers. We define the ruleset of Nim* $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P))$ *as follows:* $\mathcal{H} = \{(a_1, a_2, \ldots, a_u) : a_i \in \mathbb{N}_0\}, \mathcal{E} = \{(0, 0, \ldots, 0)\}$*, and* $\mathrm{move}(a_1, a_2, \ldots, a_u) = \{(a_1, a_2, \ldots, a_{i-1}, a_i - t, a_{i+1}, \ldots, a_u) : 1 \leq i \leq u, 1 \leq t \leq a_i\}$.

Definition 4. *We denote the Nim-sum (exclusive OR) by* $\oplus$*. That is, for any integers a and b,* $a \oplus b$ *is defined as follows:*

Let $a = \sum_{i=0}^{i=r} 2^i a_i, b = \sum_{i=0}^{i=r} 2^i b_i$*, where r satisfies* $2^r > a$ *and* $2^r > b$ *and* $a_i \in \{0, 1\}, b_i \in \{0, 1\}$*. Then,* $a \oplus b = \sum_{i=0}^{i=r} 2^i ((a_i + b_i) \bmod 2)$.

Theorem 2 (Bouton [2])**.** *A Nim position* $(a_1, a_2, \ldots, a_u)$ *in normal play is a $\mathscr{P}$-position if and only if* $a_1 \oplus a_2 \oplus \cdots \oplus a_u = 0$.

There have been numerous variants of Nim, and one of them is Welter's game [10]. This game is similar to Nim; however, any two piles cannot have the same number of stones.

Another variant of Nim is Wythoff's game. This game is played with two piles of stones, where two players take turns to remove stones from one or both piles. When removing stones from both piles, an equal number of stones must be removed from each pile. The player who removes the last stone or stones is the winner.

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Definition 5 (Wythoff's game)**.** *We define* $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P))$ *as the ruleset of Wythoff's game, as follows:* $\mathcal{H} = \{(a_1, a_2) : a_i \in \mathbb{N}_0\}, \mathcal{E} = \{(0,0)\}$*, and* $\mathrm{move}(a_1, a_2) = \{(a_1 - s, a_2) : 1 \le s \le a_1\} \cup \{(a_1, a_2 - s) : 1 \le s \le a_2\} \cup \{(a_1 - s, a_2 - s) : 1 \le s \le \min(a_1, a_2)\}$.

We use $\lfloor \cdot \rfloor$ when we describe the floor function.

Theorem 3 (Wythoff [11])**.** *A position of Wythoff's game* (a_1, a_2) *is a* $\mathscr{P}$*-position if and only if* $(a_1, a_2) \in \{(\lfloor n\phi \rfloor, \lfloor n\phi + n \rfloor), (\lfloor n\phi + n \rfloor, \lfloor n\phi \rfloor) : n \in \mathbb{N}_0\}$*, where* ϕ *is the golden ratio, that is,* $\phi = \frac{1+\sqrt{5}}{2}$.

See Refs. [1], [7], and [8] for the detailed theory of combinatorial games and impartial games.

Wythoff's game can be played with a single queen of chess on a semi-infinite chess board. Each player can move the queen any number of steps towards the upper-left corner of the grid, vertically, horizontally, or diagonally. The player who moves the queen into the upper-left corner is the winner. We refer to this game as "Corner the Queen." Obviously, there is a one-to-one correspondence between positions of this game and positions of Wythoff's game.

By substituting the queen with another piece of chess or other games, we can make variants of "Corner the Queen." For example, "Corner the Ryuoh" (Ryuoh is a piece of Shogi, i.e., Japanese chess) is the same as "the king-rook game" in Ref. [3], where "the king-rook game" is discussed as a specific case of cyclic Nimhoff. "Corner the Maharaja" has been discussed in Ref. [5], where a Maharaja is a piece which is moved either as the Queen or the Knight of Chess, and the traditional two-pile Nim can also be considered as "Corner the Rook."

Welter's game also can be considered as a piece-moving game like this: A few pieces are placed on the one-dimensional board. Two players take turns, and choose one piece and move it to one empty square on the left.

In this study, the authors use two rooks instead of the queen of Wythoff's game. The game has two aspects. One is the generalization of Nim because Nim is "Corner the Rook" and this game is "Corner Two Rooks." The other is the generalization of Welter's game because if two rooks reach the edge lines of the board, they behave as the pieces of Welter's game. The authors derive a formula for $\mathscr{P}$-positions and show that this game can be considered as a misère game of the traditional Nim game with four piles. The rest of this paper is organized as follows: In Section 2, we define the game and show a theorem of $\mathscr{P}$-positions. In Section 3, we prove the theorem as a special case of the restricted version of the game. Finally, we provide the conclusions in Section 4.

2. Rule of "Corner Two Rooks"

We deviate from chess conventions and denote the cells on a chessboard by pairs of numbers. The cell in the upper-left corner is denoted by $(0, 0)$, and the others are denoted according to a Cartesian scheme. Cell (x, y) denotes x cells to the right followed by y cells downward, as shown in **Fig. 1**.

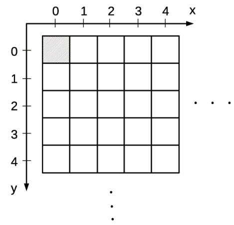

Fig. 1 Definition of coordinates in Corner Two Rooks.

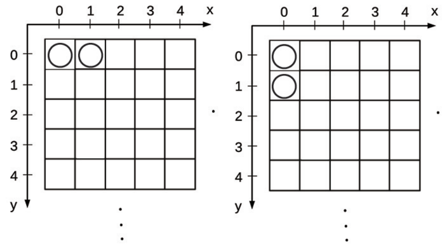

Fig. 2 Example of elements in $\mathcal{E}$ of Corner Two Rooks.

Definition 6.

(i) *We define "Corner Two Rooks." Let* $r > 1$ *and* n_0 *be fixed positive integers. Two rooks are placed on an* $n_0 \times n_0$ *chessboard, and two players take turns, and they select one of the rooks and move it. Rooks are vertically moved to the left or upward within a distance of* $r - 1$*, i.e., the reduction in a coordinate is smaller than* r.
If $r \ge n_0$*, then rooks are considered to be moved as far as possible on the board. A rook may jump over another rook but not onto another. The first player who cannot make a valid move loses.*

(ii) *We denote the positions of two rooks by* (x, y, z, w)*, where* (x, y) *is the position of one rook and* (z, w) *is the position of the other rook for any* $x, y, z, w \in \mathbb{N}_0$.

Definition 7. *For convenience sake, we define* $\mathcal{S}(x, y, z, w) = \{(x, y, z, w), (y, x, w, z), (z, w, x, y), (w, z, y, x)\}$.

The game described in Definition 6 was presented for the first time in Ref. [6]. We now define the ruleset of "Corner Two Rooks" based on Definition 1.

Definition 8 (Corner Two Rooks)**.** *We define* $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P))$ *as the ruleset of "Corner Two Rooks," as follows:* $\mathcal{H} = \{(x, y, z, w) : x, y, z, w \in \mathbb{N}_0, (x \ne z)$ *or* $(y \ne w)\}, \mathcal{E} = \mathcal{S}(0, 0, 1, 0)$*, and* $\mathrm{move}(x, y, z, w) = (\{(x - s, y, z, w) : 1 \le s \le \min(x, r - 1)\} \cup \{(x, y - s, z, w) : 1 \le s \le \min(y, r - 1)\} \cup \{(x, y, z - s, w) : 1 \le s \le \min(z, r - 1)\} \cup \{(x, y, z, w - s) : 1 \le s \le \min(w, r - 1)\}) \setminus \{(x, y, x, y), (z, w, z, w)\}$*, where* $s \in \mathbb{N}_0$.

The elements belonging to set $\mathcal{E}$ are shown in **Fig. 2**.

In this work, we only consider this game in normal play. Nevertheless, we use a theorem of misère games, and this is the reason that we are interested in this game.

Remark 1. *In Definition 8,* $\mathrm{move}(x, y, z, w)$ *does not contain* $\{(x, y, x, y), (z, w, z, w)\}$ *because a rook cannot jump onto another rook.*

If one rook can move onto another, then the game will become more simple, because we can regard it as the traditional Nim game with four piles and the limit, $r - 1$, on the number of stones that can be removed.

Definition 9 (Nim_{4r})**.** *We define the ruleset of* Nim_{4r} $(\mathcal{H}, \mathcal{E}, \mathrm{move}(P))$ *as follows:* $\mathcal{H} = \{(x, y, z, w) : x, y, z, w \in \mathbb{N}_0\}, \mathcal{E} = \{(0, 0, 0, 0)\}$*, and* $\mathrm{move}(x, y, z, w) = \{(x - s, y, z, w) : 1 \le s \le \min(x, r-1)\} \cup \{(x, y-s, z, w) : 1 \le s \le \min(y, r-1)\} \cup \{(x, y, z-s, w) 1 \le s \le \min(z, r-1)\} \cup \{(x, y, z, w-s) : 1 \le s \le \min(w, r-1)\}$.

Definition 10. *We define*

$$x_r = x \bmod r$$

and we let,

$$(x, y, z, w)_r = (x_r, y_r, z_r, w_r), \text{ and}$$

$$P_r = (x, y, z, w)_r \text{ for any } P = (x, y, z, w).$$

Theorem 4. *The set of the $\mathscr{P}$-positions of Nim$_{4r}$ (Definition 9) in normal play is* $\{(x, y, z, w) : x_r \oplus y_r \oplus z_r \oplus w_r = 0$ *and* $x, y, z, w \in \mathbb{N}_0\}$.

This can be evidently observed in Theorems 5 and 6 (see below), proposed by Sprague and Grundy.

Definition 11. *Let* $\mathcal{R}_1 = (\mathcal{H}_1, \mathcal{E}_1, \text{move}_1(P))$ *and* $\mathcal{R}_2 = (\mathcal{H}_2, \mathcal{E}_2, \text{move}_2(P))$ *be the rulesets of games. Then, we define the disjunctive sum of these games,* $\mathcal{R}_1 + \mathcal{R}_2 = (\mathcal{H}, \mathcal{E}, \text{move}(P))$*, as follows:* $\mathcal{H} = \{(P_1, P_2) : P_i \in \mathcal{H}_i\}, \mathcal{E} = \{(E_1, E_2) : E_i \in \mathcal{E}_i\}$*, and* $\text{move}(P_1, P_2) = \{(P'_1, P_2), (P_1, P'_2) : P'_i \in \text{move}_i(P_i)\}$.

Definition 12. *Let* $\mathcal{R} = (\mathcal{H}, \mathcal{E}, \text{move}(P))$ *be a ruleset and* $P \in \mathcal{H}$*. Then, we define* $\mathcal{G}(P) = \text{mex}(\{\mathcal{G}(P') : P' \in \text{move}(P)\})$*, where* $\text{mex}(A) = \min(\mathbb{N}_0 \setminus A)$ *for a set A of nonnegative integers.*

Theorem 5 (Sprague [9], Grundy [4])**.** $\mathcal{G}(P) = 0$ *if and only if P is a $\mathscr{P}$-position.*

Theorem 6 (Sprague [9], Grundy [4])**.** *Let* $P = (P_1, P_2)$ *be a position of* $\mathcal{R}_1 + \mathcal{R}_2$*. Then,* $\mathcal{G}(P) = \mathcal{G}(P_1) \oplus \mathcal{G}_(P_2)$.

In "Corner Two Rooks", the sets of $\mathscr{P}$-positions and $\mathscr{N}$-positions are similar to those of Nim$_{4r}$. In Definition 13, we characterize how they differ from Nim$_{4r}$.

Definition 13. *For* $x, y, z, w \in \mathbb{N}_0$*, let*

$$\mathcal{F}^{(1)} = \Big\{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(i, 2j, i, 2j+1), \text{ where } 0 \le i < r, 0 \le j < \left\lfloor \frac{r}{2} \right\rfloor\Big\},$$

$$\mathcal{N}_a = \{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(i, j, i, j), \text{ where } 0 \le i < r, 0 \le j < r\},$$

$$\mathcal{N}_b = \Big\{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(2i, 2j, 2i+1, 2j+1), \text{ where } 0 \le i < \left\lfloor \frac{r}{2} \right\rfloor, 0 \le j < \left\lfloor \frac{r}{2} \right\rfloor\Big\},$$

$$\mathcal{N}_c = \Big\{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(2i+1, 2j, 2i, 2j+1), \text{ where } 0 \le i < \left\lfloor \frac{r}{2} \right\rfloor, 0 \le j < \left\lfloor \frac{r}{2} \right\rfloor\Big\}$$

and

$$\mathcal{F}^{(0)} = \mathcal{N}_a \cup \mathcal{N}_b \cup \mathcal{N}_c.$$

Let

$$\mathcal{P} = (\{(x, y, z, w) : x_r \oplus y_r \oplus z_r \oplus w_r = 0 \text{ and } x, y, z, w \in \mathbb{N}_0\} \cup \mathcal{F}^{(1)}) \setminus \mathcal{F}^{(0)},$$

and

$$\mathcal{N} = (\{(x, y, z, w) : x_r \oplus y_r \oplus z_r \oplus w_r \neq 0 \text{ and } x, y, z, w \in \mathbb{N}_0\} \cup \mathcal{F}^{(0)}) \setminus \mathcal{F}^{(1)}.$$

Example 1. *Examples of elements belonging to set* $\mathcal{F}^{(1)}$ *are shown in* **Fig. 3***. Examples of the elements of set* $\mathcal{F}^{(0)}$ *are presented in* **Figs. 4** *and* **5***.*

Note that in these examples, rooks are side by side or diagonally opposite, but by Definition 13 they are side by side or diagonally opposite mod *r.*

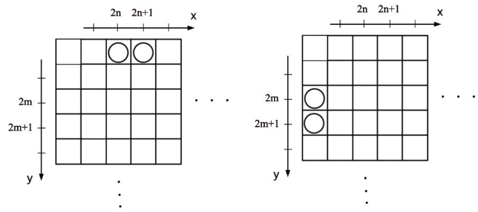

Fig. 3 Example of elements of $\mathcal{F}^{(1)}$.

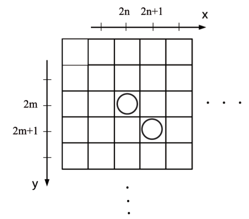

Fig. 4 Example of elements of $\mathcal{N}_b$.

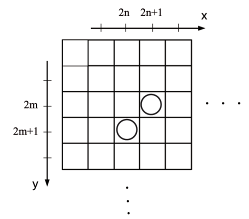

Fig. 5 Example of elements of $\mathcal{N}_c$.

Theorem 7. *When r is an even number,* $\mathcal{P}$ *and* $\mathcal{N}$ *are the sets of $\mathscr{P}$-positions and $\mathscr{N}$-positions, respectively, in "Corner Two Rooks".*

The authors present the proof of this theorem in Section 3 using a theorem for misère games.

3. Corner Two Rooks as a Misère Game of Traditional Nim

In this section, the authors present the proof of Theorem 7. First, we make a variant of the traditional Nim game with four piles with a limit r that is the restriction on the number of stones to be removed. Then, we prove that the misère version of this variant and "Corner Two Rooks" have the same set of $\mathscr{P}$-positions, and this leads to the proof of Theorem 7. We briefly discuss the misère version of the game. The method of determining the winner and loser depends on the play convention. The last player to move wins under the normal play convention but loses under the misère play convention. Thus far, we have been treating games under the normal play convention, but we shall now move to the misère version, which is crucial in this section.

We define a variant of Nim$_{4r}$ (Definition 9).

Definition 14 (Nim$_{4rb}$)**.** *We define the ruleset of Nim$_{4rb}$* $(\mathcal{H}, \mathcal{E}, \text{move}(P))$ *as follows:* $\mathcal{H} = \{(x, y, z, w) : x, y, z, w \in \mathbb{N}_0\}, \mathcal{E} = \{(x, y, z, w) : (x = z)$ *and* $(y = w)\}$*, and* $\text{move}(x, y, z, w) = \{(x - s, y, z, w) : 1 \le s \le \min(x, r-1)\} \cup \{(x, y - s, z, w) : 1 \le s \le \min(y, r-1)\} \cup \{(x, y, z - s, w) : 1 \le s \le \min(z, r-1)\} \cup \{(x, y, z, w - s) : 1 \le s \le \min(w, r-1)\}$

Lemma 1. *The set of the $\mathscr{P}$-positions of Nim$_{4rb}$ in normal play is the same as the set of the $\mathscr{P}$-positions of Nim$_{4r}$ in normal play.*

Proof. If $(x, y) = (z, w)$, then (x, y, z, w) is a $\mathscr{P}$-position of Nim_{4r}, then we obtain this theorem. □

Lemma 2. *The set of $\mathscr{P}$-positions of "Corner Two Rooks" is the same as the set of $\mathscr{P}$-positions of Nim_{4rb} in misère play.*

Proof. Let $(\mathcal{H}_1, \mathcal{E}_1, \mathrm{move}_1(P))$ be the ruleset of "Corner Two Rooks" and $(\mathcal{H}_2, \mathcal{E}_2, \mathrm{move}_2(P))$ be the ruleset of Nim_{4rb}. Then, $\mathcal{H}_2 = \mathcal{H}_1 \cup \{(x, y, z, w) : (x = z) \text{ and } (y = w)\} = \mathcal{H}_1 \cup \mathcal{E}_2$. As we consider misère play of Nim_{4rb}, every position in $\mathcal{E}_2$ is an $\mathscr{N}$-position.

Now, for every $P \in \mathcal{H}_1 \cap \mathcal{H}_2$, $\mathrm{move}_2(P) = \mathrm{move}_1(P) \cup \mathcal{A}$, where $\mathcal{A}$ is a subset of $\mathcal{E}_2$. As every element of $\mathcal{A}$ is an $\mathscr{N}$-position, it follows from Theorem 1 that the player who has a winning strategy in P does not change between "Corner Two Rooks" and Nim_{4rb} in misère play. Therefore, the sets of $\mathscr{P}$-positions are the same. □

We require Theorem 8 (see below) for the misère play game. See Refs. [7] and [8] for the detailed theory of misère games.

Definition 15. *Let P and Q be the positions of an impartial game and $\mathcal{A}$, $\mathcal{B}$ be two sets of positions of the game. Then, we use the following notations:*

(i) If $Q \in \mathrm{move}(P)$, we write $P \to Q$.
(ii) If $Q \notin \mathrm{move}(P)$, we write $P \overset{\times}{\to} Q$.
(iii) If $\mathrm{move}(P) \cap \mathcal{A} \neq \emptyset$, we write $P \to \mathcal{A}$.
(iv) If $\mathrm{move}(P) \cap \mathcal{A} = \emptyset$, we write $P \overset{\times}{\to} \mathcal{A}$.
(v) If $P \overset{\times}{\to} \mathcal{B}$ for any $P \in \mathcal{A}$, we write $\mathcal{A} \overset{\times}{\to} \mathcal{B}$.

Theorem 8 (Yamasaki [12])**.** *Let G be an impartial game with end position $\mathcal{E}$. Suppose that there are two sets of positions, $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$, that satisfy the following conditions:*

(a) $\mathcal{F}^{(0)} \cap \mathcal{F}^{(1)} = \emptyset, \mathcal{F}^{(0)} \supset \mathcal{E}$.
(b) $\mathcal{F}^{(0)} \setminus \mathcal{E} \to \mathcal{F}^{(1)}$ and $\mathcal{F}^{(1)} \to \mathcal{F}^{(0)}$.
(c) For $i = 0, 1$, $\mathcal{F}^{(i)} \overset{\times}{\to} \mathcal{F}^{(i)}$.
(d) If $P \notin \mathcal{F} = \mathcal{F}^{(0)} \cup \mathcal{F}^{(1)}$, $P \to \mathcal{F}^{(0)}$ if and only if $P \to \mathcal{F}^{(1)}$.
(e) If $P \in \mathcal{F} = \mathcal{F}^{(0)} \cup \mathcal{F}^{(1)}$, $P' \notin \mathcal{F}$ and $P \to P'$, then $P' \to \mathcal{F}$.

Then, the set of the $\mathscr{P}$-positions of the misère play game of G is

$$\mathcal{F}^{(1)} \cup (P \setminus \mathcal{F}^{(0)}).$$

Next, we use Theorem 8 for Nim_{4rb}.

Theorem 9. *We suppose that r is an even number. Then, sets $\mathcal{P}$ and $\mathcal{N}$ are the sets of the $\mathscr{P}$-positions and $\mathscr{N}$-positions of the misère play version of Nim_{4rb}, respectively.*

Proof. It is sufficient to prove that sets $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ in Definition 13 satisfy conditions (a), (b), (c), (d), and (e) of Theorem 8 for Nim_{4rb}. We check the conditions one by one.

(a) $\mathcal{F}^{(0)} \cap \mathcal{F}^{(1)} = \emptyset$ is directly obtained from the parity of the sum of four coordinates, and $\mathcal{F}^{(0)} \supset \mathcal{E}$ follows from Definition 13 and the definition of $\mathcal{E}$.

(b) If $x_r = 2i + 1$, then one can reduce to $x' = x - 1$, which satisfies $x'_r = 2i$, and if $x_r = 2i$ and $x \geq r$, then one can reduce to $x' = x - (r - 1)$, which satisfies $x'_r = 2i + 1$. We have a similar argument for other variables, y, z, and w.
Therefore, if $P = (x, y, z, w) \in \mathcal{F}^{(0)} \setminus \mathcal{E}$, we have a position P' such that $P \to P'$ and $P'_r \in \mathcal{S}(2i, j, 2i + 1, j)$ for some i, j. That is, $P \to \mathcal{F}^{(1)}$. (Note that if $(x, y, z, w)_r \in \mathcal{S}(i, j, i, j)$, then one of the variables must be larger than $r - 1$ because $(x, y, z, w) \notin \mathcal{E}$ and therefore, even if both i and j are even, $(x, y, z, w) \to P' \in \mathcal{F}^{(1)}$).
Conversely, if $P = (x, y, z, w) \in \mathcal{F}^{(1)}$, then we have a P' such that $P \to P'$ and $P'_r \in \mathcal{S}(2i, j, 2i, j)$ for some i, j. That is, $P \to \mathcal{F}^{(0)}$.

(c) We prove that $P \overset{\times}{\to} \mathcal{F}^{(i)}$ follows from $P \in \mathcal{F}^{(i)}$ for $i \in \{0, 1\}$.
Clearly, if $P \in \mathcal{N}_a$, $P \overset{\times}{\to} \mathcal{N}_a$. Similarly, if $P \in \mathcal{N}_b$, $P \overset{\times}{\to} \mathcal{N}_b$, and if $P \in \mathcal{N}_c$, $P \overset{\times}{\to} \mathcal{N}_c$.
For $P = (x, y, z, w)$, we define $p(P) = (|x_r - z_r|, |y_r - w_r|)$. Then, $p(P) = (0, 0)$ for $P \in \mathcal{N}_a$, and $p(P) = (1, 1)$ for $P \in \mathcal{N}_b$ and $P \in \mathcal{N}_c$. As a move cannot simultaneously change the two values of $p(P)$, we have $\mathcal{N}_a \overset{\times}{\to} \mathcal{N}_b$, $\mathcal{N}_a \overset{\times}{\to} \mathcal{N}_c$, $\mathcal{N}_b \overset{\times}{\to} \mathcal{N}_a$, and $\mathcal{N}_c \overset{\times}{\to} \mathcal{N}_a$.
We define $q(P) = ((x_r + y_r) \bmod 2, (z_r + w_r) \bmod 2)$. Then, $q(P) = (0, 0)$ for $P \in \mathcal{N}_b$, and $q(P) = (1, 1)$ for $P \in \mathcal{N}_c$. As a move cannot simultaneously change the two values of $q(P)$, we have $\mathcal{N}_b \overset{\times}{\to} \mathcal{N}_c$ and $\mathcal{N}_c \overset{\times}{\to} \mathcal{N}_b$.
If $P \in \mathcal{F}^{(1)}$, then clearly we have $P \overset{\times}{\to} \mathcal{F}^{(1)}$.

(d) We prove that $(P \to \mathcal{F}^{(0)}) \Leftrightarrow (P \to \mathcal{F}^{(1)})$ follows from $P \notin \mathcal{F} = \mathcal{F}^{(0)} \cup \mathcal{F}^{(1)}$. We suppose that $P \notin \mathcal{F}$.
First we consider the case that $P \to (x, y, z, w) \in \mathcal{F}^{(0)}$. Without loss of generality, we assume that $P = (x + k, y, z, w)$ with $0 < k < r$. If $x_r = 2i$, then we have $k \neq 1$ from $P \notin \mathcal{F}^{(1)}$. As r is an even number, $2i < r - 1$. Hence, we have $P \to (x + 1, y, z, w) \in \mathcal{F}^{(1)}$ (Note that $x + 1 \bmod r \neq 0$ because $2i < r - 1$). If $x_r = 2i + 1$, then we have $k \neq r - 1$ from $P \notin \mathcal{F}^{(1)}$. Therefore, we have $k < r - 1$ and we can reduce $x + k$ to $x - 1$. Then, we have $P \to (x - 1, y, z, w) \in \mathcal{F}^{(1)}$.
For the case that $P \to (x, y, z, w) \in \mathcal{F}^{(1)}$, we can show $P \to (x, y, z, w) \in \mathcal{F}^{(0)}$ in a similar manner.

(e) We prove that $P' \to \mathcal{F}$ follows from $P \in \mathcal{F}, P' \notin \mathcal{F}$, and $P \to P'$.
Let $P = (x, y, z, w)$ and $P' = (x', y, z, w)$. If $x \geq r$, then we have $P' \to (x - r, y, z, w)$. As $(x, y, z, w)_r = (x - r, y, z, w)_r$, we have $(x - r, y, z, w) \in \mathcal{F}$. We have a similar argument for the other variables, y, z, and w. Therefore, we need to consider only the case where the coordinate to be reduced is smaller than r.
For $P \in \mathcal{N}_a$ such that $P \to P' \notin \mathcal{F}$, we assume without any loss of generality that $P = (x, y, z, w)$ and $P' = (x', y, z, w)$. Let $t = x - x'$. Then, we have $x \leq z$ from $x_r = z_r$ and $x < r$. Further, $z - t \geq 0$, and we have $P' \to (x', y, z - t, y) \in \mathcal{N}_a$.
For $P = (x, y, z, w) \in \mathcal{N}_b \cup \mathcal{N}_c$ such that $P \to P' \notin \mathcal{F}$, we assume without any loss of generality that $P' = (x', y, z, w)$. There are four cases such that $(x_r, x'_r) = (2i, 2i'), (x_r, x'_r) = (2i, 2i' + 1), (x_r, x'_r) = (2i + 1, 2i')$ and $(x_r, x'_r) = (2i + 1, 2i' + 1)$. As we assumed $x < r$ and $P' \notin F$, $i > i'$. When $(x_r, x'_r) = (2i, 2i')$, we can reduce z such that $z_r = 2i + 1$ to z' such that $z'_r = 2i' + 1$ and, in a similar manner, when $(x_r, x'_r) = (2i + 1, 2i' + 1)$, we can reduce z to z' such that $z'_r = 2i'$. When $(x_r, x'_r) = (2i, 2i' + 1)$, $z_r = 2i + 1$ and, as r is an even number, $2i + 1 < r$. Then $2i + 1 - 2i' < r$ and therefore, one can reduce z to $z' = z - (2i + 1 - 2i')$, which satisfies $z'_r = 2i'$, in one move. When $(x_r, x'_r) = (2i + 1, 2i')$, as $i > i'$, one can reduce z to z' such that $z'_r = 2i' + 1$. Therefore, $P' \to (x', y, z', w) \in \mathcal{N}_b \cup \mathcal{N}_c \subset \mathcal{F}$.
For $P = (x, y, z, w) \in \mathcal{F}^{(1)}$ such that $P \to P' \notin \mathcal{F}$, we

assume without any loss of generality that $(x, y, z, w)_r = (i, 2j, i, 2j + 1)$. Similar to the above-mentioned cases, we can show that there is a position P'' such that $P' \to P''$ and $P'' \in \mathcal{F}^{(1)} \subset \mathcal{F}$.

Conditions (a), (b), (c), (d) and (e) are satisfied, and we finish the proof. □

Now, we can prove Theorem 7.

Proof of Theorem 7. According to Lemma 2, the set of $\mathscr{P}$-positions of Nim_{4rb} is the same as the set of $\mathscr{P}$-positions of "Corner Two Rooks." Therefore, we obtain the proof of this theorem from Theorem 9. □

Definition 16. *For $x, y, z, w \in \mathbb{N}_0$, let*

$$\mathcal{N}_1 = \Big\{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(r-1, 2j, r-1, 2j+1), \text{ where } 0 \le j < \left\lfloor \frac{r}{2} \right\rfloor\Big\},$$

$$\mathcal{PN}_1 = \{(x, y, z, w) : (x, y, z, w) \in \mathcal{N}_1, \text{ and If } x_r = z_r = r-1, \text{ then } x = z. \text{ If } y_r = w_r = r-1, \text{ then } y = w\},$$

$$\mathcal{F}^{(1)}_{odd} = (\mathcal{F}^{(1)} \setminus \mathcal{N}_1) \cup \mathcal{PN}_1,$$

$$\mathcal{P}_0 = \{(x, y, z, w) : (x, y, z, w)_r \in \mathcal{S}(r-1, j, r-1, j), \text{ where } 0 \le j < r\},$$

$$\mathcal{NP}_0 = \{(x, y, z, w) : (x, y, z, w) \in \mathcal{P}_0, \text{and If } x_r = z_r = r-1, \text{ then } x = z. \text{ If } y_r = w_r = r-1, \text{ then } y = w\},$$

$$\mathcal{F}^{(0)}_{odd} = (\mathcal{F}^{(0)} \setminus \mathcal{P}_0) \cup \mathcal{NP}_0,$$

$$\mathcal{P}_{odd} = (\{(x, y, z, w) : x_r \oplus y_r \oplus z_r \oplus w_r = 0\} \cup \mathcal{F}^{(1)}_{odd}) \setminus \mathcal{F}^{(0)}_{odd}$$

and

$$\mathcal{N}_{odd} = (\{(x, y, z, w) : x_r \oplus y_r \oplus z_r \oplus w_r \neq 0\} \cup \mathcal{F}^{(0)}_{odd}) \setminus \mathcal{F}^{(1)}_{odd}.$$

Theorem 10. *We suppose that r is an odd number. Then, sets $\mathcal{P}_{odd}$ and $\mathcal{N}_{odd}$ are the sets of the $\mathscr{P}$-positions and $\mathscr{N}$-positions of the misère play version of Nim_{4rb}, respectively.*

Proof.

(a) Similar to Theorem 9.

(b) For the case $P = (x, y, z, w) \in \mathcal{F}^{(1)}_{\mathrm{odd}}$, we can show that $P \to \mathcal{F}^{(0)}_{\mathrm{odd}}$ in a similar manner to the proof of Theorem 9. For the case $P = (x, y, z, w) \in (\mathcal{N}_a \setminus \mathcal{P}_0) \cup \mathcal{N}_b \cup \mathcal{N}_c$, we can also show $P \to \mathcal{F}^{(1)}_{\mathrm{odd}}$ in a similar manner.
When $P = (x, y, z, w) \in \mathcal{NP}_0, (x, y, z, w)_r \in \mathcal{S}(r-1, j, r-1, j)$. If $j = r-1$, then from the definition of $\mathcal{NP}_0$, $x = z$ and $y = w$, which is a contradiction. Thus, $j < r-1$. Therefore, for each case $j = 2j'$ and $j = 2j'+1$, there exists P' such that $P \to P'$ and $P'_r \in \mathcal{S}(r-1, 2j', r-1, 2j'+1) \subset \mathcal{PN}_1$.

(c) Similar to Theorem 9.

(d) We assume $P \to (x, y, z, w) \in \mathcal{NP}_0$. Without loss of generality, $P = (x + k, y, z, w)$ where $(0 < k < r)$. For the case $x_r = z_r = r-1$, if $k = 2i + 2 < r$, then $2 \le r - 1 - k + 2 < r$ and, therefore, $P \to (x + k, y, z - (r - 1 - k + 2), w)$ and $(x + k, y, z - (r - 1 - k + 2), w)_r \in \mathcal{S}(2i + 1, j, 2i, j)$. If $k = 2i + 1 < r - 1$, then $1 \le r - 1 - k < r - 1$ and, therefore, $P \to (x+k, y, z-(r-1-k), w)$ and $(x+k, y, z-(r-1-k), w)_r \in \mathcal{S}(2i, j, 2i + 1, j)$. For the other cases, we can prove that if $P \to \mathcal{F}^{(0)}_{\mathrm{odd}}$, then $P \to \mathcal{F}^{(1)}_{\mathrm{odd}}$ in a similar manner to the proof of Theorem 9.
Next, we assume $P \to (x, y, z, w) \in \mathcal{PN}_1$. Without loss of generality, $P = (x+k, y, z, w)$ where $(0 < k < r)$. For the case $x_r = z_r = r-1$, if $k = 2i + 2 < r$, then $2 \le r - 1 - k + 2 < r$ and, therefore, $P \to (x + k, y, z - (r - 1 - k + 2), w)$. For the other cases, we can prove that if $P \to \mathcal{F}^{(0)}_{\mathrm{odd}}$, then $P \to \mathcal{F}^{(1)}_{\mathrm{odd}}$ in a similar manner to the proof of Theorem 9.

(e) Let $\mathcal{F}_{\mathrm{odd}} = \mathcal{F}^{(0)}_{\mathrm{odd}} \cup \mathcal{F}^{(1)}_{\mathrm{odd}}$. If $P' \in \mathcal{P}_0 \cup \mathcal{N}_1$, then, from the definitions of $\mathcal{P}_0$ and $\mathcal{N}_1$, there is no P such that $P \to P'$ and $P \in \mathcal{F}_{\mathrm{odd}}$. Therefore, we assume $P' \notin \mathcal{P}_0 \cup \mathcal{N}_1$.
We assume $P = (x, y, z, w)$ and $P' = (x', y, z, w)$. If $x \ge r$, then we have $P' \to (x - r, y, z, w)$. As $(x, y, z, w)_r = (x - r, y, z, w)_r$, we have $(x - r, y, z, w) \in \mathcal{F}_{\mathrm{odd}}$, or, $x_r = z_r = r - 1$ and $x = z$. We need to consider the latter case. When $x_r = z_r = r - 1$ and $x = z$, $x'_r = 2i < r - 1$ or $x'_r = 2i + 1 < r - 1$ and therefore, there exists $z' < r - 1$ which satisfies $(x'_r, z'_r) = (2i, 2i + 1)$ or $(x'_r, z'_r) = (2i + 1, 2i)$. Thus, if $y_r = w_r$, then $P' \to (x', y, z', w) \in \mathcal{F}^{(1)}_{\mathrm{odd}} \subset \mathcal{F}_{\mathrm{odd}}$. If $y_r = 2j, w_r = 2j + 1$, then $P' \to (x', y, z - (x - x'), w) \in \mathcal{S}(i, 2j, i, 2j + 1) \subset (\mathcal{F}^{(1)} \setminus \mathcal{N}_1) \subset \mathcal{F}_{\mathrm{odd}}$. We have a similar argument for other variables. Therefore, we need to consider only the case that the coordinate to be reduced is smaller than r and we can show that $P' \to \mathcal{F}^{(0)}_{\mathrm{odd}} \cup \mathcal{F}^{(1)}_{\mathrm{odd}}$ follows from $P \in \mathcal{F}^{(0)}_{\mathrm{odd}} \cup \mathcal{F}^{(1)}_{\mathrm{odd}}, P' \notin \mathcal{F}^{(0)}_{\mathrm{odd}} \cup \mathcal{F}^{(1)}_{\mathrm{odd}}$, and $P \to P'$ in a similar manner to the proof of Theorem 9.

Conditions (a), (b), (c), (d) and (e) are satisfied, and we finish the proof. □

Next, we study the case that r is infinitely large or $r \ge n_0$ and two rooks are placed on an $n_0 \times n_0$ chessboard.

Definition 17. *For $x, y, z, w \in \mathbb{N}_0$, let*

$$\mathcal{F}'^{(1)} = \{(x, y, z, w) : (x, y, z, w) \in \mathcal{S}(i, 2j, i, 2j + 1), \text{ where } i, j \in \mathbb{N}_0\},$$

$$\mathcal{N}'_b = \{(x, y, z, w) : (x, y, z, w) \in \mathcal{S}(2i, 2j, 2i + 1, 2j + 1), \text{ where } i, j \in \mathbb{N}_0\},$$

$$\mathcal{N}'_c = \{(x, y, z, w) : (x, y, z, w) \in \mathcal{S}(2i + 1, 2j, 2i, 2j + 1), \text{ where } i, j \in \mathbb{N}_0\}$$

and

$$\mathcal{F}'^{(0)} = \mathcal{N}'_b \cup \mathcal{N}'_c.$$

Let

$$\mathcal{P}' = (\{(x, y, z, w) : x \oplus y \oplus z \oplus w = 0 \text{ and } x, y, z, w \in \mathbb{N}_0\} \cup \mathcal{F}'^{(1)}) \setminus \mathcal{F}'^{(0)}$$

and

$$\mathcal{N}' = (\{(x, y, z, w) : x \oplus y \oplus z \oplus w \neq 0 \text{ and } x, y, z, w \in \mathbb{N}_0\} \cup \mathcal{F}'^{(0)}) \setminus \mathcal{F}'^{(1)}.$$

Theorem 11. *$\mathcal{P}'$ and $\mathcal{N}'$ are the sets of the $\mathscr{P}$-positions and $\mathscr{N}$-positions, respectively, of "Corner Two Rooks" without any restriction on the movement of rooks.*

Proof. When r is infinitely large or $r \ge n_0$ and two rooks are placed on an $n_0 \times n_0$ chessboard, Theorem 7 is the same as this theorem. □

4. Conclusion

We conclude by summarizing the impact of this study on the research field of combinatorial games. In this article, we describe three games: (1), (2), and (3), and study the relation between them.

(1) "Corner Two Rooks": A game presented by the authors. It is a normal game.
(2) The classical Nim with four piles: It is also a normal game.
(3) A variant of game (2): It is also introduced by the authors.

In this article, the authors showed that games (2) and (3) have the same set of $\mathcal{P}$-positions, so their mathematical structure is almost the same, and game (1) and the misère game of (3) have the same set of $\mathcal{P}$-positions, using Yamasaki's theorem for misère games. As far as the impact of this study on combinatorial game theory is concerned:

(a) The authors demonstrated a new aspect of the theory of classical Nim by proving that its misère version is closely related to a new game, "Corner Two Rooks".
(b) Traditionally, a misère version of a game is more complicated than the normal version, since useful theorems that are valid in the normal version are not valid in the misère version. For example, the disjunctive sum of $\mathcal{P}$-positions is not always a $\mathcal{P}$-position in the misère version. In this work, the authors used the misère version of game (3) to study game (1). This is a very unusual method, and there is a possibility that this method could be useful for the analysis of other games.
(c) This study includes a beautiful application of Yamasaki's theorem, and the authors were able to show the importance of this theorem in the field of misère games.

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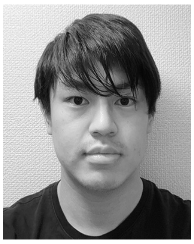

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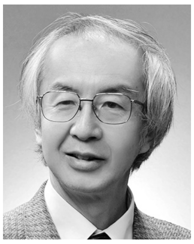

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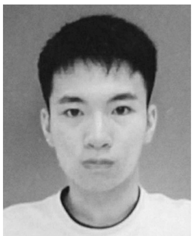

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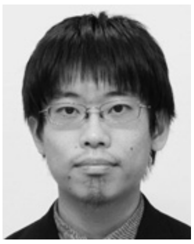

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