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Enumeration of Associative Magic Squares of Order 7

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Abstract: An associative magic square is a magic square such that the sum of any 2 cells at symmetric positions with respect to the center is constant. The total number of associative magic squares of order 7 is enormous and thus, it is not realistic to obtain the number by simple backtracking. As a recent result, Ripatti reported the number of semi-magic squares of order 6 (the magic squares of 6×6 without diagonal sum conditions) in 2018. In this research, with reference to Ripatti's method of enumerating semi-magic squares, we have calculated the total number of associative magic squares of order 7. There are exactly 1,125,154,039,419,854,784 associative magic squares of order 7 excluding symmetric patterns.

1. Introduction

A magic square of order n is an $n \times n$ square grid such that the sums of the numbers in each row, column, and diagonal are equal. The semi-magic square and associative magic square are special kinds of magic squares. A semi-magic square is a magic square without the diagonal sum condition. An associative magic square is a magic square such that the sum of any 2 cells at symmetric positions with respect to the center is constant. **Figure 1** shows examples of these squares.

The known numbers of these squares are summarized in **Table 1**. These squares remain of the same kind even after rotating and reflecting their entries. Thus, the table represents numbers of essentially different squares relative to these operations. The history of magic squares is long. It is said that a magic square of order 3, called lo shu, was described in China around 2200 BC. 880 magic squares of order 4 were found by Bernard Frenenicle de Bessy in 1693 [2], and Kathleen Ollerenshaw and Hermann Bondi proved that no other magic square of order 4 exists [4]. The number of magic squares of order 5 was calculated by Richard Schroeppel by using a backtracking algorithm in 1973, and the results were published by Martin Gardner in 1976 [5]. The number of magic squares of order 5 was calculated in 1973, but even after more than 40 years, the number of magic squares of order 6 remains unknown.

According to Walter Trump's website [7], Mutsumi Suzuki calculated the number of associative magic squares of order 5 to be 48,544. In 1919, Charles Planck proved that there are no associative magic squares of order 6 [3]. Although it is known that there are many associative magic squares of order 7, the exact number was not known until this report.

The number of 7×7 associative magic squares was estimated to

Fig. 1 Example of magic squares.

Table 1 Number of magic squares.

n	semi-magic	magic	associative magic
3	9	1	1
4	68,688	880	48
5	579,043,051,200	275,305,224	48,544
6	94,590,660,245,399,996,601,600	(unsolved)	0
7	(unsolved)	(unsolved)	**(this research)**

be within the range of $(1.125151 \pm 0.000051) \times 10^{18}$ with a probability of 99% by Walter Trump, who used a method combining Monte Carlo and backtracking [7]. Since there are approximately 10^{18} associative squares of order 7, we cannot calculate the exact number in a realistic amount of time with simple backtracking algorithms, which require a computational time at least proportional to the number of solutions.

On the other hand, Ripatti recently reported the number of semi-magic squares of order 6 in 2018 [1]. Ripatti divided the square into two parts and enumerated each part. Then, he combined the enumerations of each part. This method proved to be faster than simple backtracking.

In this paper, we extend Ripatti's method for semi-magic squares to associative magic squares and propose an algorithm to calculate the number of associative magic squares of order 7. Section 2 describes the properties of associative squares. Next, Section 3 outlines Ripatti's method of counting 6×6 semi-magic squares. Section 4 describes our algorithm to enumerate associative magic squares of order 7, and Section 5 shows the results of

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the calculation.

2.   Preliminary

In this paper, we refer to an $n \times n$ matrix such that natural numbers $1, 2, \cdots, n^2$ appear once only as a square (of order n). Let X_{ij} be the i-th row, j-th column element of the square. An associative magic square of order 7 is a magic square such that the sum of any 2 cells at symmetric positions from the center equals 50. An example of a 7×7 associative magic square is shown in **Fig. 2**.

The sum of the numbers in each row of an associative magic square is $(1+2+\cdots+49)/7 = 175$. Each associative magic square represents exactly 8 associative magic squares under rotation and reflection. We can calculate the total number of associative magic squares and then divide it by 8 to get the number of associative magic squares up to a reflection or rotation.

2.1   Properties of Associative Squares

A 7×7 square (not necessarily a magic square) such that the sum of any two elements at symmetrical positions from the center element is constant ($X_{ij} + X_{(8-i)(8-j)} = 7^2 + 1 = 50$) has the following properties.

(1) The central element X_{44} is always 25.
(2) The element at the counterpart position can be determined as follows:
$$X_{ij} = 50 - X_{(8-i)(8-j)} (i, j = 1, 2, \cdots, 7)$$
(3) The sum of the numbers on each diagonal must be 175.
$$\sum_{i=1}^{7} X_{ii} = \sum_{i=1}^{7} X_{(8-i)i} = 175$$
(4) The sums of the 4-th row and column must each be 175.
$$\sum_{i=1}^{7} X_{i4} = \sum_{i=1}^{7} X_{4i} = 175$$
(5) The sum of the k-th row is 175 if and only if the sum of the $(8-k)$-th row is 175.
$$\sum_{i=1}^{7} X_{ki} = 175 \Leftrightarrow \sum_{i=1}^{7} X_{(8-k)i} = 175 (k = 1, 2, 3)$$

From (1) and (2), all elements of the square are determined once we have decided on the values of the 24 cells shown in

1	41	17	38	20	21	37
40	3	18	44	5	34	31
48	26	35	27	28	4	7
11	14	8	25	42	36	39
43	46	22	23	15	24	2
19	16	45	6	32	47	10
13	29	30	12	33	9	49

Fig. 2   Example of an associative magic square of order 7.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24				

Fig. 3   24 key cells of associative magic squares.

Fig. 3. The sums of all the rows, columns, and diagonals should be 175 in order for a square X satisfying the above symmetry to be a magic square. Moreover, for X to be a magic square, it is sufficient that only 1st, 2nd, 3rd rows and columns of X from (3), (4) and (5) satisfy the constraints.

3.   Previous Method

In this section, we outline the method [1] that Ripatti used for counting semi-magic squares of order 6. A semi-magic square of order 6 is a 6×6 square grid such that the sums of the numbers in any row or column (not necessarily a diagonal) are equal. An example of a 6×6 semi-magic square is shown in **Fig. 4**. The sum of the numbers of each row or column of a 6×6 semi-magic square is $(1 + 2 \cdots + 36)/6 = 111$. The number of semi-magic squares of order 6 is 94,590,660,245,399,996,601,600, and as in the case of associative magic squares of order 7, it is not realistic to count them with backtracking.

A semi-magic square can be transformed into $6! \times 6! = 518,400$ different semi-magic squares by rearranging the rows and columns. There are 6! arrangements of rows and 6! arrangements of columns.

Ripatti defined canonical semi-magic squares as representatives of these 518,400 semi-magic squares. We can count the canonical semi-magic squares and then multiply the number by 518,400 to get the total number of semi-magic squares.

Next, Ripatti divided the square into the upper half and lower half as shown in **Fig. 5**.

Let the *profile* of the upper half of the square be:

$$\left(\sum_{i=1}^{3} X_{i1}, \sum_{i=1}^{3} X_{i2}, \sum_{i=1}^{3} X_{i3}, \cdots, \sum_{i=1}^{3} X_{i6} \right)$$

Let the *profile* of the lower half of the square be:

$$\left(111 - \sum_{i=4}^{6} X_{i1}, 111 - \sum_{i=4}^{6} X_{i2}, \cdots, 111 - \sum_{i=4}^{6} X_{i6} \right)$$

9	15	32	21	26	8
17	31	3	33	2	25
29	5	13	10	35	19
20	36	7	14	4	30
24	1	34	6	28	18
12	23	22	27	16	11

Fig. 4   Example of a 6×6 semi-magic square.

A	A	A	A	A	A
A	A	A	A	A	A
A	A	A	A	A	A
B	B	B	B	B	B
B	B	B	B	B	B
B	B	B	B	B	B

Fig. 5   Division of a 6×6 square.

Fig. 6 Profile of a semi-magic square.

For example, the profile of the upper half of **Fig. 6** is $(70, 42, 57, 47, 53, 64)$, while that of the lower half is $(111 - 41, 111 - 69, 111 - 54, 111 - 64, 111 - 58, 111 - 47) = (70, 42, 57, 47, 53, 64)$. In accordance with these definitions, the combination of the upper half and lower half is a square such that the sum of the elements of each column is 111 when the two halves have the same profile. Therefore, the combination of the upper half and lower half squares is a semi-magic square if and only if the combination satisfies all of the following conditions.

- Each of the upper half and lower half squares satisfies the semi-magic row constraints (the sum of the elements of each row is equal to 111).
- The profiles of the upper and lower squares are the same.
- Each element appears only once in the combination of the upper half and lower half.

It is possible to count the number of semi-magic squares according to the following procedure.

(1) Create all sets (U, L) (U is a set of 18 elements in the upper half of the square, and L is a set of the 18 elements of the lower half). Perform the following procedure for each (U, L). ($U \cup L = \{1, 2, \cdots, 36\}$)

(2) Prepare an array of counters $N_U[p]$ for a respective profile p, and initialize its elements to 0. Generate all the upper halves of the square such that the sum of the elements of each row is 111 using U numbers; then compute the profile value p and increment $N_U[p]$ for each of the upper half. Thus, we can calculate that there are $N_U[p]$ patterns for the upper half satisfying the constraints for each profile.

(3) As well as (2), prepare an array of counters $N_L[p]$ for a respective profile p, and initialize its elements to 0. Generate all the lower halves of the square such that the sum of each row is 111 using L numbers; then compute the profile value p and increment $N_L[p]$ for each of the lower half.

(4) The number of semi-magic squares of order 6 for one pair (U, L) is $\sum_p (N_U[p] \times N_L[p])$.

In this way, we can divide the problem of enumerating semi-magic squares into two problems: counting the upper half of the squares and counting the lower half. If we use simple backtracking without splitting the square, we need to count $N_U[p] \times N_L[p]$ semi-magic squares one by one. However, when counting separately, we count only $N_U[p]$ upper halves and $N_L[p]$ lower halves in order to count $N_U[p] \times N_L[p]$ semi-magic squares.

Straightforwardly, the number of pairs (U, L) is $_{36}C_{18}$, but by considering constraints such as $\sum_{x \in U} x = \sum_{x \in L} x = 111 \times 3$, Ripatti suggested that it is sufficient to consider only 9,366,138

(U, L) pairs.

According to Ripatti, we can efficiently count the upper halves of the squares and lower halves by using the family of sets of numbers that fit on individual lines of semi-magic squares, $S = \{s \mid s \subset \{1, 2, \cdots, 36\}, \; |s| = 6, \; \sum_{a \in s} a = 111\}$.

It is possible to calculate efficiently if we count only the combinations of the upper half squares and lower half squares which become canonical semi-magic squares. Ripatti also used other speed-up techniques and handled corner cases but we will omit them here.

Speeding up the calculations by parallelization is effective since this counting method is completely independent for each (U, L) pair. Ripatti's method took over 5 months on 10 threads to report that the number of semi-magic squares of order 6 was 94,590,660,245,399,996,601,600.

4. Our Method

4.1 Overview

We extend the existing method described in Section 3 so that it can be used to count associative magic squares of order 7. First, we describe row and column rearrangements that transform associative magic squares into different ones. Next, we propose one division of the 7×7 square to count parts separately. We define a profile well suited for the division and show that we can divide up the problem of counting associative magic squares into two. Finally, we present the detailed calculation procedure for counting associative magic squares of order 7.

4.2 Associative Magic Square Transformations

Associative magic squares of order 7 can be transformed into other associative magic squares by symmetrical swapping of rows and columns with respect to the center. Such swappings are shown below.

(1) Swap the 1st and 7th rows.
(2) Swap the 2nd and 6th rows.
(3) Swap the 3rd and 5th rows.
(4) Swap the 1st and 2nd rows, and swap 6th and 7th rows.
(5) Swap the 2nd and 3rd rows, and swap 5th and 6th rows.
(6) Swap the 1st and 7th columns.
(7) Swap the 2nd and 6th columns.
(8) Swap the 3rd and 5th columns.
(9) Swap the 1st and 2nd columns, and swap 6th and 7th columns.
(10) Swap the 2nd and 3rd columns, and swap 5th and 6th columns.

For example, as shown in **Fig. 7**, the associative magic square on the right is obtained by swapping rows (4) of the left square.

Since the above rearrangements are symmetrical replacements of rows and columns, two elements in the symmetrical position before swapping remain symmetrical after swapping. As stated in (3) in Section 2.1, the sums of the numbers on the diagonals equal 175 when the sum of any two symmetrical elements is constant. The sum of each row and each column after transformation is the same as before, since the set of seven numbers in each row and each column remains the same before and after the replacements. Therefore, associative magic squares remain associative

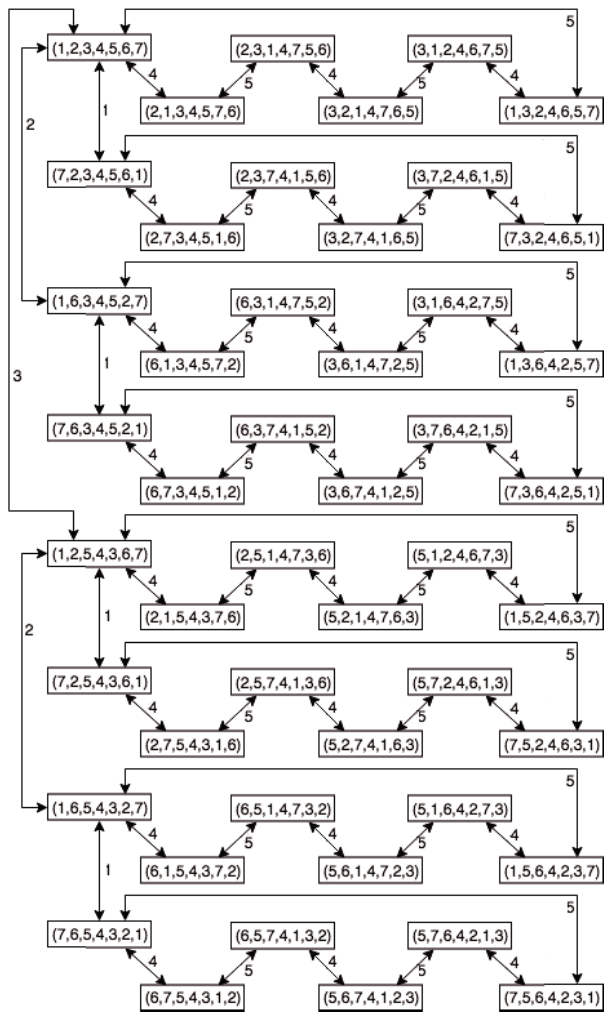**Fig. 7**   Swapping rows (4) of an associative magic square.

Fig. 8   48 permutations of rows in associative magic squares of order 7.

after symmetrical rearrangements of rows and columns such as given above.

Figure 8 shows how symmetric squares are transformed by the above row swapping. Here, $(r_1, r_2, r_3, \cdots, r_7)$ represents an associative magic square such that the i-th row is replaced by the r_i-th row of the original associative magic square. The original associative magic square is represented by $(1, 2, 3, 4, 5, 6, 7)$. Here, $(r_1, r_2, r_3, \cdots, r_7)$ equals $(r_1, r_2, r_3, 4, 8 - r_3, 8 - r_2, 8 - r_1)$, because the two rows in the symmetrical position remain symmetrical with row swapping (1) through (5), and the fourth row doesn't move. Thus, three pairs $(1, 7), (2, 6), (3, 5)$ are assigned to one of r_1, r_2, r_3, respectively. There are 2^3 ways to choose which of the two numbers of each pair is assigned to r_1, r_2, r_3, which is expressed as the vertical transformations shown

(a) 3 upper rows and 4 lower rows

(b) Checkered pattern

(c) 1 center row and 6 outer rows

(d) 3 center rows and 4 outer rows

Fig. 9   Examples of dividing the square.

in Fig. 8. There are 3! ways to choose pairs to be assigned to one of $(r_1, r_7), (r_2, r_6), (r_3, r_5)$, which is expressed as the horizontal transformations shown in Fig. 8. Therefore, an associative magic square can be transformed $2^3 \times 3! = 48$ ways by rearranging the rows. We can also transform an associative magic square into 48 associative magic squares by rearranging the columns. Thus, an associative magic square of order 7 can be transformed into $48 \times 48 = 2304$ associative magic squares in total.

In Section 4.5.2, we defined a canonical associative magic square of order 7 that represents the 2304 associative magic squares made by such transformations. We can count all the associative magic squares by calculating the number of only canonical associative magic squares and multiplying that number by 2304.

4.3   Square Division

Since there are 49 cells in the 7×7 square, there are 2^{49} ways to divide the square into two. Some examples are shown below.

If we divide the square into an upper half and lower half like the semi-magic square division shown in **Fig. 9** (a) to count associative magic squares, it is difficult to consider the upper half and lower half squares independently because the arrangement of elements in the upper half complexly affects the arrangement of elements in the lower half, wherein $X_{ij} + X_{(8-i)(8-j)} = 50$. Therefore, it is preferable to divide up a square so that two cells at symmetrical positions are included in the same group.

If we divide the square into two alternating groups, A and B, as shown in Fig. 9 (b), two cells at symmetrical positions are put into the same group. However, each row and each column of the square are divided into two groups. It is difficult to consider groups A and B independently because the arrangement of elements in group A has a complicated effect on the arrangement of elements in group B such that the sum of the elements of each row and column is equal to 175. Therefore it is better to avoid dividing the rows and columns in the two groups as much as possible.

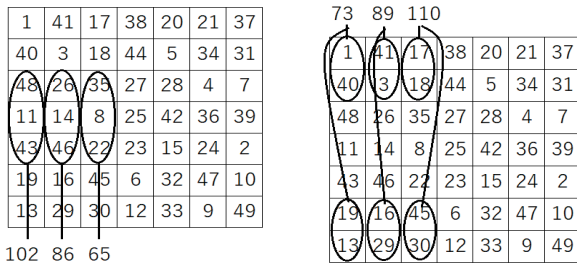As shown in Fig. 9 (c), when dividing the square into one cen-

Fig. 10   Profiles of 7×7 associative magic square.

ter row and six outer rows, symmetrical cells are included in the same group, and the rows are not divided. However, the number of cells in the six outer rows is too large. Therefore, there is little difference in problem size between counting associative squares of order 7 directly and counting 6 outer rows that satisfy some constraints. As a result, we cannot efficiently calculate with such divisions. We should mostly divide the square into two halves in order to reduce the sizes of the two counting subproblems obtained by the division.

From the above, the division shown in Fig. 9 (d) is superior to many of the other divisions. Therefore, we will consider counting associative magic squares of order 7 by dividing up the square into 3 center rows and 4 outer rows.

4.4 Profiles of Divided Groups

Considering the arrangements of the numbers in the center part and outer part such that the sums of any two symmetrical cells are the same, from the considerations given in Section 2.1, a combination of arrangements of the center part and the outer part becomes an associative magic square if and only if the sums of the elements of the individual 1st, 2nd, and 3rd rows and columns are 175, and each element appears only once in the combination. In the existing method of the semi-magic squares of order 6, it was necessary to define the profile as a vector of 6 partial sums of the columns, however we can define the profiles of associative magic squares as a vector of 3 partial sums for the left three columns because of their symmetric constraints. We define the profile of the center parts as

$$\left(\sum_{i=3}^{5} X_{i1}, \ \sum_{i=3}^{5} X_{i2}, \ \sum_{i=3}^{5} X_{i3} \right)$$

and define profile of the outer parts as

$$(175 - X_{11} + X_{21} + X_{61} + X_{71},$$
$$175 - X_{12} + X_{22} + X_{62} + X_{72},$$
$$175 - X_{13} + X_{23} + X_{63} + X_{73})$$

For example, the profile of the center part of the square shown in **Fig. 10** (a) is $(102, 86, 65)$, and the profile of the outer part of the same square shown in Fig. 10 (b) is $(175-73, 175-89, 175-110)$. The sums of the 1st, 2nd and 3rd columns of a combination of center and outer rows are each 175 if and only if the combination consists of center and outer parts that have the same profile.

Therefore, a combination of the center part and outer part of a square is an associative magic square if and only if it satisfies all of the following conditions.

- For each center part and outer part of the square, the sum of any 2 symmetrical elements is constant.
- The sum of the elements of the 3rd row of the square is equal to 175.
- The sums of the elements of the 1st and 2nd rows of the square are each equal to 175.
- The center part and the outer part of the square have the same profile.
- Each element appears exactly once in the combination.

4.5 Procedure for Counting Associative Magic Squares of Order 7

From the discussion in Section 4.4, we can calculate the total number of associative magic squares by using the following procedure.

(1) Create all sets (S_A, S_B) (S_A is a set of 21 elements in the center parts of the square, and S_B is a set of 28 elements in the outer parts.). Perform the following procedure for each (S_A, S_B). ($S_A \cup S_B = \{1, 2, \cdots, 49\}$)

(2) Prepare an array of counters $N_A[p]$ for the respective profile p, and initialize its entries to 0. Using S_A numbers, generate all the center parts of the square such that the sum of the elements of the 3rd row of the square is 175 and the sum of any two symmetrical elements is 50, satisfying canonical associative magic square constraints. For each of the generated center parts of the square, compute the profile value p and increment $N_A[p]$. Thus, we can calculate that there are $N_A[p]$ patterns for the center part satisfying the constraints for each profile.

(3) The same as in (2) above, prepare an array of counters $N_B[p]$ for respective profile p, and initialize its entries to 0. Using S_B numbers, generate all the outer parts of the square such that the sums of the elements of the 1st and 2nd rows are each 175 and the sum of any two symmetrical elements is 50, satisfying the canonical associative magic square constraints. For each of the outer parts, compute the profile value p and increment $N_B[p]$.

(4) The number of canonical associative magic squares of order 7 for one pair (S_A, S_B) is $\sum_p (N_A[p] \times N_B[p])$.

The details of (1) are in Section 4.5.1, while the definitions of canonical associative magic squares are in Section 4.5.2. Moreover, the details of (2) are in Section 4.5.3, and the details of (3) are described in Section 4.5.4.

In this way, we can divide the problem of enumerating associative magic squares of order 7 into two problems: counting the center parts of the square and counting the outer parts. If we enumerate them with simple backtracking without splitting the square, we need to explore $N_A[p] \times N_B[p]$ patterns of associative magic squares one by one. However, when counting halves separately, we explore only $N_A[p]+N_B[p]$ patterns to count $N_A[p] \times N_B[p]$ associative magic squares.

4.5.1 Number of (S_A, S_B) Pairs

We will discuss the number of (S_A, S_B) pairs in the procedure of (1) in Section 4.5. Considering simply, the number of pairs (S_A, S_B) is $_{49}C_{21}$ (about 3.9×10^{13}). But, there are $_{48}C_{20}$ ways, if we consider that X_{44} must be 25. Since the sum of any two

symmetrical elements about the center is constant, if S_A includes x, then S_A must include $50 - x$. Therefore, it is sufficient to divide 24 numbers $1, 2, \cdots, 24$ into (S_A, S_B). Thus, we only have to calculate $_{24}C_{10} = 1,961,256$ (S_A, S_B) pairs.

4.5.2 Canonical Associative Magic Squares of Order 7

As described in Section 4.2, we define a canonical associative magic square of order 7 that represents the 2304 associative magic squares made by the transformations, and count only the canonical associative magic squares. We can calculate the number of associative magic squares by multiplying the number of canonical squares by 2304 and thus reduce the number of squares that need to be counted by a factor of 2304.

In the procedure shown in Section 4.5, it takes much more computation time to count the outer parts than the center parts because of the difference in the number of included cells. Therefore, we face a bottleneck in counting the outer parts that satisfy some conditions. We can prune the search to count the outer parts efficiently by imposing the constraints of canonical associative magic squares as much as possible on the outer parts. From the above, we define canonical associative magic squares as those that satisfy all of the following conditions.

(1) $X_{11} < X_{17}, X_{12} < X_{16}, X_{13} < X_{15}$
(2) $X_{11} < X_{12} < X_{13}$
(3) $row_1 < row_7, row_2 < row_6, row_3 < row_5$
(4) $row_3 < row_1 < row_2$
However, $row_i = \min\{X_{i1}, X_{i2}, \cdots, X_{i7}\}$.

(1) and (2) are restrictions on column swapping, and these restrictions constrain the first row. (3) and (4) are restrictions on row swapping, but it is not possible to put all these restrictions on the four outer rows. We make row_3 small instead, and the following formula holds:

$$row_3 = \min\{row_1, row_2, row_3, row_5, row_6, row_7\}$$

The 3rd row of the canonical squares always contains the smallest number, except for numbers in the 4th row. Thus, the center parts always contain the element one. The number of (S_A, S_B) pairs that need to be calculated as described in Section 4.5.1, is found to be $_{23}C_9 = 817,190$ using this property of canonical squares. By defining the canonical associative magic squares in this way, the counting of the outer parts becomes approximately 384 times faster, and the number of (S_A, S_B) pairs that need to be calculated is approximately 0.4 times the total.

4.5.3 Counting the Center Parts

Here, we describe the (2) of the procedure in Section 4.5. From Fig. 3 in Section 2.1, if the numbers in the 3rd row of the square and X_{41}, X_{42}, X_{43} are determined, the remaining elements in the center part are also determined.

We define R, a family of sets of 7 numbers that can be put in rows other than the 4th row of the associative magic of order 7, as

$$R = \{r \mid r \subset \{1, 2, \cdots, 24, 26, 27, \cdots, 49\}, \ |r| = 7,$$
$$\sum_{a \in r} a = 175, \forall x, y \in r, x + y \neq 50\}$$

We have generated R in preparation for counting the center parts. The family size of R is 452,188. When we use S_A numbers, R_3,

the family of possible sets of 7 numbers in the 3rd row of an associative square of order 7, can be easily calculated as

$$R_3 = \{r \mid r \in R, r \subset S_A\}$$

For each $r \in R_3$, there are 7! ways for all permutations of r to choose the elements of the 3rd row, and the elements in the 5th row are determined by the elements in the symmetrical 3rd row. But, we exclude the $row_3 < row_5$ arrangement from the constraints of the canonical form. There are $6 \times 4 \times 2$ ways to choose the elements of X_{41}, X_{42}, X_{43} from the unused numbers in S_A. We generate all the arrangements of elements in the center parts, calculate of profile p for each arrangement, and increment $N_A[p]$.

4.5.4 Counting the Outer Parts

Here, we describe (3) of the procedure in Section 4.5. From Fig. 3 in Section 2.1, if the elements in the 1st and 2nd rows are determined, the remaining elements in the outer parts are also determined.

Besides counting the center parts, R_1, we can easily calculate the family of sets of 7 elements in the first row of the associative magic square,

$$R_1 = \{r \mid r \in R, r \subset S_B\}$$

Next, we define R_{26}, a family of 14-number subsets of S_B excluding the numbers in the 1st and 7th rows, as

$$R_{26} = \{r \mid r \subset \{1, 2, \cdots, 24, 26, 27, \cdots, 49\}, |r| = 14,$$
$$\forall x \in r, \ \exists y \in r, \ x + y = 50\}$$

Furthermore, we can pre-generate R_2 for each $S_{26} \in R_{26}$, a family of 7-number subsets of S_{26} that can be put on the 2nd row of the associative magic square of order 7, defined as

$$R_2 = \{r \mid r \subset S_{26}, \ |r| = 7, \ \sum_{a \in r} a = 175,$$
$$\forall x, y \in r, \ x + y \neq 50\}$$

By pre-generating R_2 for all $S_{26} \in R_{26}$, it is possible to count the outer parts by the following procedure. For each $r \in R_1$, there are 7! ways for all permutations of r to choose the elements of the 1st row, and the elements in the 7th row are determined by the elements in the symmetrical 1st row. But, we exclude arrangements violating the constraints of the canonical form. Let S_{26} be the set of unused numbers of S_B (excluded r and the numbers in the 7th row). For each $r' \in R_2$ calculated from $S_{26} \in R_{26}$, there are 7! ways for all permutations of r' to choose the elements of the 2nd row, and the elements in the 6th row are determined by the elements in the symmetrical 2nd row. We generate all the arrangements of elements in the outer parts, calculate the profile p for each arrangement, and increment $N_B[p]$.

5. Experimental Results

5.1 Results

We wrote a C++ program implementing the procedure described in section 4. We assigned $1, 2, \cdots, 817190$ ID numbers to each pair of (S_A, S_B). Our program counts associative magic squares of order 7 for each (S_A, S_B) designated with ID numbers.

Table 2 Calculation results.

(S_A, S_B) ID	number of associative magic	calc. time	computer
1–50000	100798108317305280	14.5 days	2
50001–100000	91535720218951104	14.4 days	1
100001–150000	88372685889123552	14.2 days	1
150001–200000	83733351186221856	14.1 days	1
200001–250000	81588443264793504	14.0 days	2
250001–300000	79361704382078592	13.8 days	1
300001–350000	68614934779440864	13.9 days	1
350001–400000	60333826371280992	13.9 days	1
400001–450000	58972609900819872	13.6 days	2
450001–500000	56989665917916192	13.6 days	1
500001–550000	58076327642080032	13.6 days	1
550001–600000	58605580160376480	13.7 days	1
600001–650000	56406391669618560	12.9 days	2
650001–700000	56103389221682304	13.5 days	1
700001–750000	54683346217110336	13.0 days	1
750001–817190	70977954281055264	14.8 days	1

Table 3 Known numbers of squares including the results of this research [7].

n	semi magic	magic	associative magic
3	9	1	1
4	68,688	880	48
5	579,043,051,200	275,305,224	48,544
6	94,590,660,245,399,996,601,600	$(1.775399(42) \times 10^{19})$	0
7	$(4.2848(17) \times 10^{38})$	$(3.79809(50) \times 10^{34})$	1,125,154,039,419,854,784
8	$(1.0806(12) \times 10^{59})$	$(5.2225(18) \times 10^{54})$	$(2.5228(14) \times 10^{27})$
9	$(2.9008(22) \times 10^{84})$	$(7.8448(38) \times 10^{79})$	$(7.28(15) \times 10^{40})$
10	$(1.4626(16) \times 10^{115})$	$(2.4149(12) \times 10^{110})$	0

* $1.775399(42) \times 10^{19}$ and other estimated values signifies that the exact number of the squares is within range $(1.775399 \pm 0.000042) \times 10^{19}$ with a probability of 99%.

As the experimental environment, two computers were used: a PC with an Intel Core i7-4960X 3.6GHz CPU and 64GB RAM running 64bit Windows 7 is computer 1, and an Intel Core i5 1.8GHz CPU and 8GB RAM running 64bit macOS High Sierra is computer 2.

The calculation can be speeded up by parallelization because the counting method is completely independent for each (S_A, S_B) pair. We divided 817,190 (S_A, S_B) pairs into 16 groups and calculated all associative magic squares of order 7 of one group in one thread. The calculation was executed on 12 threads on the Windows computer and 4 threads on the Mac computer. The calculation took about 2 weeks, and the results are shown in **Table 2**.

We found that the number of associative magic squares of order 7 is 1,125,154,039,419,854,784 up to a reflection or rotation.

We did not spend much time worrying about how to to divide the 817,190 problems into 16 groups, but we did achieve an effective distribution, because the maximum calculation time was 14.8 days and the minimum calculation time was 12.9 days.

Additionally, we submitted our results to the On-Line Encyclopedia of Integer Sequences (OEIS), and it was accepted on December 10, 2018. It is currently posted on the website [6].

5.2 Verification

Walter Trump [7] estimated the number of associative magic squares of order 7 to be within the range $(1.125151 \pm 0.000051) \times 10^{18}$ with a probability of 99%. Our result, 1,125,154,039,419,854,784, is within the range of this estimate.

Walter Trump confirmed the number of 7×7 associative magic squares of one (S_A, S_B) with backtracking. Trump also confirmed our results with his own program based on our method.

As described in Section 4.2, a certain associative magic square

can be transformed into 2304 associative magic by swapping rows and columns. When we add a 90-degree rotation to it, a certain associative magic can be transformed to $2304 \times 2 = 4608$ associative magic squares. Therefore, the number of associative magic squares of order 7 up to a reflection or rotation is a multiple of $576(= 4608/8)$. Our result also has this property.

Our results have been confirmed by a probabilistic estimation, Trump's implementation, and the properties of associative magic squares.

6. Concluding Remarks

We propose a method to count the total number of associative magic squares of order 7 by extending Ripatti's method of counting 6×6 semi-magic squares to 7×7 associative squares. The proposed method divides the 7×7 square into two and divides the problem into two smaller ones. It is important to divide the 7×7 matrix into a center part and outer part and consider them independently. The proposed method counts only canonical associative magic squares which are representative of 2304 other associative magic squares and these canonical associative magic squares depend on how the square is divided up. Our calculation shows that the total number of associative magic squares of order 7 is 1,125,154,039,419,854,784, up to a reflection or rotation.

Table 3 summarizes results on known squares up to $n = 10$, including our own. Charles Planck proved that there is no associative magic square of order N (N is even number that is not a multiple of 4) [3]. The table includes number of squares estimated by Walter Trump for squares whose exact number is unknown [7].

Topics for future work include calculating the number of higher-order associative magic squares and other magic squares whose total number is unknown. However, the number of asso-

ciative magic squares of higher order would be too large to count with our method. The exact number of 6×6 magic squares is very interesting, but counting may prove difficult by just using a simple extension of our method. We may also consider other kinds of number assignment or constraint satisfaction problems.

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