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Simple Folding is Really Hard

Hugo A. Akitaya[1,a) Erik D. Demaine[2,b) Jason S. Ku[2,c)

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Abstract: Simple folding (folding along one line at a time) is a practical form of origami used in manufacturing such as sheet metal bending. We prove strong NP-completeness of deciding whether a crease pattern can be simply folded, both for orthogonal paper with assigned orthogonal creases and for square paper with assigned or unassigned creases at multiples of 45°. These results settle a long standing open problem, where weak NP-hardness was established for a subset of the models considered here, leaving open the possibility of pseudopolynomial-time algorithms. We also formalize and generalize the previously proposed simple folding models, and introduce new infinite simple-fold models motivated by practical manufacturing. In the infinite models, we extend our strong NP-hardness results, as well as polynomial-time algorithms for rectangular paper with assigned or unassigned orthogonal creases (map folding). These results motivate why rectangular maps have orthogonal but not diagonal creases.

Keywords: computational geometry, computational origami, simple folds, strong NP-completeness

1. Introduction

Origami as a craft is centuries old, but in recent years it has exploded into an exquisite art form, a rich mathematical and computational field, and a branch of mechanical engineering exploring applications [*1]. Perhaps the most researched subset of origami studies flat foldings—folded states that lie in the plane, with multiple overlapping layers. If we unfold such a folding, we obtain a straight line planar graph formed by the creases called *crease pattern*. Given a crease pattern, optionally *assigned* by each crease labeled either mountain (the paper folds backwards) or valley (the paper folds forwards), the *flat foldability problem* asks whether the crease pattern comes from some flat folding. This decision problem is known to be NP-complete for both assigned and unassigned crease patterns [4].

In this paper we study *simple foldability*, deciding whether a 2D crease pattern can be folded by a sequence of simple folds. Informally, a simple fold can only rotate paper around a single axis before returning the paper back to the plane. This restriction is motivated by practical sheet-metal bending, where a single robotic tool can fold the sheet material at once. We build on the work of Arkin et al. [2]. They introduce many models of simple folds, proving that deciding simple foldability is weakly NP-complete for some of them, and that simple foldability can be solved in polynomial time for rectangular paper with paper-aligned orthogonal creases. We abbreviate this restriction on the input (rectangular paper and paper-aligned orthogonal creases) as ⊞ crease patterns, and will abbreviate other restrictions similarly. We also introduce a new model of simple folding, namely the infinite simple folds model where simple folds must fold at least one layer everywhere the paper intersects the fold axis. Akitaya et al. [1] describe an exponential method to obtain all possible folding sequences using simple folds under a stronger model that allows paper intersection during the folding motion.

We prove strong NP-completeness for every model proved weakly NP-complete in Ref. [2], namely that simple foldability is hard for:

(1) orthogonal paper with paper-aligned orthogonal creases (abbreviated ⊞) with crease assignment in the one-layer, some-layers, and all-layers models, even to approximate the number of possible simple-folds; and

(2) square paper with paper-aligned creases at multiples of 45° (abbreviated ⊠) with crease assignment in the some-layers and all-layers models.

Additionally we prove strong NP-completeness deciding simple foldability of:

(3) ⊠ crease patterns without crease assignment in the some-layers and all-layers models; and

(4) ⊞ crease patterns with or without crease assignment in the infinite one-layer and some-layers models.

We also point out some errors in the NP-complete reduction in Arkin et al. to simple foldability of orthogonal polygons with unassigned crease patterns, but we do not comment further as the result is subsumed by result (3) above. In the last section, we extend the polynomial-time result from Ref. [2] to the infinite simple folds models, proving the infinite and non-infinite models are equivalent for ⊞ crease patterns. **Table 1** shows the computational complexity of simple-foldable decidability in various models.

1 Tufts University, Medford, MA 02155, USA
2 MIT Computer Science and Artificial Intelligence Laboratory, Cambridge, MA 02139, USA
a) hugo.alves_akitaya@tufts.edu
b) edemaine@mit.edu
c) jasonku@mit.edu

*1 For example, the latest 6OSME conference (Tokyo, 2014) embodies all of these directions. The recent mechanical engineering effort has been the topic of an NSF program (ODISSEI) and several sessions at ASME conferences.

Table 1 Computational complexity of simple folding problems, either open, solvable in polynomial time (poly), or strongly/weakly NP-complete (strong/weak). **Bold** results are new in this paper. Rows list simple folding models while the columns list restrictions on the input: orthogonal paper/orthogonal creases , square paper/45° creases , or rectangular paper/orthogonal creases).

Model	Assigned			Unassigned		
One-layer	weak → **strong**	open	poly	open	open	poly
Some-layers	weak → **strong**	weak → **strong**	poly	open	**strong**	poly
All-layers	weak → **strong**	weak → **strong**	poly	open	**strong**	poly
Inf. One-layer	**strong**	open	**poly**	**strong**	open	**poly**
Inf. Some-layers	**strong**	open	**poly**	**strong**	open	**poly**
Inf. All-layers	open	open	**poly**	open	open	**poly**

2. Definitions

In general, we are guided by the terminology laid out in Ref. [2], though for this paper we restrict our discussion to folding two-dimensional paper. We will operate in $\mathbb{R}^3$ containing a folding plane $\mathbb{P}$ congruent to $\mathbb{R}^2$ with a surface normal vector $\hat{n}$. We call the direction $\hat{n}$ *above* and $-\hat{n}$ *below*. A two-dimensional *paper* P is a connected polygon in $\mathbb{P}$, possibly with holes. We denote the boundary of P by ∂P. We call the side of a paper pointing in the $\hat{n}$ direction the *top* and the opposite side the *bottom*. A *crease* is a line segment on a paper. A *crease pattern* (P, Σ) is a paper P and a set of creases Σ contained in the paper, no two of which intersect except at a common endpoint. A *facet* of a crease pattern is a connected open set in $P \setminus \Sigma$ whose boundary is a subset of $\partial P \cup \Sigma$. Two crease pattern facets are *adjacent* if their boundaries share a common crease.

A *flat fold isometry* (P, Σ, f) is a crease pattern (P, Σ) together with an isometric embedding f of the paper into $\mathbb{P}$ such that (1) each facet of the crease pattern is mapped to a congruent copy, (2) connectivity is preserved between facets and creases, and (3) for every pair of adjacent facets, exactly one of the facets is reflected in the embedding. If a crease pattern with n creases has a flat fold isometry, we call the crease pattern *locally flat-foldable*, which is checkable in polynomial time in n [4]. We denote the preimage of $U \subset f(P)$ as $f^{-1}(U) \subset P$. Two facets F_1 and F_2 *overlap* if $f(F_1) \cap f(F_2) \neq \emptyset$.

A *flat folding* (P, Σ, f, λ) is a flat fold isometry (P, Σ, f) together with a *layer ordering* [5], that, in the case of convex facets, can be described by a directed graph on the facets. We represent such a graph by a function λ mapping from a pair of overlapping facets to $\{-1, 1\}$ so that: (1) if F_1 and F_2 are overlapping facets, $\lambda(F_1, F_2) = -\lambda(F_2, F_1)$; (2) if F_1, F_2, F_3 are such that F_1 and F_2 are adjacent and F_3 overlap the crease $c = \partial F_1 \cap \partial F_2$, i.e., $f(F_3) \cap f(c) \neq \emptyset$, then $\lambda(F_1, F_3) = \lambda(F_2, F_3)$; (3) if F_1, F_2, F_3, F_4 are pairwise overlapping and such that F_1 and F_2 (resp., F_3 and F_4) are adjacent and the crease between F_1 and F_2 overlap with the crease between F_2 and F_3, i.e., $f(c_1) \cap f(c_2) \neq \emptyset$, where $c_1 = \partial F_1 \cap \partial F_2$ (resp., $c_2 = \partial F_3 \cap \partial F_4$), then $\lambda(F_1, F_3) + \lambda(F_1, F_4) + \lambda(F_2, F_3) + \lambda(F_2, F_4) \in \{-4, 0, 4\}$. The conditions above derive from Justin's conditions, as called in Ref. [5]. We say that F_1 is *above* (resp., *below*) F_2 if $\lambda(F_1, F_2)$ is -1 (resp., 1). If a crease pattern has a flat folding, we call the crease pattern *globally flat-foldable*.

The *flat-foldability decision problem* takes as input a locally flat-foldable crease pattern and asks if it is globally flat foldable. If no other information is given, the problem is called *unassigned*. A common variant of the decision problem also provides in the input an assignment $\alpha : \Sigma \to \{\mathrm{M}, \mathrm{V}\}$ of the creases to either mountain or valley, and the question asks if a flat folding exists satisfying the assignment according to the following definitions. A crease $c = \partial F_1 \cap \partial F_2$ such that $f(F_1)$ is reflected is called *mountain* (M) if $\lambda(F_1, F_2) = 1$ or a *valley* (V) otherwise. This definition adheres to the intuition that a valley brings the top surfaces of F_1 and F_2 together while a mountain brings the bottom surfaces together. Arbitrarily assigning mountain or valley to the creases of a flat fold isometry may be consistent with zero, one, or multiple flat foldings. If α is given, the decision problem is called *assigned*.

A *simple folding* $(P, \Sigma_2, f_2, \lambda_2)$ of an input flat folding, $(P, \Sigma_1, f_1, \lambda_1)$, is itself a flat folding parameterized by a *fold axis* (a directed line $\ell \in \mathbb{P}$) and a *folded region* (a subset $U \subsetneq P$) satisfying the following conditions.

(1) Points on the boundary of the folded region are either in the boundary of the paper or the preimage of the fold axis, i.e., $\partial U \subset \partial P \cup f_1^{-1}(\ell \cap f_1(P))$;

(2) Everything in the folded region moves to a reflected point across the fold axis ℓ;

(3) The creases of the new flat folding contain the creases of the old one, i.e., $\Sigma_1 \subsetneq \Sigma_2$.

(4) Points not in the folded region are unchanged, i.e., $f_2(p) = f_1(p)$ for $p \in P \setminus U$ and $\lambda_2(F_1, F_2) = \lambda_1(F_1, F_2)$ for facets F_1 and F_2 in $P \setminus (U \cup \Sigma_2)$;

(5) The output layer ordering of the folded region is exactly the opposite of the input layer ordering, i.e., if F_u and F_p are facets of (P, Σ_2, f_2) respectively containing u and p such that $u, p \in U$, $f_2(u) = f_2(p)$, then $\lambda_2(F_u, F_p) = -\lambda_1(F'_u, F'_p)$ where F_u and F_p are the facets of (P, Σ_1, f_1) respectively containing u and p.

(6) The folded region is either completely above or completely below points not in the folded region in the input flat folding, according to the direction of ℓ, i.e., if F_u and F_p are facets of (P, Σ_1, f_1) respectively containing u and p such that $u \in U$, $p \in f_1^{-1}(f_1(u)) \setminus U$ and $f_1(u)$ is on the right (resp., left) side of ℓ, then $\lambda_1(F_u, F_p) = 1$ (resp., $\lambda_1(F_u, F_p) = -1$).

(7) The folded region is either completely above or completely below points not in the folded region in the resulting flat folding, according to the direction of ℓ, i.e., if F_u and F_p are facets of (P, Σ_2, f_2) respectively containing u and p such

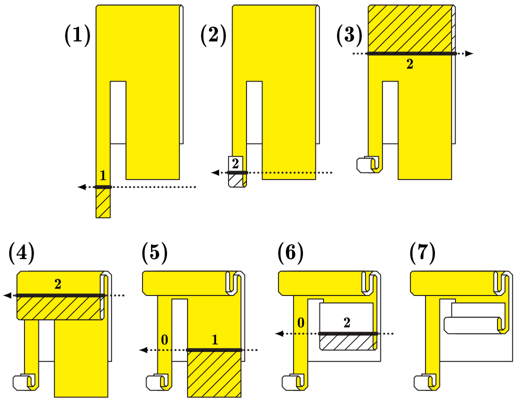

Fig. 1 Example folding steps demonstrating the differences between simple folding models. L is a directed dotted line in the direction of a, U is textured, and the fold line $f^{-1}(L)\cap\partial U$ is a thick line with the number of layers # specified.

Table 2 Definitions for different models of simple folding according to restrictions on the number of layers that must be folded along the fold axis. Example steps are shown in **Fig. 1**.

Model	**Restriction on #**	Foldable Example Steps
Some-layers	no restriction	(1), (2), (3), (4), (5), (6)
One-layer	$\#(q) \in \{0, 1\}$	(1), (5)
All-layers	$\#(q) \in \{0, \#_+(q)\}$	(1), (2), (3)
Infinite Some-layer	$\#(q) \geq 1$	(1), (3), (4)
Infinite One-layer	$\#(q) = 1$	(1)
Infinite All-layers	$\#(q) = \#_+(q)$	(1), (3)

that $u \in U$, $p \in f_2^{-1}(f_2(u)) \setminus U$ and $f_2(u)$ is on the right (resp., left) side of ℓ, then $\lambda_2(F_u, F_p) = -1$ (resp., $\lambda_2(F_u, F_p) = 1$).

A *simple fold* is then a rotation of a folded region in a flat folding about a fold axis back into the plane to form a new flat folding. Conditions (1) and (2) ensure the rotation is isometric; condition (3) ensures that existing creases do not unfold; conditions (4) and (5) ensure that folding occurs exactly in the folded region and the layer orderings before and after the simple fold are consistent; conditions (6) and (7) ensure that the paper does not intersect itself.

We define different models of simple folding that limit the choice of U. Let $L = \ell \cap f(P)$ be the intersection of fold axis ℓ and input flat folding $(P, \Sigma_1, f_1, \lambda_1)$, and let $\#_+(q) = |f^{-1}(q) \setminus (\partial P \cup \Sigma_1)|$ be the number of foldable layers at $q \in L$. Then the function $\# : L \to \{0, \ldots, \#_+\}$ denotes the number of layers that are folded in a simple fold at every point along the fold axis, specifically $\#(q) = |(f^{-1}(q) \cap \partial U) \setminus (\partial P \cup \Sigma_1)|$ for $q \in L$. **Table 2** defines our models of simple folding based on restrictions on # that limit the choice of folded region. Of particular interest is the infinite all-layers model which corresponds to folding everything on one side of the fold axis to the other side, a model which has practical applications in manufacturing. For instance, Balkcom and Mason [3] describe a robotic system restricted to such model of simple folds.

Given locally flat-foldable crease pattern (P, Σ), we say that it is *simply-foldable* or equivalently *flat-foldable via a sequence of simple folds* in some model, if the crease pattern can be folded by a sequence of m simple folds into a series of flat foldings $S_i = (P, \Sigma_i, f_i, \lambda_i)$ for $i \in \{1, \ldots, m\}$ such that S_1 is the original unfolded paper with $\Sigma_1 = \emptyset$, each flat folding S_{i+1} is a simple folding of S_i, and $S_m = (P, \Sigma_m = \Sigma, f_m, \lambda_m)$ is a flat folding of the input.

If it is hard to decide simple-foldability, a natural question arises: how close can we estimate the number of possible simple folds that can be performed? Define MaxFold, the natural optimization version of the decision problem asking for the maximum number of simple folds that can be folded given a locally flat-foldable crease pattern (P, Σ), or formally, the maximum length sequence of simple folds to fold any simply-foldable crease pattern (P, Σ') with $\Sigma' \subset \Sigma$.

3. Results

(1) Orthogonal Paper/Orthogonal Creases

In this section we prove that the simple-foldability decision problem of an orthogonal piece of paper with a M/V assigned paper-aligned orthogonal crease pattern is strongly NP-complete in the one-layer, some-layer, and all-layer models of simple folding. This result is the same as Theorem 6.3 from Ref. [2], but proves strong NP-completeness because we reduce from a strongly NP-complete problem. Additionally, we prove that it is hard even to approximate the associated natural optimization problem.

Theorem 1. *The assigned simple-foldability decision problem for orthogonal paper with paper-aligned orthogonal creases is strongly NP-complete in the one-layer, some-layers, and all-layers models.*

Proof. The proof is by reduction from 3-Partition. Given an instance of 3-Partition with integers $A = \{a_1, \ldots, a_n\}$ to be partitioned into $n/3$ triples each with sum $\left(\sum_{a\in A} a\right)/(n/3) = t$, construct an orthogonal polygon with M/V assigned paper-aligned orthogonal creases as shown in **Fig. 2** (the width of the polygon is one everywhere). We assume each a_i is close to $t/3$ and divisible by $2n$: if not, add a large number to each and multiply by $2n$ so that they are.

There are five main functional sections of the polygon, as shown in **Fig. 3**. On the left is the Bar, a section whose convex hull is a $5 : 2\infty$ rectangle of paper without creases that is very long ($\infty = 10nt$). Attached to the middle of the Bar is a $\frac{5n}{3} + \frac{1}{2}$ long strip extending to the right which we call the Arm. The Staircase encodes the a_is in order as a series of steps with height equal to their value plus one. Step i contains two creases c_{2i-1}, c_{2i} that when both folded raise the Bar by exactly $2a_i$. The Wrapper section is a horizontal rectangle of length $2n/3$ with vertical valley creases d_i (d_1 being the right most crease) dividing the Wrapper into unit squares. The Cage on the right bounds a polygonal area whose the left vertical edge we call the Column.

The construction forces the Bar to wrap inside the Cage $n/3$ times, each time shifted up by distance $2t$ (note that ∞ is chosen large to ensure that the Staircase never intersects the Cage polygon while wrapping). To prove the claim, we first prove the Wrapper must fold its vertical creases in order from right to left. If this were not the case, then there exists some first crease d_i to be folded whose right neighbor d_{i-1} has not yet been folded. But d_i has at least two squares of unfolded paper to its left that will cover d_{i-1} when folded, making d_{i-1} impossible to fold using simple folds without violating the M/V assignment, contradicting our model. Because the Wrapper executes its folds from right to

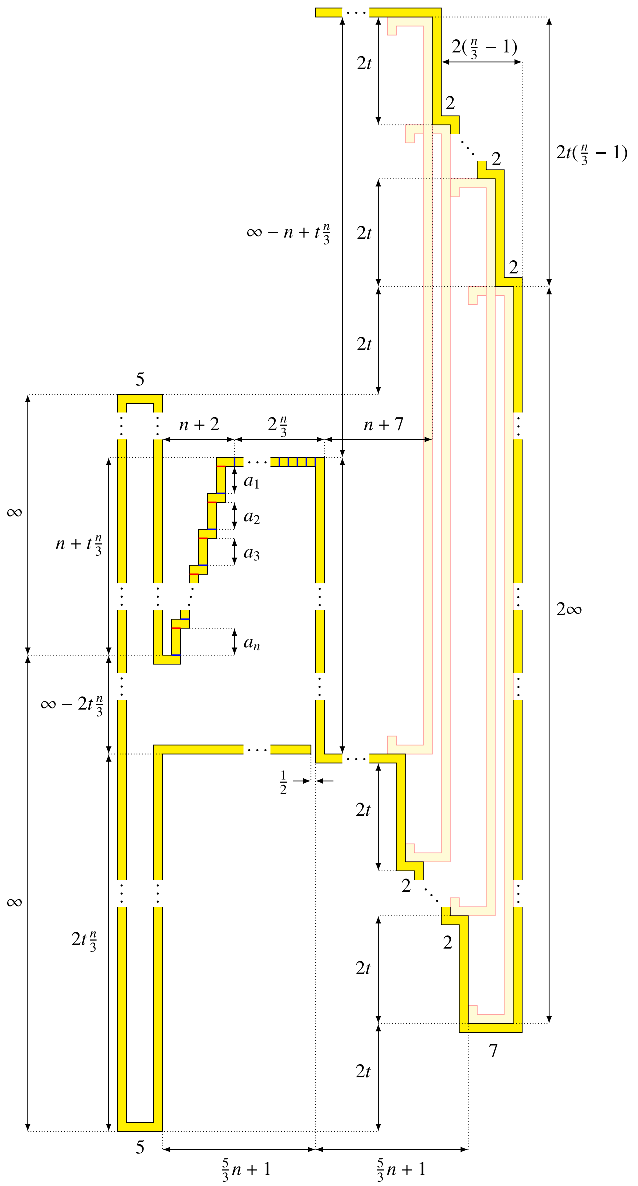

Fig. 2 An orthogonal simple polygon with orthogonally aligned mountain-valley creases (drawn in red and blue respectively) constructed from an instance of 3-Partition that can be folded using simple folds if and only if the instance of 3-Partition has a solution.

left, the Bar must pass through the Cage $n/3$ times sequentially from the rightmost slot to the leftmost, with each subsequent slot shifted up by $2t$.

If the 3-Partition instance has a positive solution, then the polygon has a simple folding: just pleat the creases associated with the a_is in one of the satisfying triples, then fold the Bar through the Cage along the next Wrapper creases, and repeat. Because all folds in the Wrapper are all valley, the Arm will go around the Column and never cross it. Further, if the polygon has a simple folding, the 3-Partition instance has a positive solution because the Staircase must be folded on both creases from exactly three a_is between each wrap. To prove this, all a_is are close to $t/3$ so in order to shift by $2t$, exactly three a_i sections must be flipped from their original orientation. Further, because each a_i is divisible by $2n$, no one unit section between a_is can flip if the total height is to be raised by t, since t is also divisible by $2n$. So the a_is flipped at each step correspond to triplets of the 3-Partition instance that sum to t.

Folded in this way, each simple fold can be performed in the one-layer and some-layers models because the construction only ever folds through one layer of paper at a time. And because creases only ever exist in a single layer, the all-layers model also applies. The reduction is polynomial because the entire constructed polygon is bounded by a $30nt \times 4n$ rectangle. Lastly, the problem is in NP because given a certificate of the crease folding order, each fold can be simulated and checked in polynomial time. □

The optimization version of the decision problem is even hard to approximate.

Theorem 2. *Given an orthogonal paper with paper-aligned orthogonal creases admitting a maximum sequence of m simple folds, approximating* MaxFold *to within a factor of $m^{1-\varepsilon}$ for any constant $\varepsilon > 0$ is NP-complete in the some-layers and all-layers models.*

Proof. Construct a crease pattern similar to Fig. 2, but with the Wrapper modified in the following way: add δ horizontal lines of creases all the way through the Wrapper, with each horizontal line composed of $2n/3 + 1$ collinear creases alternating M/V assignment in each section between vertical creases, splitting each of the $2n/3$ vertical creases into $\delta + 1$ collinear vertical valley creases.

For a positive instance of 3-Partition, the proof of Theorem 1 implies that the $8n/3$ original creases may be folded as simple folds, then allowing δ more simple folds to be performed by folding along each line of horizontal of creases from top to bottom in the some-layers and all layers models. None of the added horizontal creases can be folded before all vertical creases in the Wrapper are folded due to M/V alternation along the line. This construction is thus simple-foldable via a sequence of $m = 8n/3 + \delta$ simple folds.

For a negative instance of the 3-Partition problem, there exists at least one line of vertical Wrapper creases that cannot be folded, reducing the number of possible simple folds to strictly less than $8n/3$.

Setting $\delta = (8n/3)^{1/\varepsilon} - 8n/3$, Theorem 1 implies it is NP-hard to distinguish the case where m folds are possible from the case where at most $8n/3 = m^{\varepsilon}$ are possible. The reduction is polynomial since both δ and m are $O(n^{1/\varepsilon})$ for constant ε. □

(2) Assigned Square Paper/45° Creases

Arkin et al. adapt their Partition reduction to square paper with M/V assigned paper-aligned creases at multiples of 45° by constructing an approximation of their orthogonal construction from a square. Unfortunately their modification cannot be applied to our 3-Partition reduction in the all-layers model because their construction requires folds along the long construction end which will overlap other parts of the paper during construction.

Instead, we use a similar idea to construct an orthogonal polygon approximation from a square but with a different turn gadget that enforces the order of construction while only making folds local to the gadget that works in both the some-layers and all-layers models.

Theorem 3. *The assigned simple-foldability decision problem*

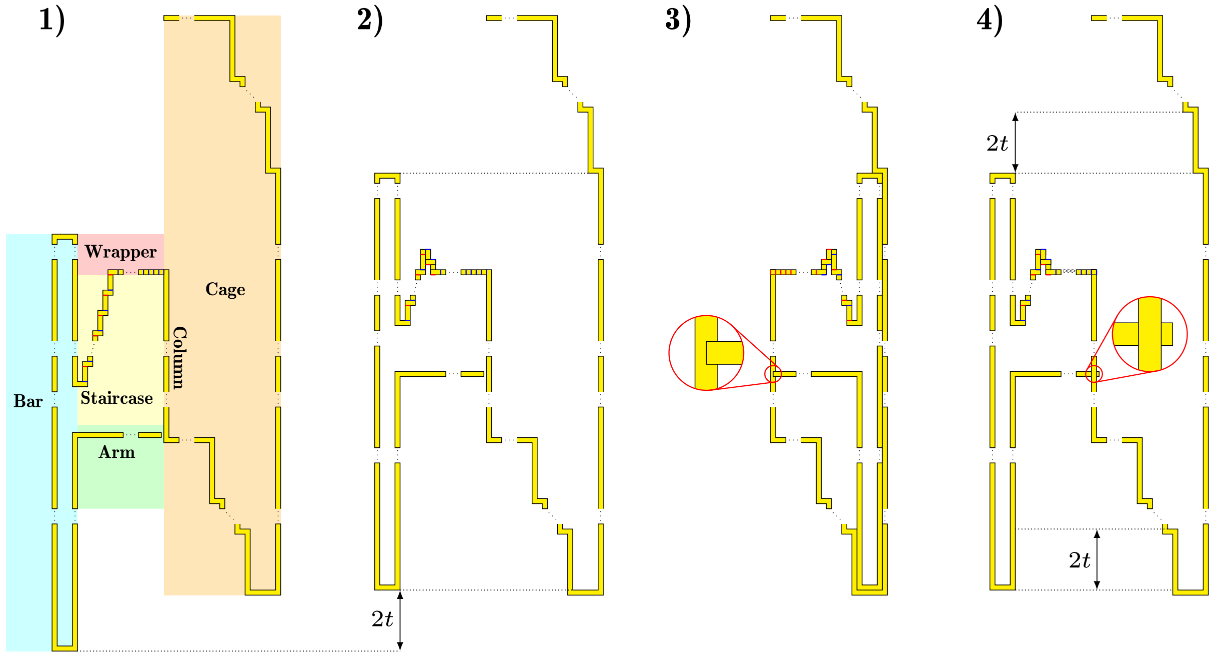

Fig. 3 Process to check the Partition solution: 1) pleat variables to change height of bar by $2t$, 2) fold along the rightmost wrapper crease around the column, 3) fit the bar through the cage folding the bar to the left along the next wrapper crease, 4) repeat until $n/3$ triples adding to $2t$ have been checked.

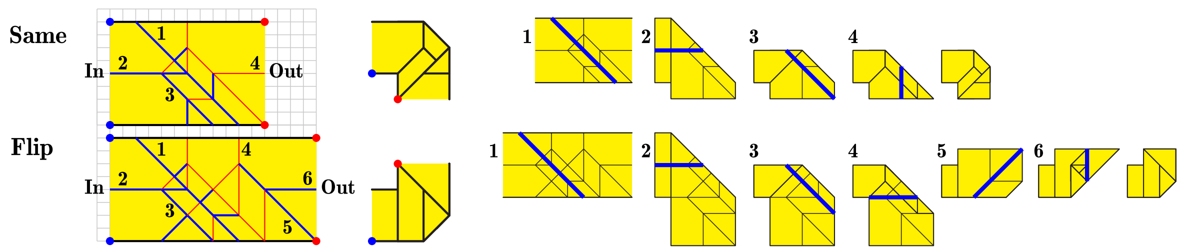

Fig. 4 Turn gadgets for the assigned case. Red/blue lines represent the M/V assignment.

for square paper with paper-aligned creases at multiples of $45°$ $\boxtimes$ *is strongly NP-complete in the some-layers and all-layers models.*

Proof. The proof is by reduction from the decision problem in Theorem 1. Given such an orthogonal polygon with M/V assigned paper-aligned orthogonal creases, we construct a crease pattern on a square that folds using simple folds if and only if the original orthogonal crease pattern is simply-foldable.

We start by constructing a long rectangle from the starting square of appropriate aspect ratio in the same way as Ref. [2], double the width of the orthogonal polygon we want to create. Then we use turn gadgets to shape the long rectangle into the target orthogonal polygon. **Figure 4** depicts crease patterns for our turn gadgets, Same and Flip, along with drawings depicting their valid flat foldings. We call creases located on the horizontal center line halfway between the edges of the paper *axial* creases. These crease patterns have the property that the axial crease extending the right edge (the output) cannot be folded unless all creases in the gadget have already been folded.

When folded, both gadgets align the edges of the original long rectangle to one side. Having both the Same and Flip gadgets allows us to combine them in one long strip to turn right or left no matter which side the original edges are on. The Same gadget turns the paper to the same side as the original edges, while the Flip gadget turns the paper to the other side. If chained in a sequence, turning in the same direction as the previous turn necessitates a Same gadget, while the Flip gadget turns the paper in the opposite direction.

The construction is as follows. We trace the path of the target orthogonal polygon starting at the cage end. Wherever a turn is needed, apply the appropriate turn gadget. The creases of the target crease pattern are overlaid to be foldable only after the appropriate section has been folded in half. If the orthogonal polygon is simply-foldable, we can then fold the remaining creases.

Now we prove the orthogonal polygon can be folded if the square crease pattern is simply-foldable. Before any section can be folded along axial creases, all creases behind the axial crease must have already been folded. The gadgets can be folded using only valley folds, so the paper will never self intersect. Further, creases local to a turn gadget do not overlap any other paper because gadgets are far from each other. In particular, no crease of the target crease pattern may fold before the cage is constructed. Once the cage has been constructed, no Wrapper crease may fold until the Bar has been constructed completely because any uncreased paper will be too large to fit through the cage.

The reduction is polynomial because the side of the input

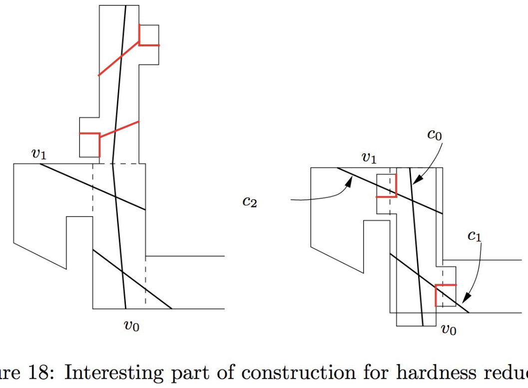

Fig. 5 Figure 18 from Ref. [2]. Corrections marked in red creating reflections of c_1 and c_2 on the covering flap, and trimming the covering flaps so that c_1 and c_2 do not intersect v_0 or v_1 within the covering flap.

square is bounded by $O(nt)$ and the number of creases is bounded by $O(n^2)$. Lastly, the problem is in NP because given a certificate of the crease folding order, each fold can be simulated and checked in polynomial time. □

(3) M/V Unassigned Square Paper/45° Creases

M/V unassigned crease patterns are naturally less restrictive than M/V assigned crease patterns. This freedom can make collision avoidance easier, providing a choice of folding direction at each crease. However when proving hardness for M/V unassigned crease patterns, one cannot use crease direction to enforce fold ordering or layering and must restrict them using other techniques. Arkin et al. provide a weakly NP-hard reduction for orthogonal polygons with unconstrained creases without crease assignment in Theorem 7.1, but their proof has two errors discussed next.

The first error in the proof of Theorem 7.1 in Ref. [2] is that Arkin et al. claim that their reduction for the M/V assigned case can be used directly to prove hardness of the M/V unassigned all-layers model, saying, "in the all-layers case, as soon as two layers of paper overlap they are 'stuck' together." However, this claim is not true under their definition of the all-layers model.

The second error is a fixable problem in the creases shown in **Fig. 5**. Their construction modifies their Partition reduction by adding pleats to force the folding direction of creases v_1 and v_2, claiming the added cross pleats must fold first to enforce v_0 and v_1 to fold in the same direction. However, pleating c_0 before c_1 and c_2 locks the latter two creases to paper containing no creases, preventing them from ever folding. Adding mirrored creases on the cover fixes this problem. Further, the positions of creases c_1 and c_2 lock the layers containing v_0 and v_1 to overlapping uncreased paper meant to enforce folding direction. Trimming problematic extra paper can fix the proof.

We do not elaborate further as we prove stronger results that subsume Theorem 7.1, namely Theorems 4 and 5.

Theorem 4. *The unassigned simple-foldability decision problem for orthogonal paper with paper-aligned creases at multiples of 45° is strongly NP-complete in the some-layers and all-layers models.*

Proof. The proof of Theorem 1 still holds using the same construction with unassigned crease patterns except for two points: (1) the argument ensuring that the creases of the Wrapper fold in order from right to left does not apply without crease assignment; and (2) the argument ensuring that the bar folds through the cage each time requires every vertical fold in the Wrapper to either be all mountain or all valley.

To fix problem (1), we modify the Wrapper paper to be two units tall and replace the Wrapper creases with the creases shown in **Fig. 6**. These creases have the property that the new creases $\Sigma_{i+1} \setminus \Sigma_i$ added in any sequence of simple folds resulting in a simple folding of the Wrapper are uniquely defined in the all-layers model, namely each simple fold S_i can only be folded if all simples folds S_j for $j < i$ have already been folded. In the some-layers model, the order on simple folds is not quite unique since some strict subset of creases in some of the $\Sigma_{i+1} \setminus \Sigma_i$ above may fold out of order, but it remains that for any crease in Σ_i to fold, some nonempty subset of Σ_j must have already been folded for $j < i$, enforcing the new Wrapping creases to fold in order from right to left. The ordering is given in the figure and follows from the observation that a simple fold can only occur when the subset of creases to be folded divides the paper, with creases collinear in the flat folding.

To fix (2), we must ensure that the vertical Wrapper creases are either all mountain or all valley in any flat folding reachable as a sequence of simple folds. The Arm in the original construction (not useful for the original theorem) is included to enforce this requirement. After the rightmost vertical Wrapper crease has been folded first, the Arm will overlap the Column. Since the order is enforced by (1), the second vertical Wrapper crease must fold next, while on the right of the Column. If it folds with assignment opposite the first, the Arm would intersect the Column contradicting the simplicity of the fold. This argument holds inductively for the remainder of the vertical Wrapper creases, so they must all fold with the same assignment in any sequence of simple folds.

Having addressed these two problems, the arguments of Theorem 4 directly apply under both the some-layers and all-layers models, since the order and assignment of the Wrapper creases are forced in both models. □

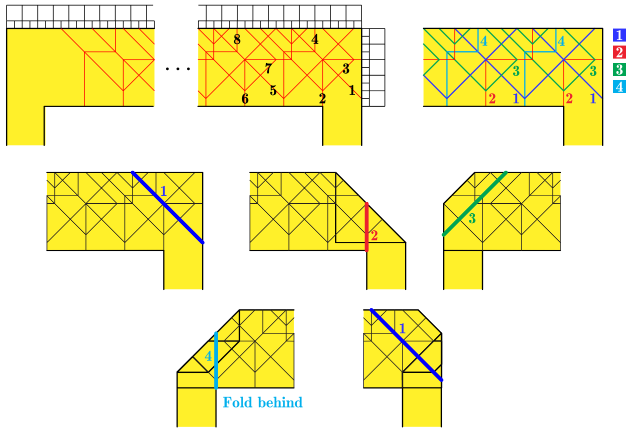

Fig. 6 (Top-left) Crease pattern for the Wrapper in the unassigned model. Red lines show unassigned creases. (Top-right) Creases are colored according to their folding order. (Bottom) Folding sequence showing the creases that are being folded.

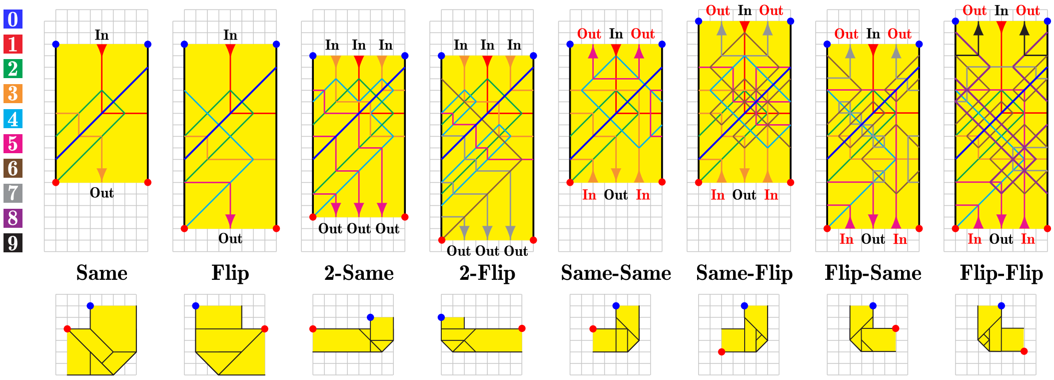

Fig. 7 Unassigned turn gadgets. Creases must be folded according to color order on left. Input and output creases are labeled with arrow heads, forward signals in black and return signals in red.

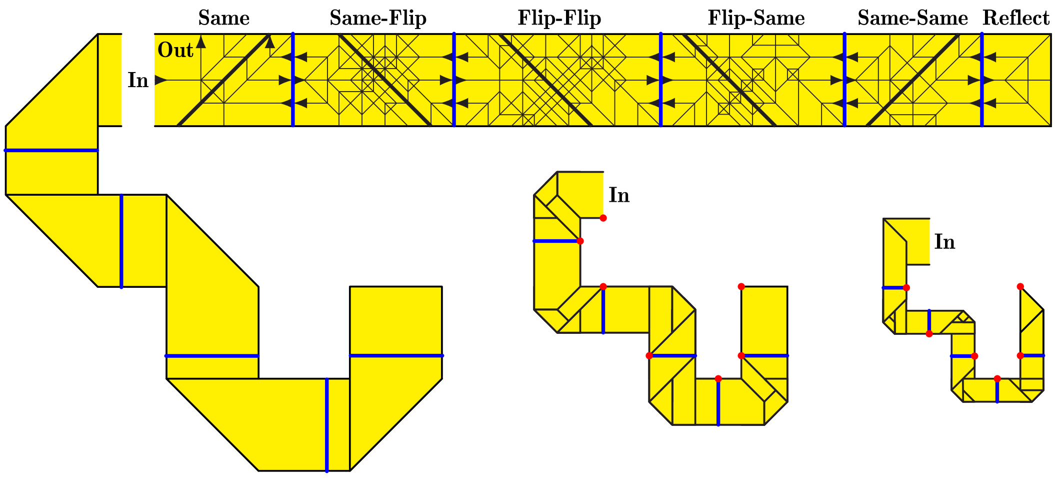

Fig. 8 Example collection of turn gadgets connected in series demonstrating forward and return signal propagation.

Theorem 4 goes beyond Theorem 7.1 from Ref. [2] by both proving a stronger notion of hardness and restricting creases to only multiples of 45°. The following is an even stronger result, showing that the problem is still hard even when the orthogonal polygon is a square.

Theorem 5. *The unassigned simple-foldability decision problem for square paper with paper-aligned creases at multiples of 45° is strongly NP-complete in the some-layers and all-layers models.*

Proof. We will use the same techniques from the proof of Theorem 3 to build an approximation (small corners missing) of an orthogonal polygon from a square, propagating a signal along the paper to force construction parts of the orthogonal polygon, and then invoke the proof of Theorem 4. However, since both the Arm and the Cage are necessary for the arguments of the latter proof, we will need to enforce construction of the entire orthogonal polygon before Wrapper creases can execute, not just the Cage. We force the entire orthogonal polygon to be constructed, first by propagating a signal throughout the length of the polygon, and then back to the Wrapper using the eight turn gadgets shown in **Fig. 7**.

Just as for the assigned gadgets in Fig. 4, the relevant creases in each gadget have a fixed order that ensure the output crease(s) of a signal may only fold if the input crease(s) have already been folded. When chained together, these signals enforce the order in which turns are constructed and completed.

We split the gadgets into three groups: Simple turns (Same, Flip), Double turns (2-Same, 2-Flip) and 2-Way turns (Same-Same, Same-Flip, Flip-Same, Flip-Flip). Simple and Double turns encode only a forward signal and are completed once the output has been folded, the only difference being that the Double turns are folded in half twice. Alternatively 2-Way turns encode both forward and return signals, respective outputs only foldable if respective inputs have been folded, the return signal folding after the forward signal. The forward signal is propagated along the center axial crease as in the Same gadgets, while the return signal is propagated on the sides. The naming of the 2-Way gadgets are analogous to the Simple gadgets: Same-Flip meaning the original edge of the long rectangle is on the same side as the turn when propagating the signal forward, with the original edge opposite the turn upon the return. An example assembling many of these gadgets coupled in a series is shown in **Fig. 8**.

Note that we can trivially connect Double and 2-Way turns together, while Single turns may also interface with them by adding additional folds as shown in Fig. 8. Figure 8 also depicts a Reflection gadget that turns the forward input signal around, propagating from the center to the return outputs on the outside. In this example, the only creases foldable using simple folds from the start are the set of diagonal creases C shown as bold lines in the crease pattern, the folded result shown in the left diagram.

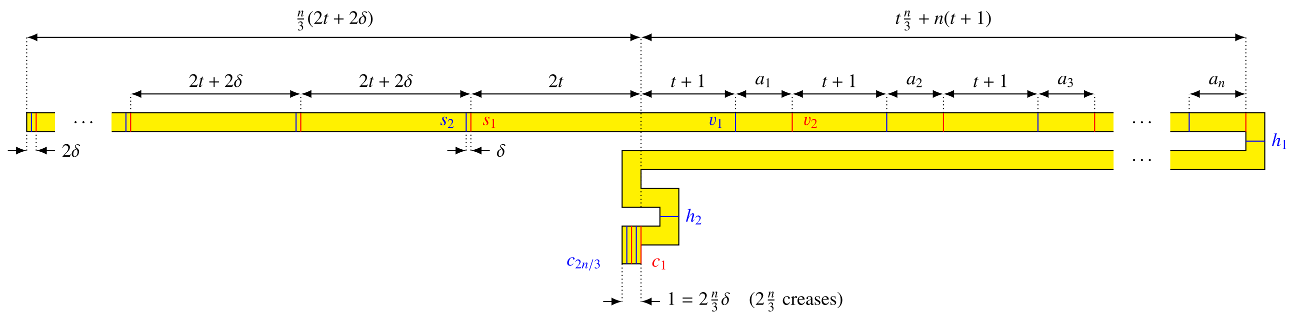

Fig. 9 An orthogonal simple polygon with mountain-valley assigned paper-aligned orthogonal creases (drawn in red and blue respectively) constructed from an instance of 3-Partition that can be folded in the infinite one-layer model if and only if the instance of 3-Partition has a solution.

Note that these creases don't all have to be folded at the same time, but each must be folded before a forward signal may past through them. Because the inputs and outputs of each gadget are chained, the first simple fold not in C that may fold contains the crease labeled "In", which when folded will unlock a series of simple folds to propagate the forward signal to the reflect gadget as shown in the middle diagram. The return signal folds may then be executed, ending with the simple fold labeled "Out". The final flat folding is shown in the right diagram.

Now we follow the same construction from the proof of Theorem 3, constructing an appropriately long rectangle of width four units to be shaped into an approximation of the orthogonal polygon in Fig. 3, with Wrapper modified to be two units high as in Fig. 6. We begin construction from the tip of the Arm using Double turns all the way to the top of the Staircase, allowing appropriate space between gadgets so that the constructed polygon has the correct dimensions. The paper will not overlap where creases are folded because the constructed polygon is always three units away from the rest of the polygon already constructed. The Wrapper can be constructed double the width by switching to Single turns on the ends. The construction proceeds to fold the rest of the cage using 2-Way gadgets with a reflect gadget at the end, with the return signal ending by folding the right edge of the wrapper.

The crease pattern resulting from this construction can only fold in the order enforced by the chain of connected gadgets, by the analysis of the gadgets above. Recall that the first crease to fold in the modified Wrapper from Fig. 6 is a diagonal crease terminating on the right edge of the wrapper which will reflect across the last crease of the return signal, and won't be foldable unless the entire orthogonal polygon approximation has been constructed. Then the same argument as the proof of Theorem 4 proves the claim directly. □

(4) Infinite, Orthogonal Paper/Orthogonal Creases ⊞

In the infinite one-layer or some-layers models, a simple fold must fold one (or more) layer(s) everywhere in the intersection of the fold axis and the valid flat folding. This is more restrictive than the one-layer model as foldability in the infinite one-layer model implies foldability in the one-layer model but not the reverse.

Theorem 6. *The assigned simple-foldability decision problem for orthogonal paper with paper-aligned orthogonal creases ⊞ is strongly NP-complete in the infinite one-layer and infinite some-layers models.*

Proof. The proof is again a reduction from 3-Partition. Given an instance of 3-Partition with integers $A = \{a_1, \ldots, a_n\}$ to be partitioned into $n/3$ triples each with sum $\sum a_i/(n/3) = t$, $i \in \{1, \ldots, n\}$, construct an orthogonal polygonal paper P with paper-aligned orthogonal creases Σ and assignment $\alpha : \Sigma \to \{M, V\}$ as shown in **Fig. 9**, with width one everywhere. For our construction we assume each a_i is sufficiently close to $t/3$: if not, add a large number to each so that they are.

There are three functional sections of the polygon. The paper above crease h_1, called the *Pleater*, encodes the integers on the right, and the sets to be satisfied on the left using a pair of creases for each. The paper between creases h_1 and h_2, called the *Base*, is uncreased paper used to exploit the one-layer infinite model. Without loss of generality, we assume the Base remains fixed during folding. The paper below crease h_2, called the *Checker*, can only be completely folded if the input 3-Partition instance has a solution. The $2n$ creases on the right of the Pleater encode each a_i with two vertical creases, one mountain and one valley separated by distance a_i, each pair separated from the others by distance $t+1$. Call this set of creases V containing creases v_i labeled $i \in \{1, \ldots, 2n\}$ increasing from left to right. The $2n/3$ creases on the left of the Pleater come in pairs bounding small distance $\delta = \frac{3}{2n}$, each pair separated from each other by $2t+2\delta$. Call this set of creases S containing creases s_j labeled $j \in \{1, \ldots, 2n/3\}$ from right to left. Lastly, let C be the set of $2n/3$ creases in the Checker alternating M/V, containing creases c_j labeled $j \in \{1, \ldots, 2n/3\}$ from right to left.

First, if the 3-Partition instance has a solution, then (P, Σ, α) is foldable under the infinite one-layer model. Fold explicitly using the following procedure. First fold the two horizontal creases h_1 and h_2. Then choose a triple of a_is in the 3-Partition solution and pleating their corresponding creases v_{2i-1} and v_{2i}. These three pairs are foldable under the infinite one-layer model by folding first v_{2i} then v_{2i-1} for each a_i in the triple. Pleating all creases corresponding to a valid triple moves the creases in S to the right by exactly $2t$, aligning s_1 and s_2 with c_1 and c_2 respectively. Now aligned, these creases can then be pleated together, moving creases c_3, c_4, s_3, and s_4 to the locations where c_1, c_2, s_1, and s_2 used to be respectively, serving as an invariant. Repeating this process $n/3 - 1$ more times successfully folds all creases.

Second, if (P, Σ, f) is foldable under the infinite one-layer model, then there exists a solution to the 3-Partition instance.

We first prove two intermediate results: (1) each crease in C can only fold if aligned and folded with some crease in S; and (2) creases s_1 and s_2 must be the first and second creases in S to fold, and must fold aligned with creases c_1 and c_2 respectively.

Proof of (1). By construction, the infinite line induced by each crease c_i will always overlap some part of the Base (which contains no creases) for any folded or partially folded configuration. Thus in order to fold c_i, some other crease must align with c_i on top of the Base. Clearly c_i cannot align with any crease in V or any other crease in C, so it must align with some crease in S. So for any valid folding, there exists a bijection between creases in C and S.

Proof of (2). Suppose for contradiction $s_i \neq s_1$ is the first crease in S to be folded. Then one of two cases apply. Either s_i folds without aligning with some crease in C, a contradiction by (1); or s_i is folded aligned with some crease in C by folding some subset of creases $V' \subset V$, with s_1 not yet folded and strictly to the right of all creases in C. But since the creases in V' cannot be unfolded, the distance between s_1 and any crease in C can only increase further, and s_1 will never align with a crease in C, a contradiction.

Further, since the horizontal position of s_1 is purely a function of the folded state of creases V and only integral distances exist between folds in V, the horizontal position of s_1 can only change by integral amounts. The only crease of C that is an integral horizontal distance from s_1 is c_1, so they must fold together. Additionally after s_1 and c_1 are folded, s_2 and c_2 are also aligned and must be the next creases to be folded. Suppose for contradiction they were not. We cannot fold any other crease in C or S since no other pair are aligned with each other; and folding some crease in V prevents s_2 from ever aligning with a crease in C, a contradiction.

Now we prove the claim. By (1), creases s_1 and s_2 fold before all other creases in S, aligned with creases c_1 and c_2 respectively. In order to align these creases, some subset of V must have been folded to shift s_1 to the right by exactly $2t$. With s_1 and s_2 so aligned, no section with length $t+1$ between a_i sections can be flipped from their original orientation or else s_1 would have shifted to the right by more than $2t$. Furthermore, since a_is are close in value to $t/3$, exactly three a_is that sum to t must have been flipped, i.e., v_{2i-1} and v_{2i} must have been folded from some triple of a_is that sum to t.

Once s_1 and s_2 have been folded, the paper now represents a smaller instance of 3-Partition with three fewer a_is that sum to t with identical structure. The remaining creases of S have shifted to the right by $2t+2\delta$ and the remaining creases of C have shifted to the right by 2δ; in particular, s_3, s_4, c_3, and c_4 are in exactly the same horizontal locations respectively that s_1, s_2, c_1, and c_2 used to be. (2) continues to apply recursively, constraining the next crease pair to fold only after new a_i triples summing to t have been identified and folded. Thus, if (P, Σ, f) is foldable in the infinite one-layer model, there exists a solution to the 3-Partition instance.

The theorem follows directly. The reduction is polynomial since the construction is bounded by a $4tn/3 \times 8$ rectangle with $2n + 4n/3$ creases. Further, solutions can be checked naively in $O(n^2)$ time by performing each simple fold in order while checking for self intersection after each fold. □

This reduction only applies in the infinite one-layer model; in the one-layer model, the constructed crease pattern folds trivially. Surprisingly, none of the above arguments relied on knowing the M/V assignment of the creases. For creases C to ever fold, creases h_1 and h_2 must be folded in the same direction; the creases in V must pleat a_i intervals with alternating crease assignment, and the same is true of the creases in S and C. Thus, the theorem also holds in the unassigned case.

Theorem 7. *The unassigned simple-foldability decision problem for orthogonal paper with paper-aligned orthogonal creases* $\boxplus$ *is strongly NP-complete in the infinite one-layer and infinite some-layers models.*

(5) Infinite, Rectangle Paper/Orthogonal Creases $\boxplus$

For assigned crease patterns on rectangular paper with paper-aligned orthogonal creases, Arkin et al. show that determining simple-foldability can be decided in polynomial time in the one-layer, some-layers, and all-layers models, noting that the answer is automatically no in the one-layer model for crease patterns containing both horizontal and vertical creases. Note that when unassigned, all rectangular paper with paper-aligned orthogonal creases $\boxplus$ can be produced by folding the horizontal folds in order alternating mountain and valley, followed by similarly pleating the vertical folds. We prove the same results apply in the infinite one-layer, infinite some-layer, and infinite all-layer models, because the corresponding non-infinite models are equivalent for $\boxplus$ crease patterns.

Theorem 8. *Concerning simple-foldability of rectangular paper with paper-aligned orthogonal creases* $\boxplus$*, the infinite (one, some, all)-layer models are equivalent to the (one, some, all)-layer models respectively.*

Proof. The only difference between the infinite and non-infinite versions of simple folds models is that the infinite versions must fold at least one layer everywhere paper exists along the fold axis, while the non-infinite versions do not. Assume for the sake of contradiction that the models are not equivalent so that in a given valid flat folding (P, f, λ) a simple fold may occur that folds paper $U \subset P$ about the fold axis ℓ for which $f(U) \cap \ell$ does not equal $f(P) \cap \ell$. Let q be a point on the boundary of the former but on the interior of the latter which exists since $f(P) \cap \ell$ is a line segment. Some p exists in the preimage $f^{-1}(q)$ that is not the endpoint of an already folded crease or else the paper would be discontinuous. Then the crease containing p bounds two facets, of which one facet F intersects U but is not contained in U or else q would not be a boundary point. But rotating $F \cap U$ without rotating $F \setminus U$ would violate isometry, a contradiction. □

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Hugo A. Akitaya was born in 1988. He received his M.S. degree from Tsukuba University in 2014 and is currently a Ph.D. candidate at Tufts University. His research interests are computational geometry, folding and graph algorithms.

Erik D. Demaine received a B.Sc. degree from Dalhousie University in 1995, and M.Math. and Ph.D. degrees from the University of Waterloo in 1996 and 2001, respectively. Since 2001, he has been a professor in computer science at the Massachusetts Institute of Technology. His research interests range throughout algorithms, from data structures for improving web searches to the geometry of understanding how proteins fold to the computational difficulty of playing games. In 2003, he received a MacArthur Fellowship as a "computational geometer tackling and solving difficult problems related to folding and bending moving readily between the theoretical and the playful, with a keen eye to revealing the former in the latter." He cowrote a book about the theory of folding, together with Joseph O'Rourke (Geometric Folding Algorithms, 2007), and a book about the computational complexity of games, together with Robert Hearn (Games, Puzzles, and Computation, 2009).

Jason S. Ku was born in 1986. Ku received Bachelor's, Master's, and Doctoral degrees in the Mechanical Engineering department at the Massachusetts Institute of Technology in 2009, 2011, and 2016 respectively. He now works there as a postdoctoral associate in computer science. He primarily studies computational geometry, particularly questions involving the relationship between two and three dimensions. His research topics include the design and fabrication of static and transformable folding structures from sheet-like material, 3D object reconstruction algorithms from lower dimensional projections, and 3D projection.