Balanced C_{12}-Trefoil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_{12} be *the* 12*-cycle*. *The* C_{12}*-trefoil* is a graph of 3 edge-disjoint C_{12}'s with a common vertex and the common vertex is called *the center* of the C_{12}-trefoil. When K_n is decomposed into edge-disjoint sum of C_{12}-trefoils, we say that K_n *has a* C_{12}*-trefoil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_{12}-trefoils, we say that K_n *has a balanced* C_{12}*-trefoil decomposition* and this number is called *the replication number*.

In this paper, it is shown that the necessary and sufficient condition for the existence of a balanced C_{12}-trefoil decomposition of K_n is $n \equiv 1 \pmod{72}$. The decomposition algorithm is also given.

2. Balanced C_{12}-trefoil decomposition of K_n

Notation. We denote a C_{12}-trefoil passing through $v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_{10} - v_{11} - v_{12} - v_1$,
$v_1 - v_{13} - v_{14} - v_{15} - v_{16} - v_{17} - v_{18} - v_{19} - v_{20} - v_{21} - v_{22} - v_{23} - v_1$,
$v_1 - v_{24} - v_{25} - v_{26} - v_{27} - v_{28} - v_{29} - v_{30} - v_{31} - v_{32} - v_{33} - v_{34} - v_1$,
by $\{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}),$
$(v_1, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}),$
$(v_1, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34})\}$.

Theorem. K_n has a balanced C_{12}-trefoil decomposition if and only if $n \equiv 1 \pmod{72}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{12}-trefoil decomposition. Let b be the number of C_{12}-trefoils and r be the replication number. Then $b = n(n-1)/72$ and $r = 34(n-1)/72$. Among r C_{12}-trefoils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{12}-trefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $6r_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/72$ and $r_2 = 33(n-1)/72$. Therefore, $n \equiv 1 \pmod{72}$ is necessary.

(Sufficiency) Put $n = 72t + 1$. Construct tn C_{12}-trefoils as follows:

$B_i^{(1)} = \{\, (i, i+1, i+6t+2, i+24t+2, i+42t+3, i+18t+2, i+57t+3, i+21t+2, i+48t+3, i+27t+2, i+12t+2, i+3t+1),$
$(i, i+2, i+6t+4, i+24t+3, i+42t+5, i+18t+3, i+57t+5, i+21t+3, i+48t+5, i+27t+3, i+12t+4, i+3t+2),$
$(i, i+3, i+6t+6, i+24t+4, i+42t+7, i+18t+4, i+57t+7, i+21t+4, i+48t+7, i+27t+4, i+12t+6, i+3t+3)\}$

$B_i^{(2)} = \{\, (i, i+4, i+6t+8, i+24t+5, i+42t+9, i+18t+5, i+57t+9, i+21t+5, i+48t+9, i+27t+5, i+12t+8, i+3t+4),$
$(i, i+5, i+6t+10, i+24t+6, i+42t+11, i+18t+6, i+57t+11, i+21t+6, i+48t+11, i+27t+6, i+12t+10, i+3t+5),$
$(i, i+6, i+6t+12, i+24t+7, i+42t+13, i+18t+7, i+57t+13, i+21t+7, i+48t+13, i+27t+7, i+12t+12, i+3t+6)\}$

$B_i^{(3)} = \{\, (i, i+7, i+6t+14, i+24t+8, i+42t+15, i+18t+8, i+57t+15, i+21t+8, i+48t+15, i+27t+8, i+12t+14, i+3t+7),$
$(i, i+8, i+6t+16, i+24t+9, i+42t+17, i+18t+9, i+57t+17, i+21t+9, i+48t+17, i+27t+9, i+12t+16, i+3t+8),$
$(i, i+9, i+6t+18, i+24t+10, i+42t+19, i+18t+10, i+57t+19, i+21t+10, i+48t+19, i+27t+10, i+12t+18, i+3t+9)\}$

...

$B_i^{(t)} = \{\, (i, i+3t-2, i+12t-4, i+27t-1, i+48t-3, i+21t-1, i+63t-3, i+24t-1, i+54t-3, i+30t-1, i+18t-4, i+6t-2),$
$(i, i+3t-1, i+12t-2, i+27t, i+48t-1, i+21t, i+$

$63t-1, i+24t, i+54t-1, i+30t, i+18t-2, i+6t-1)$,
$(i, i+3t, i+12t, i+27t+1, i+48t+1, i+21t+1, i+63t+1, i+24t+1, i+54t+1, i+30t+1, i+18t, i+6t)\}$
$(i = 1, 2, ..., n)$.

Then they comprise a balanced C_{12}-trefoil decomposition of K_n.
This completes the proof.

Example 1. *Balanced C_{12}-trefoil decomposition of K_{73}.*
$B_i = \{(i, i+1, i+8, i+26, i+45, i+20, i+60, i+23, i+51, i+29, i+14, i+4)$,
$(i, i+2, i+10, i+27, i+47, i+21, i+62, i+24, i+53, i+30, i+16, i+5)$,
$(i, i+3, i+12, i+28, i+49, i+22, i+64, i+25, i+55, i+31, i+18, i+6)\}$ $(i = 1, 2, ..., 73)$.

Example 2. *Balanced C_{12}-trefoil decomposition of K_{145}.*
$B_i^{(1)} = \{(i, i+1, i+14, i+50, i+87, i+38, i+117, i+44, i+99, i+56, i+26, i+7)$,
$(i, i+2, i+16, i+51, i+89, i+39, i+119, i+45, i+101, i+57, i+28, i+8)$,
$(i, i+3, i+18, i+52, i+91, i+40, i+121, i+46, i+103, i+58, i+30, i+9)\}$
$B_i^{(2)} = \{(i, i+4, i+20, i+53, i+93, i+41, i+123, i+47, i+105, i+59, i+32, i+10)$,
$(i, i+5, i+22, i+54, i+95, i+42, i+125, i+48, i+107, i+60, i+34, i+11)$,
$(i, i+6, i+24, i+55, i+97, i+43, i+127, i+49, i+109, i+61, i+36, i+12)\}$ $(i = 1, 2, ..., 145)$.

Example 3. *Balanced C_{12}-trefoil decomposition of K_{217}.*
$B_i^{(1)} = \{(i, i+1, i+20, i+74, i+129, i+56, i+174, i+65, i+147, i+83, i+38, i+10)$,
$(i, i+2, i+22, i+75, i+131, i+57, i+176, i+66, i+149, i+84, i+40, i+11)$,
$(i, i+3, i+24, i+76, i+133, i+58, i+178, i+67, i+151, i+85, i+42, i+12)\}$
$B_i^{(2)} = \{(i, i+4, i+26, i+77, i+135, i+59, i+180, i+68, i+153, i+86, i+44, i+13)$,
$(i, i+5, i+28, i+78, i+137, i+60, i+182, i+69, i+155, i+87, i+46, i+14)$,
$(i, i+6, i+30, i+79, i+139, i+61, i+184, i+70, i+157, i+88, i+48, i+15)\}$
$B_i^{(3)} = \{(i, i+7, i+32, i+80, i+141, i+62, i+186, i+71, i+159, i+89, i+50, i+16)$,
$(i, i+8, i+34, i+81, i+143, i+63, i+188, i+72, i+161, i+90, i+52, i+17)$,
$(i, i+9, i+36, i+82, i+145, i+64, i+190, i+73, i+163, i+91, i+54, i+18)\}$ $(i = 1, 2, ..., 217)$.

Example 4. *Balanced C_{12}-trefoil decomposition of K_{289}.*
$B_i^{(1)} = \{(i, i+1, i+26, i+98, i+171, i+74, i+231, i+86, i+195, i+110, i+50, i+13)$,
$(i, i+2, i+28, i+99, i+173, i+75, i+233, i+87, i+197, i+111, i+52, i+14)$,
$(i, i+3, i+30, i+100, i+175, i+76, i+235, i+88, i+199, i+112, i+54, i+15)\}$
$B_i^{(2)} = \{(i, i+4, i+32, i+101, i+177, i+77, i+237, i+89, i+201, i+113, i+56, i+16)$,
$(i, i+5, i+34, i+102, i+179, i+78, i+239, i+90, i+203, i+114, i+58, i+17)$,
$(i, i+6, i+36, i+103, i+181, i+79, i+241, i+91, i+205, i+115, i+60, i+18)\}$
$B_i^{(3)} = \{(i, i+7, i+38, i+104, i+183, i+80, i+243, i+92, i+207, i+116, i+62, i+19)$,
$(i, i+8, i+40, i+105, i+185, i+81, i+245, i+93, i+209, i+117, i+64, i+20)$,
$(i, i+9, i+42, i+106, i+187, i+82, i+247, i+94, i+211, i+118, i+66, i+21)\}$
$B_i^{(4)} = \{(i, i+10, i+44, i+107, i+189, i+83, i+249, i+95, i+213, i+119, i+68, i+22)$,
$(i, i+11, i+46, i+108, i+191, i+84, i+251, i+96, i+215, i+120, i+70, i+23)$,
$(i, i+12, i+48, i+109, i+193, i+85, i+253, i+97, i+217, i+121, i+72, i+24)\}$ $(i = 1, 2, ..., 289)$.

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