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Balanced (C_4, C_7)-$2t$-Foil Decomposition Algorithm of Complete Graphs

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1. Introduction

Let K_n denote *the complete graph* of n vertices. Let C_4 and C_7 be *the 4-cycle* and *the 7-cycle*, respectively. *The* (C_4, C_7)*-2t-foil* is a graph of t edge-disjoint C_4's and t edge-disjoint C_7's with a common vertex and the common vertex is called *the center of the* (C_4, C_7)*-2t-foil*. In particular, the (C_4, C_7)-2-foil is called *the* (C_4, C_7)*-bowtie*. When K_n is decomposed into edge-disjoint sum of (C_4, C_7)-$2t$-foils, we say that K_n *has a* (C_4, C_7)*-2t-foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_4, C_7)-$2t$-foils, we say that K_n *has a balanced* (C_4, C_7)*-2t-foil decomposition* and this number is called *the replication number*.
Note that (C_4, C_7)-$2t$-foil has $9t+1$ vertices and $11t$ edges.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or 3 (mod 6). This decomposition is known as *a Steiner triple system*. See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a (C_3, C_3)-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a bowtie system*.
In this sense, our balanced (C_4, C_7)-$2t$-foil decomposition of K_n is to be known as *a balanced* (C_4, C_7)*-2t-foil system*.

2. Balanced (C_4, C_7)-$2t$-foil decomposition of K_n

Theorem. K_n has a balanced (C_4, C_7)-$2t$-foil decomposition if and only if $n \equiv 1$ (mod $22t$).

Proof. **(Necessity)** Suppose that K_n has a balanced (C_4, C_7)-$2t$-foil decomposition. Let b be the number of (C_4, C_7)-$2t$-foils and r be the replication number. Then $b = n(n-1)/22t$ and $r = (9t+1)(n-1)/22t$. Among r (C_4, C_7)-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_4, C_7)-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/22t$ and $r_2 = 9(n-1)/22$. Therefore, $n \equiv 1$ (mod $22t$) is necessary.
(Sufficiency) Put $n = 22st + 1$, $T = st$. Then $n = 22T + 1$.
When $T = 1$, construct a balanced (C_4, C_7)-2-foil decomposition of K_{23} as follows:
$B_i = \{(i, i+5, i+13, i+6), (i, i+1, i+3, i+7, i+10, i+20, i+9)\}$ $(i = 1, 2, ..., 23)$.
First, consider a sequence $S : g_1, g_2, g_3, ..., g_T$.
When $T = 2$, put $S : g_1, g_2$ with $g_1 = 21, g_2 = 19$.
When $T = 3$, put $S : g_1, g_2, g_3$ with $g_1 = 28, g_2 = 30, g_3 = 29$.
When $T = 4$, put $S : g_1, g_2, g_3, g_4$ with $g_1 = 39, g_2 = 41, g_3 = 38, g_4 = 37$.
When $T = 5$, put $S : g_1, g_2, g_3, g_4, g_5$ with $g_1 = 51, g_2 = 47, g_3 = 49, g_4 = 48, g_5 = 46$.
When $T \equiv 2$ (mod 4), $T \geq 6$, put $T = 4p+2$ and $S : g_1, g_2, g_3, ..., g_{4p+2}$ with $S_1 : g_1, g_3, g_5, ..., g_{2p-1}$, $S_2 : g_2, g_4, g_6, ..., g_{2p}$, $S_3 : g_{2p+1}$, $S_4 : g_{2p+2}, g_{2p+3}, g_{2p+4}, ..., g_{4p+2}$ such as
$S_1 : 10T-2, 10T-4, 10T-6, ..., 10T-2p$
$S_2 : 10T+1, 10T-1, 10T-3, ..., 10T-2p+3$
$S_3 : 10T-2p+1$ $S_4 : 10T-2p-1, 10T-2p-2, 10T-2p-3, ..., 9T+1$.
When $T \equiv 3$ (mod 4), $T \geq 7$, put $T = 4p+7$ and $S : g_1, g_2, g_3, ..., g_{4p+7}$ with $S_1 : g_1, g_{2p+3}, g_{4p+5}, g_{4p+6}, g_{4p+7}$, $S_2 : g_2, g_3, g_4, ..., g_{2p+2}$, $S_3 : g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+4}$, $S_4 : g_{2p+5}, g_{2p+7}, g_{2p+9}, ..., g_{4p+3}$ such as $S_1 : 10T+$

$1, 10T-2p-3, 9T+5, 9T+3, 9T+1$ S_2 : $10T-1, 10T-2, 10T-3, ..., 10T-2p-1$ S_3 : $10T-2p-5, 10T-2p-7, 10T-2p-9, ..., 9T+2$ S_4 : $10T-2p-2, 10T-2p-4, 10T-2p-6, ..., 9T+7$.
When $T \equiv 0 \pmod 4$, $T \geq 8$, put $T = 4p+4$ and S : $g_1, g_2, g_3, ..., g_{4p+4}$ with S_1 : $g_1, g_3, g_5, ..., g_{2p-1}$, $S_2 : g_2, g_4, g_6, ..., g_{2p+2}$, S_3 : g_{2p+1} , $S_4 : g_{2p+3}, g_{2p+4}, g_{2p+5}, ..., g_{4p+4}$ such as S_1 : $10T-2, 10T-4, 10T-6, ..., 10T-2p$ $S_2 : 10T+1, 10T-1, 10T-3, ..., 10T-2p+1$ $S_3 : 10T-2p-1$ $S_4 : 10T-2p-2, 10T-2p-3, 10T-2p-4, ..., 9T+1$.
When $T \equiv 1 \pmod 4$, $T \geq 9$, put $T = 4p+9$ and S : $g_1, g_2, g_3, ..., g_{4p+9}$ with S_1 : $g_1, g_{2p+5}, g_{4p+7}, g_{4p+8}, g_{4p+9}$, S_2 : $g_2, g_3, g_4, ..., g_{2p+3}$, S_3 : $g_{2p+4}, g_{2p+6}, g_{2p+8}, ..., g_{4p+6}$, S_4 : $g_{2p+7}, g_{2p+9}, g_{2p+11}, ..., g_{4p+5}$ such as $S_1 : 10T+1, 10T-2p-3, 9T+5, 9T+3, 9T+1$ S_2 : $10T-1, 10T-2, 10T-3, ..., 10T-2p-2$ S_3 : $10T-2p-5, 10T-2p-7, 10T-2p-9, ..., 9T+2$ S_4 : $10T-2p-4, 10T-2p-6, 10T-2p-8, ..., 9T+7$.
Next, construct n (C_4, C_7)-$2T$-foils as follows: $B_i = \{(i, i+T+1, i+15T+2, i+2T+1), (i, i+1, i+3T+2, i+10T+2, i+15T+3, i+20T+3, i+g_1)\} \cup \{(i, i+T+2, i+15T+4, i+2T+2), (i, i+2, i+3T+4, i+10T+3, i+15T+5, i+20T+4, i+g_2)\} \cup \{(i, i+T+3, i+15T+6, i+2T+3), (i, i+3, i+3T+6, i+10T+4, i+15T+7, i+20T+5, i+g_3)\} \cup ... \cup \{(i, i+2T, i+17T, i+3T), (i, i+T, i+5T, i+11T+1, i+17T+1, i+21T+2, i+g_T)\}$ $(i = 1, 2, ..., n)$.
Last, decompose each (C_4, C_7)-$2T$-foil into s (C_4, C_7)-$2t$-foils. Then they comprise a balanced (C_4, C_7)-$2t$-foil decomposition of K_n.

Corollary. K_n has a balanced (C_4, C_7)-bowtie decomposition if and only if $n \equiv 1 \pmod{22}$.

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