Regular Paper

On the Church-Rosser Property of Non-E-overlapping and Strongly Depth-preserving Term Rewriting Systems

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A term rewriting system (TRS) is said to be depth-preserving if for any rewrite rule and any variable appearing in the both sides, the maximal depth of the variable occurrences in the left-hand-side is greater than or equal to that of the variable occurrences in the right-hand-side, and to be strongly depth-preserving if it is depth-preserving and for any rewrite rule and any variable appearing in the left-hand-side, all the depths of the variable occurrences in the left-hand-side are the same. This paper shows that there exist non-E-overlapping and depth-preserving TRS's which are not Church-Rosser, but all the non-E-overlapping and strongly depth-preserving TRS's are Church-Rosser.

1. Introduction

A term rewriting system (TRS) is a set of directed equations (called rewrite rules). A TRS is Church-Rosser (CR) if any two interconvertible terms reduce to some common term by applications of the rewrite rules. This CR property is important in various applications of TRS's and has received much attention so far[1)~3),5)~8)]. Although the CR property is undecidable for general TRS's, many sufficient conditions for ensuring this property have been obtained[1),2),5)~8)].

However, for nonlinear and nonterminating TRS's, only a few results on the CR property have been obtained. Our previous papers[5),6)] may be pioneering ones which have first given nontrivial conditions for the CR property, though these conditions can be applied only to subclasses of right-linear TRS's. On the other hand, if we omit the right-linearity condition, then it has been shown that only the non-E-overlapping condition is insufficient for ensuring the CR property of TRS's[2)]. For example, $R_0 = \{f(x,x) \to a,\ g(x) \to f(x,g(x)),\ c \to g(c)\}$, where x is a variable and f, g, a, c are function symbols, is non-E-overlapping, but not CR.

In this paper, we consider the CR property of nonlinear, nonterminating and depth-preserving TRS's. Here, a TRS is depth-preserving if for each rule $\alpha \to \beta$ and any variable x appearing in both α and β, the maximal depth of the x occurrences in α is greater than or equal to that of the x occurrences in β[3)]. For example, TRS $R_1 = \{h(k(x),x) \to f(x,g(x))\}$, where x is a variable, is depth-preserving, since the maximal depth of the x occurrences of the left-hand-side is 2 and that of the right-hand-side 2. Note that R_0 is not depth-preserving.

We first show that only the non-E-overlapping and depth-preserving properties are insufficient for ensuring the CR property. That is, the following non-E-overlapping and depth-preserving TRS R_2 is not CR:

$$R_2 = \{f(x,x) \to a,\ c \to h(c,g(c)),\ h(x,g(x)) \to f(x,h(x,g(c)))\}$$

where x is a variable and f, g, h, a, c are function symbols.

Next, we introduce the notion of strongly depth-preserving property (stronger than the depth-preserving one). A TRS R is strongly depth-preserving if R is depth-preserving and for each $\alpha \to \beta$ and for any variable x appearing in α, all the depths of the x occurrences in α are the same. For example, TRS $R_3 = \{h(g(x),g(x)) \to f(x,h(x,g(c)))\}$ is strongly depth-preserving, since R_3 is depth-preserving and all the depths of x occurrences of the left-hand-side are 2. In this paper, we prove that all the non-E-overlapping and strongly depth-preserving TRS's are CR (Theorem 1).

This paper is organized as follows. Section 2 is devoted to definitions. In Section 3, we show that the above TRS R_2 is not CR. Some assertions to prove Theorem 1 are given in Section 4 and proven in Section 5, so that we obtain Theorem 1. Some concluding remarks are given in Section 6.

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2. Definitions

The following definitions and notations are similar to those in Refs. 2), 5). Let X be a set of variables, F be a finite set of function symbols and T be the set of terms constructed from X and F.

For a term M, we use $O(M)$ to denote the set of occurrences (positions) of M, and M/u to denote the subterm of M at occurrence u, and $M[u \leftarrow N]$ to denote the term obtained from M by replacing the subterm M/u by term N. The set of occurrences $O(M)$, where $M \in T$, is partially ordered by the prefix ordering: $u \leq v$ iff $\exists w.\ uw = v$. In this case, we denote w by v/u. If $u \leq v$ and $u \neq v$, then $u < v$. If $u \not\leq v$ and $v \not\leq u$, then u and v are said to be *disjoint* and denoted $u|v$. Let $V(M)$ be the set of variables in M, $O_x(M)$ be the set of occurrences of variable $x \in V(M)$, and $O_X(M) = \cup_{x \in V(M)} O_x(M)$ i.e., the set of variable occurrences in M. $\bar{O}(M)$ is the set of non-variable occurrences, i.e., $\bar{O}(M) = O(M) - O_X(M)$. We use $N[u \leftarrow M/u \mid u \in U]$ to denote $N[u_1 \leftarrow M/u_1, u_2 \leftarrow M/u_2, \cdots, u_n \leftarrow M/u_n]$ where $U = \{u_1, u_2, \cdots, u_n\}$, and $u_1, \cdots, u_n$ are pairwise disjoint. Here, $N[u_1 \leftarrow M/u_1, u_2 \leftarrow M/u_2, \cdots, u_n \leftarrow M/u_n] = (N[u_1 \leftarrow M/u_1, u_2 \leftarrow M/u_2, \cdots, u_{n-1} \leftarrow M_{n-1}/u_{n-1}])[u_n \leftarrow M/u_n]$ if $n > 1$.

For a term M, $H(M) = Max\{|u| \mid u \in O(M)\}$. $H(M)$ is called "height of M". The depth of occurrence $u \in O(M)$ is defined by $|u|$.

Example. $H(f(g(x))) = 2, H(a) = 0, H(g(x)) = 1$.

A rewrite rule is a directed equation $\alpha \rightarrow \beta$ such that $\alpha \in T - X, \beta \in T$ and $V(\alpha) \supseteq V(\beta)$. A term-rewriting system (TRS) is a set of rewrite rules.

A term M reduces to a term N if $M/u = \sigma(\alpha)$ and $N = M[u \leftarrow \sigma(\beta)]$ for some $\alpha \rightarrow \beta \in R$ and $\sigma : X \rightarrow T$. We denote this reduction by $M \overset{u}{\rightarrow} N$. In this notation u may be omitted (i.e., $M \rightarrow N$) and $\rightarrow^*$ is the reflexive-transitive closure of $\rightarrow$. Let $M \overset{u}{\leftrightarrow} N$ be $M \overset{u}{\rightarrow} N$ or $N \overset{u}{\rightarrow} M$.

A parallel reduction $M \leftrightarrow\!\!\!\mapsto N$ is defined as follows: $M \leftrightarrow\!\!\!\mapsto N$ iff $\exists U \subseteq O(M)$ such that $\forall u, v \in U \;\; u \neq v \Rightarrow u|v$, $\forall u \in U\ M/u \overset{\varepsilon}{\leftrightarrow} N/u$ and $N = M[u \leftarrow N/u \mid u \in U]$. In this case, let $R(M \leftrightarrow\!\!\!\mapsto N) = U$. (Note. $U = \phi$ is allowed.) Let $\leftrightarrow\!\!\!\mapsto^*$ be the reflexive-transitive closure of $\leftrightarrow\!\!\!\mapsto$.

We assume that $\gamma : M_0 \leftrightarrow\!\!\!\mapsto M_1 \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto M_n$ in the following definitions, which will be used in Sections 4 and 5.

Let $R(\gamma) = \bigcup_{0 \leq i < n} R(M_i \leftrightarrow\!\!\!\mapsto M_{i+1})$ and $MR(\gamma)$ be the set of *minimal* occurrences in $R(\gamma)$ under the prefix ordering.

For $u \in O(M_0)$, if there exists no $v \in R(\gamma)$ such that $v \leq u$, then γ is said to be *u-invariant*.

Let $\delta : N_0 \leftrightarrow\!\!\!\mapsto N_1 \cdots \leftrightarrow\!\!\!\mapsto N_k$ where $N_0 = \sigma(\alpha)$ or $N_k = \sigma(\alpha)$ for some $\alpha \rightarrow \beta \in R$ and $\sigma : X \rightarrow T$. Then, δ is said to be *α-keeping* if δ is u-invariant for all $u \in \bar{O}(\alpha)$, that is, δ is α-keeping iff all reductions of δ occur in the variable parts of α.

We denote by $\gamma[i, j]$ the subsequence $M_i \leftrightarrow\!\!\!\mapsto M_{i+1} \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto M_j$ of γ where $i \geq 0$ and $j \leq n$.

If $M_n = N_0$, then the composition of γ and $\delta : N_0 \leftrightarrow\!\!\!\mapsto N_1 \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto N_k$, i.e., $M_0 \leftrightarrow\!\!\!\mapsto M_1 \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto M_n (= N_0) \leftrightarrow\!\!\!\mapsto N_1 \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto N_k$ is denoted by $(\gamma; \delta)$.

Let $u \in MR(\gamma)$. Then, the *cut* sequence of γ at u is $\gamma/u = (M_0/u \leftrightarrow\!\!\!\mapsto M_1/u \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto M_n/u)$.

We denote by $\gamma[\xi'/\xi]$ the sequence obtained from reduction sequence γ by replacing subsequence or cut sequence (or cut subsequence) ξ of γ by sequence ξ'.

Let γ^R be the reverse sequence of γ, i.e., $\gamma^R : M_n \leftrightarrow\!\!\!\mapsto \cdots \leftrightarrow\!\!\!\mapsto M_1 \leftrightarrow\!\!\!\mapsto M_0$.

The number of parallel reduction steps of γ is $|\gamma|_p = n$.

Note. If $\delta : M \leftrightarrow\!\!\!\mapsto M$, then $|\delta|_p = 1$.

Example. Let $\delta : f(c, c) \leftrightarrow\!\!\!\mapsto f(g(c), g(c)) \leftrightarrow\!\!\!\mapsto a$, then $|\delta|_p = 2$.

Let $net(\gamma)$ is the sequence obtained from γ by removing all $M_i \leftrightarrow\!\!\!\mapsto M_{i+1}$ satisfying that $M_i = M_{i+1}$, $0 \leq i < n$.

Example. Let $\delta : g(c) \leftrightarrow\!\!\!\mapsto g(g(c)) \leftrightarrow\!\!\!\mapsto a \leftrightarrow\!\!\!\mapsto a$, then $net(\delta) : g(c) \leftrightarrow\!\!\!\mapsto g(g(c)) \leftrightarrow\!\!\!\mapsto a$.

We use $|\delta|_{np}$ to denote $|net(\delta)|_p$.

Definition of ⟨$H(\gamma)$ — the height of reduction sequence⟩

$H(\gamma) = Max\{H(M_i) \mid 0 \leq i \leq n\}$

Example. Let $\delta : g(c) \leftrightarrow\!\!\!\mapsto g(g(c)) \leftrightarrow\!\!\!\mapsto a$, then $H(\delta) = H(g(g(c))) = 2$.

Definition of ⟨peak[4)]⟩

Reduction sequence γ is said to be a *peak* if $\gamma : M_0 \overset{\varepsilon}{\longleftarrow} M_1 \leftrightarrow\!\!\!\mapsto^* M_{n-1} \overset{\varepsilon}{\longrightarrow} M_n$ and the subsequence $M_1 \leftrightarrow\!\!\!\mapsto^* M_{n-1}$ is ε-invariant.

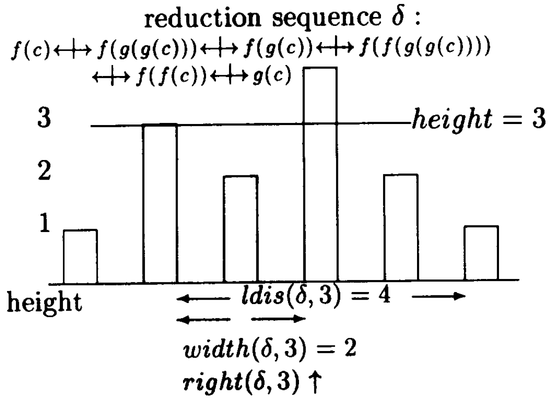


Fig. 1 Example of *ldis*, *right* and *width*.

Definitions of ⟨$left(\gamma,h)$, $right(\gamma,h)$, $ldis(\gamma,h)$, $width(\gamma,h)$⟩

$$left(\gamma,h)=\begin{cases} Min\{i \mid H(M_i)=h\} \\ \quad \text{if } \exists i\ (0\le i\le n) \text{ such that } H(M_i) \\ \quad = h \text{ and } \forall j(0\le j<i)\ H(M_j)<h \\ \bot \quad \text{otherwise} \end{cases}$$

$$right(\gamma,h)=\begin{cases} Max\{i \mid H(M_i)=h\} \\ \quad \text{if } \exists i\ (0\le i\le n) \text{ such that } H(M_i) \\ \quad = h \text{ and } \forall j(i<j\le n)\ H(M_j)<h \\ \bot \quad \text{otherwise} \end{cases}$$

$$ldis(\gamma,h)=\begin{cases} n-left(\gamma,h) \\ \quad \text{if } left(\gamma,h)\ne\bot \\ \bot \quad \text{otherwise} \end{cases}$$

$$width(\gamma,h)=\begin{cases} right(\gamma,h)-left(\gamma,h) \\ \quad \text{if } left(\gamma,h)\ne\bot \wedge right(\gamma,h)\ne\bot \\ right(\gamma,h)-left(\gamma,h') \\ \quad \text{if } left(\gamma,h)=\bot \wedge right(\gamma,h)\ne\bot \wedge h' \\ \quad = Min\{h' \mid h'>h \wedge left(\gamma,h')\ne\bot\} \\ right(\gamma,h')-left(\gamma,h) \\ \quad \text{if } left(\gamma,h)\ne\bot \wedge right(\gamma,h)=\bot \wedge h' \\ \quad = Min\{h' \mid h'>h \wedge right(\gamma,h')\ne\bot\} \\ \bot \quad \text{otherwise} \end{cases}$$

We write $P(\gamma,h)\downarrow$ if $P(\gamma,h)\ne\bot$ and otherwise $P(\gamma,h)\uparrow$ for $P\in\{left, right, ldis, width\}$.

In **Fig. 1**, we illustrate *width* and *ldis* by examples.

Example. Let δ : $f(c)\leftrightarrow\!\!\!\leftrightarrow f(g(g(c)))\leftrightarrow\!\!\!\leftrightarrow f(g(c))\leftrightarrow\!\!\!\leftrightarrow f(f(g(g(c))))\leftrightarrow\!\!\!\leftrightarrow f(f(c))\leftrightarrow\!\!\!\leftrightarrow g(c)$. This δ is illustrated in Fig. 1. Then, we have $left(\delta,1)=0$, $left(\delta,2)\uparrow$, $left(\delta,3)=1$, $ldis(\delta,1)=5$, $ldis(\delta,3)=4$, $right(\delta,1)=5$, $right(\delta,3)\uparrow$, $width(\delta,1)=right(\delta,1)-left(\delta,1)=5$, $width(\delta,2)=3$, $width(\delta,3)=2$.

Definitions of ⟨$K_{ldis}(\gamma)$, $K_{width}(\gamma)$, $K_{right}(\gamma)$⟩

$K_{ldis}(\gamma)=\{(h, ldis(\gamma,h)) \mid ldis(\gamma,h)\downarrow\}$
$K_{width}(\gamma)=\{(h, width(\gamma,h)) \mid width(\gamma,h)\downarrow\}$
$K_{right}(\gamma)=\{(h, right(\gamma,h)) \mid right(\gamma,h)\downarrow\}$

Example. For δ : $f(c)\leftrightarrow\!\!\!\leftrightarrow f(g(g(c)))\leftrightarrow\!\!\!\leftrightarrow f(g(c))\leftrightarrow\!\!\!\leftrightarrow f(f(g(g(c))))\leftrightarrow\!\!\!\leftrightarrow f(f(c))\leftrightarrow\!\!\!\leftrightarrow g(c)$ in the previous example, we have $K_{ldis}(\delta)=\{(1,5), (3,4), (4,2)\}$, $K_{width}(\delta)=\{(1,5), (2,3), (3,2), (4,0)\}$ and $K_{right}(\delta)=\{(1,5), (2,4), (4,3)\}$.

We define an ordering $<_s\subseteq N\times N$ (where $N=\{0,1,2,\cdots\}$) as follows: $(a,b)<_s(a',b')\Leftrightarrow(a<a'\bigwedge b\le b')\bigvee(a=a'\bigwedge b<b')$. Let $\le_s$ be $<_s\bigcup=$. We use $\ll_s$ to denote the multiset ordering of this ordering $<_s$. Let $\underline{\ll}_s$ be $\ll_s\bigcup=$. We use $\{\cdots\}_m$ to denote a multiset, e.g., $\{1,1,2\}_m$. We use $\ll_w$ to denote the multiset ordering of a lexicographic ordering $<$ (i.e., $(a,b)<(a',b')\Leftrightarrow a<a'\bigvee(a=a'\bigwedge b<b')$). Let $\underline{\ll}_w$ be $\ll_w\bigcup=$. Note that if $(a,b)<_s(a',b')$, then $(a,b)<(a',b')$, but the converse does not necessarily hold. And if $A\ll_s B$, then $A\ll_w B$. The orderings of $>_s$ and $>$ are well-founded, so that $\gg_s$ and $\gg_w$ are well-founded[1].

In Sections 4 and 5, we will define orderings of reduction sequences γ,δ by using K_Y where $Y\in\{ldis, right, width\}$ and $\ll_s$ (or $\ll_w$).

3. Depth-preserving TRS's

In this paper, we consider the CR property of non-E-overlapping and depth-preserving TRS's. Now we give these definitions, and introduce the class of strongly depth-preserving TRS's which is a subclass of depth-preserving TRS's.

Definition of ⟨E-overlapping TRS R⟩

A TRS R is said to be E-overlapping iff there exists an ε-invariant reduction sequence $\sigma(\alpha_1/u)\leftrightarrow\!\!\!\leftrightarrow^*\sigma'(\alpha_2)$ for some $\alpha_1\to\beta_1,\alpha_2\to\beta_2\in R$, $u\in\bar{O}(\alpha_1)$ and mappings $\sigma,\sigma':X\to T$ where $u=\varepsilon$ implies that $(\alpha_1\to\beta_1)\ne(\alpha_2\to\beta_2)$. In this case, the pair $(\sigma(\alpha_1)[u\leftarrow\sigma'(\beta_2)],\sigma(\beta_1))$ is called an *E-critical pair*. A TRS R is non-E-overlapping if there exist no E-critical pairs.

Definition of ⟨depth-preserving TRS R[3]⟩

A TRS R is *depth-preserving* if $\forall\alpha\to\beta\in R\,\forall x\in V(\alpha)\bigcap V(\beta)\quad Max\{|v| \mid v\in O_x(\beta)\}\le Max\{|u| \mid u\in O_x(\alpha)\}$.

Example. $R_2=\{f(x,x)\to a,\ c\to h(c,g(c)),\ h(x,g(x))\to f(x,h(x,g(c)))\}$ (where x is a variable) given in Section 1 is depth-preserving, since for the first and the second rules, the right-hand-sides contain no variables, and for the third rule, the maximal depth of the x occurrences of the left-hand-side $h(x,g(x))$ is 2 and that of the right-hand-side $f(x,h(x,g(c)))$ is 2.

Definition of ⟨strongly depth-preserving TRS R⟩

A TRS R is *strongly depth-preserving* if R is depth-preserving and $\forall\alpha \to \beta \in R\ \forall x \in V(\alpha)\ \forall u,v \in O_x(\alpha)\quad |u| = |v|$.

Example. Let $R_4 = \{f(x,x) \to a,\ c \to g(c),\ g(x) \to f(x,x)\}$ and $R_5 = \{f(x,x,x) \to h(x,x,x,x,g(c)),\ c \to g(c)\}$ where x is a variable. Both R_4 and R_5 are strongly depth-preserving. (Note that both R_4 and R_5 are duplicating [6].)

In this section, we show that the TRS

$$R_2 = \{f(x,x) \to a,\ c \to h(c,g(c)),\ h(x,g(x)) \to f(x,h(x,g(c)))\}$$

given in Section 1 is non-E-overlapping and depth-preserving, but not CR.

Obviously, R_2 is non-E-overlapping, since there is no pair $(\alpha_1/u, \alpha_2)$ satisfying that the root (topmost) symbols of α_1/u and α_2 are the same for $\alpha_1 \to \beta_1,\ \alpha_2 \to \beta_2 \in R_2$ and $u \in \bar{O}(\alpha_1)$, except that $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $u = \varepsilon$. It has already been explained in the above that R_2 is depth-preserving.

We can show that TRS R_2 is not CR. Note that $c \to h(c,g(c)) \to f(c,h(c,g(c))) \to f(h(c,g(c)),h(c,g(c))) \to a$ and $c \to^* h(a,g(a))$.

Thus, $a \leftrightarrow\!\!\!\!+\!\!\!\to^* h(a,g(a))$ holds, but we can show that a and $h(a,g(a))$ are not joinable.

To prove this, we assume to the contrary that a and $h(a,g(a))$ are joinable. Since a is in normal forms (i.e., there exists no reduction from a), there must exist a sequence $h(a,g(a)) \to f(a,h(a,g(c))) \to^* a$. So, let $\gamma : h(a,g(a)) \to^* a$ be a sequence with the minimal number of reduction steps. Since there is only one reduction from $h(a,g(a))$, the first reduction of γ is $h(a,g(a)) \to f(a,h(a,g(c)))$, so that the remaining subsequence is $f(a,h(a,g(c))) \to^* a$. It follows that there exists a sequence $h(a,g(c)) \to^* a$ in this subsequence, since $f(a,h(a,g(c))) \to^* f(a,a) \to a$ must hold. If $h(a,g(c)) \to^* a$ is possible, then the third rule must be eventually used, i.e., $h(a,g(c)) \to^* h(a,g(a))$ and $h(a,g(a)) \to^* a$. This contradicts to the minimality of γ. Hence, $h(a,g(a))$ and a are not joinable.

Thus, R_2 is not CR. Note that R_2 is also non-duplicating, since for each rule the number of x occurrences of the left-hand-side is greater than or equal to that of the right-hand-side.

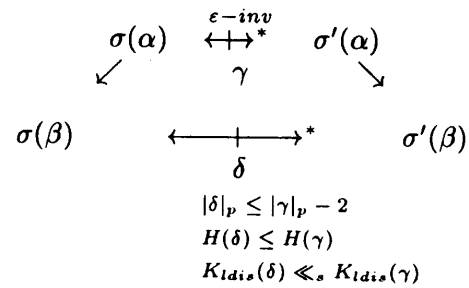


Fig. 2 Assertion $S(n)$.

In the following sections, we will prove that all the non-E-overlapping and strongly depth-preserving TRS's are CR. Henceforth, we are dealing with a fixed TRS R and assume that R is non-E-overlapping and strongly depth-preserving.

4. Assertions

We use the following six assertions $S(n)$, $S'(n)$, $P(k)$, $P'(k)$, $Q(k)$ and $Q'(k)$ (where $n \geq 2, k \geq 0$) to prove that non-E-overlapping and strongly depth-preserving TRS R is CR.

Assertions $S(n)$ and $S'(n)$ are similar to the Elimination lemma in Ref. 4). Assertion $Q(k)$ ensures that TRS R is CR.

Assertion $S(n)$

Let $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\alpha) \to \sigma'(\beta)$ for some rule $\alpha \to \beta \in R$ and mappings σ, σ' where $|\gamma|_p = n$ and the subsequence $\bar{\gamma} : \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\alpha)$ is ε-invariant.

Then $\exists\delta : \sigma(\beta) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\beta)$ such that the following conditions (i)–(iii) hold:

(i) $|\delta|_p \leq n - 2$

(ii) If β is a variable, then $H(\delta) < H(\gamma)$. Otherwise, δ is ε-invariant and $H(\delta) \leq H(\gamma)$.

(iii) $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$.

(See **Fig. 2**.)

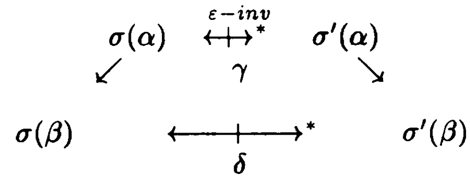
Assertion $S'(n)$

Let $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\alpha) \to \sigma'(\beta)$ for some rule $\alpha \to \beta \in R$ and mappings σ, σ' where $|\gamma|_p = n$ and the subsequence $\bar{\gamma} : \sigma(\alpha) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\alpha)$ is ε-invariant.

Then $\exists\delta : \sigma(\beta) \leftrightarrow\!\!\!\!+\!\!\!\to^* \sigma'(\beta)$ such that the following conditions (i)–(iii) hold:

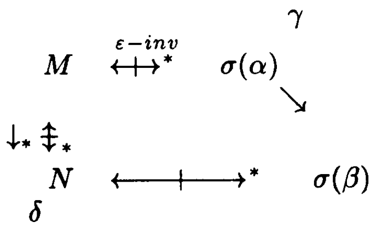
(i) $|\delta|_p = |\gamma|_p, |\delta|_{np} \leq |\gamma|_{np} - 2$

(ii) If β is a variable, then $H(\delta) < H(\gamma)$. Otherwise, δ is ε-invariant and $H(\delta) \leq H(\gamma)$.



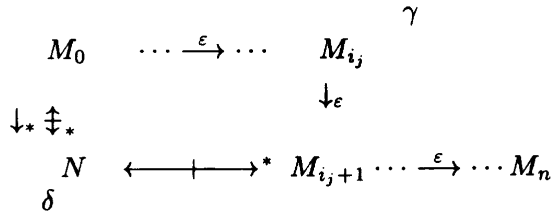
$|\delta|_p = |\gamma|_p$, $|\delta|_{np} \leq |\gamma|_{np} - 2$
$H(\delta) \leq H(\gamma)$
$K_{ldis}(\delta) \lneqq_s K_{ldis}(\gamma)$, $K_{right}(\delta) \lneqq_s K_{right}(\gamma)$

Fig. 3 Assertion $S'(n)$.



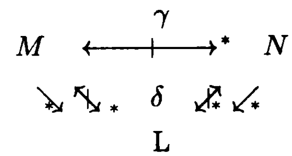
$H(\delta) \leq H(\gamma)$

Fig. 4 Assertion $P(k)$.



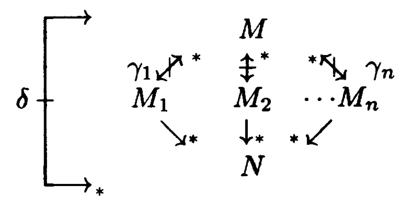
$H(\delta) \leq H(\gamma[0, i_j + 1])$

Fig. 5 Assertion $P'(k)$.



$H(\delta) \leq k$

Fig. 6 Assertion $Q(k)$.



$H(\delta) \leq k$

Fig. 7 Assertion $Q'(k)$.

(iii) $K_{ldis}(\delta) \lneqq_s K_{ldis}(\gamma)$ and $K_{right}(\delta) \lneqq_s K_{right}(\gamma)$.

(See **Fig. 3**.)

Note that γ satisfies the same condition in $S(n)$ and $S'(n)$.

Assertion $P(k)$

Let $\gamma : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* \sigma(\alpha) \rightarrow \sigma(\beta)$ for some rule $\alpha \rightarrow \beta \in R$ and mapping σ where $H(\gamma) \leq k$ and the subsequence $\bar{\gamma} : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* \sigma(\alpha)$ is ε-invariant.

Then, there exists $\delta : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* N \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* \sigma(\beta)$ for some N such that the following conditions (i)–(iii) hold:

(i) $H(\delta) \leq H(\gamma)$
(ii) $M \rightarrow^* N$
(iii) for the subsequence $\delta' : N \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* \sigma(\beta)$ of δ, either $H(\delta') < H(\gamma)$ or δ' is ε-invariant.

(See **Fig. 4**.)

Assertion $P'(k)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\!+\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\!+\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\!+\!\!\!\rightarrow M_n$ where $H(\gamma) \leq k$, the number of ε-reductions in γ is l (> 0) and each ε-reduction is $M_i \xrightarrow{\varepsilon} M_{i+1}$ for some i $(0 \leq i < n)$. Let $M_{i_1} \xrightarrow{\varepsilon} M_{i_1+1}, \cdots, M_{i_l} \xrightarrow{\varepsilon} M_{i_l+1}$ be the ε-reductions of γ, $0 \leq i_1 < i_2 \cdots < i_l < n$. Then, there exist i_j $(1 \leq j \leq l)$ and $\delta : M_0 \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* N \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* M_{i_j+1}$ for some N such that the following conditions (i)–(iii) hold:

(i) $H(\delta) \leq H(\gamma[0, i_j + 1])$
(ii) $M_0 \rightarrow^* N$
(iii) for the subsequence $\delta' : N \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* M_{i_j+1}$ of δ, either $H(\delta') < H(\gamma[0, i_j + 1])$ holds or $i_j = i_l$ and δ' is ε-invariant.

(See **Fig. 5**.)

Assertion $Q(k)$

Let $\gamma : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* N$ where $H(\gamma) \leq k$.

Then, $\exists \delta : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* L \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* N$ for some L such that $H(\delta) \leq k$, $M \rightarrow^* L$ and $N \rightarrow^* L$ (see **Fig. 6**).

Assertion $Q'(k)$

Let $\gamma_i : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* M_i$, where $H(\gamma_i) \leq k$, $1 \leq i \leq n$ and $n \geq 2$.

Then, $\exists \delta : M \leftrightarrow\!\!\!\!+\!\!\!\rightarrow^* N$ for some N such that $H(\delta) \leq k$ and $\forall i$ $(1 \leq i \leq n)$ $M_i \rightarrow^* N$ (see **Fig. 7**).

To prove these assertions, we use the following properties of *left*, *right*, *width*.

Property 1

Let $\gamma : M_0 \leftrightarrow\!\!\!\!+\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\!+\!\!\!\rightarrow \cdots \leftrightarrow\!\!\!\!+\!\!\!\rightarrow M_n$.

1. $\forall i$ $(0 \leq i \leq n)$ $\exists j$ $(0 \leq j \leq i)$ such that $H(M_j) \geq H(M_i)$, $left(\gamma, H(M_j)) \downarrow$ and $left(\gamma, H(M_j)) = j$.
2. $\forall i$ $(0 \leq i \leq n)$ $\exists j$ $(i \leq j \leq n)$ such that $H(M_j) \geq H(M_i)$, $right(\gamma, H(M_j)) \downarrow$ and $right(\gamma, H(M_j)) = j$.

Property 2

Let γ and δ be parallel reduction sequences. Let $Y \in \{ldis, right, width\}$.

$K_Y(\delta) \lesseqqgtr_s K_Y(\gamma)$ iff the following condition (p.2) holds:

(p.2) $\forall(h,l) \in K_Y(\delta) \quad \exists(h',l') \in K_Y(\gamma)$ $(h,l) \leq_s (h',l')$.

Property 3

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_n$. Let $u \in MR(\gamma)$ and $\bar{\gamma} = \gamma[i,j]/u$ where $0 \leq i < j \leq n$. Let $\delta : L_i \leftrightarrow\!\!\!\rightarrow L_{i+1} \cdots \leftrightarrow\!\!\!\rightarrow L_j$ where $L_i = M_i/u, L_j = M_j/u$, $|\delta|_p = |\bar{\gamma}|_p$ and $H(\delta) \leq H(\bar{\gamma})$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$.

1 If $K_{ldis}(\delta) \lesseqqgtr_s K_{ldis}(\bar{\gamma})$, then $K_{ldis}(\gamma') \lesseqqgtr_s K_{ldis}(\gamma)$.

2 If $K_{right}(\delta) \lesseqqgtr_s K_{right}(\bar{\gamma})$, then $K_{right}(\gamma') \lesseqqgtr_s K_{right}(\gamma)$.

3 If $K_{ldis}(\delta) \lesseqqgtr_s K_{ldis}(\bar{\gamma})$ and $K_{right}(\delta) \lesseqqgtr_s K_{right}(\bar{\gamma})$, then $K_{width}(\gamma') \lesseqqgtr_s K_{width}(\gamma)$.

Property 3'

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_n$ and $\bar{\gamma} = \gamma[i,j]$ where $0 \leq i < j \leq n$. Let $\delta : L_i \leftrightarrow\!\!\!\rightarrow L_{i+1} \cdots \leftrightarrow\!\!\!\rightarrow L_{j'}$ where $L_i = M_i, L_{j'} = M_j$, $|\delta|_p < |\bar{\gamma}|_p$ and $H(\delta) \leq H(\bar{\gamma})$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$.

1 If $K_{ldis}(\delta) \lesseqqgtr_s K_{ldis}(\bar{\gamma})$, then $K_{ldis}(\gamma') \ll_s K_{ldis}(\gamma)$.

2 If $K_{right}(\delta) \lesseqqgtr_s K_{right}(\bar{\gamma})$, then $K_{right}(\gamma') \ll_s K_{right}(\gamma)$.

Property 4

Let γ be a parallel reduction sequence. Then, $K_{ldis}(net(\gamma)) \lesseqqgtr_s K_{ldis}(\gamma)$ and $K_{right}(net(\gamma)) \lesseqqgtr_s K_{right}(\gamma)$.

Property 5

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_n$ and $\bar{\gamma} = \gamma[0,i]$ where $0 \leq i \leq n$. Let $\delta : L_0 \leftrightarrow\!\!\!\rightarrow L_1 \cdots \leftrightarrow\!\!\!\rightarrow L_j$ where $0 \leq j$, $L_j = M_i$ and $H(\delta) < H(\bar{\gamma})$. Let $\gamma' = \gamma[\delta/\bar{\gamma}]$. Then, $K_{ldis}(\gamma') \ll_w K_{ldis}(\gamma)$ and $K_{width}(\gamma') \ll_w K_{width}(\gamma)$.

For the proofs of these properties, see Appendix.

5. Proof of Assertions

We are now ready to prove these assertions. We first prove $S(n) \bigwedge S'(n)$ by induction on $n \geq 2$. Then we will prove that $P(k) \Rightarrow P'(k)$ and $Q(k) \Rightarrow Q'(k)$. Using these results, we will finally prove $P(k) \bigwedge Q(k)$ by induction on $k \geq 0$.

Proof of $S(n) \bigwedge S'(n)$

Basis: the case of $n = 2$.

Let $\gamma : \sigma(\beta) \leftarrow \sigma(\alpha) \rightarrow \sigma(\beta)$. Then obviously $\delta : \sigma(\beta) = \sigma(\beta)$ satisfies the required conditions of $S(n)$ and $\delta' : \sigma(\beta) \leftrightarrow\!\!\!\rightarrow \sigma(\beta) \leftrightarrow\!\!\!\rightarrow \sigma(\beta)$ satisfies the conditions of $S'(n)$.

Induction step: the case of $n > 2$.

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \leftrightarrow\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\rightarrow M_{n-1} \leftrightarrow\!\!\!\rightarrow M_n$ where $M_0 = \sigma(\beta), M_1 = \sigma(\alpha), M_{n-1} = \sigma'(\alpha)$ and $M_n = \sigma'(\beta)$. For the subsequence $\bar{\gamma} = \gamma[1, n-1] : \sigma(\alpha) = M_1 \leftrightarrow\!\!\!\rightarrow^* \sigma'(\alpha) = M_{n-1}$, let $\Gamma_\gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i$ for some $u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$. We prove $S(n) \bigwedge S'(n)$ by induction on $weight(\Gamma_\gamma)$ which is defined as follows:

$$weight(\Gamma_\gamma) = \{(H(\gamma_i), |\gamma_i|_{np}) \mid \gamma_i \in \Gamma_\gamma\}_m$$

where we use $\ll_s$ as the ordering of $weight(\Gamma_\gamma)$'s.

Basis: the case of $weight(\Gamma_\gamma) = \phi$, i.e., $\bar{\gamma}$ is α-keeping.

For each $x \in V(\beta)$, we choose a redex occurrence $u_x \in O_x(\alpha)$. Let σ_j, $1 \leq j \leq n-1$, be mappings satisfying that $\sigma_j(x) = M_j/u_x$ for all $x \in V(\beta)$. Let $N_j = \sigma_j(\beta)$ where $1 \leq j \leq n-1$. Then, we have $\delta : \sigma(\beta) = N_1 \leftrightarrow\!\!\!\rightarrow N_2 \cdots \leftrightarrow\!\!\!\rightarrow N_{n-1} = \sigma'(\beta)$.

We first prove that δ satisfies the required conditions of $S(n)$. (i) holds by $|\delta|_p = n-2$. By the depth-preserving property,

$$H(N_j) \leq \max(H(M_j), H(\beta)) \qquad (*\text{s1})$$

holds for $1 \leq j \leq n-1$. Hence

$$H(\delta) \leq H(\gamma). \qquad (*\text{s2})$$

If β is a variable, then obviously $H(\delta) < H(\gamma)$. Otherwise, δ is ε-invariant. Thus, (ii) holds.

It remains to prove (iii) $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$.

We first note that if $left(\delta, h) \downarrow$, then $H(\sigma\ (\beta)) \leq h \leq H(\delta)$ holds. And $left(\delta, H(\sigma\ (\beta))) = 0 = left(\gamma, H(\sigma(\ \beta)))$, so that $ldis(\delta, H(\sigma(\beta))) = n-2 < ldis(\gamma, H(\sigma(\beta))) = n$ holds by (i). Thus, by Property 2, we only need to prove that for any h where $H(\sigma(\beta)) < h \leq H(\delta)$ if $left(\delta, h)\downarrow$, then $\exists h' \geq h$ such that $ldis(\delta, h) \leq ldis(\gamma, h')$.

To prove this, let $left(\delta, h) = i > 0$ for $h > H(\sigma(\beta))$. (Note that $H(N_{i+1}) = h$.) Then, $ldis(\delta, h) = n-2-i$. By (*s1) and $h > H(\sigma(\beta))$, $H(N_{i+1}) = h \leq H(M_{i+1})$ holds, so that $\exists h' \geq h$ $left(\gamma, h') \downarrow$ and $left(\gamma, h') \leq i+1$ by Property 1.1. Hence, $ldis(\gamma, h') \geq n-(i+1) > (n-2) - i = ldis(\delta, h)$, as required.

Thus, (iii) holds. Hence, $S(n)$ holds.

We next prove $S'(n)$. Let $\delta' = (N_1 \leftrightarrow\!\!\!\rightarrow N_1)$; δ; $(N_{n-1} \leftrightarrow\!\!\!\rightarrow N_{n-1}) : N_1 \leftrightarrow\!\!\!\rightarrow N_1 \leftrightarrow\!\!\!\rightarrow N_2 \cdots \leftrightarrow\!\!\!\rightarrow N_{n-1} \leftrightarrow\!\!\!\rightarrow N_{n-1}$. Then, (i) holds by $|\delta'|_p = n$ and $|\delta'|_{np} \leq |\gamma|_{np} - 2$. Obviously $H(\delta') \leq H(\gamma)$ by (*s2). If β is a variable, then obviously $H(\delta') < H(\gamma)$. Otherwise, δ' is ε-invariant. Thus, (ii) holds. It remains to prove (iii) $K_{ldis}(\delta') \lesseqqgtr_s K_{ldis}(\gamma)$ and

$K_{right}(\delta') \lessgtr_s K_{right}(\gamma)$.

We first prove $K_{ldis}(\delta') \leqq_s K_{ldis}(\gamma)$.

The proof is similar to that of $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$. That is we can easily prove that $ldis(\delta', H(\sigma(\beta))) = ldis(\gamma, H(\sigma(\beta)))$. And for any $h > H(\sigma(\beta))$ if $left(\delta', h) \downarrow$, then there exists $h' \geq h$ such that $ldis(\delta', h) \leq ldis(\gamma, h')$, since if $left(\delta', h) = i > 0$, then $H(N_i) = h \leq H(M_i)$ holds by (*s1) and $h > H(\sigma(\beta))$, so that $\exists h' \geq h$ such that $ldis(\gamma, h') \geq n - i = ldis(\delta', h)$ by Property 1.1. Hence, $K_{ldis}(\delta') \leqq_s K_{ldis}(\gamma)$ holds by Property 2.

We next prove $K_{right}(\delta') \leqq_s K_{right}(\gamma)$.

Note that if $right(\delta', h) \downarrow$, then $H(\sigma'(\beta)) \leq h \leq H(\delta')$ holds. And $right(\delta', H(\sigma'(\beta))) = n = right(\gamma, H(\sigma'(\beta)))$. Thus, by Property 2 we only need to prove that for any h where $H(\sigma'(\beta)) < h \leq H(\delta')$, if $right(\delta', h) \downarrow$, then $\exists h' \geq h$ such that $right(\gamma, h') \geq right(\delta', h)$. This proof is straightforward , since if $right(\delta', h) = i < n$, then $i > 0$ and $H(N_i) = h \leq H(M_i)$ holds by (*s1), so that $\exists h' \geq h$ such that $right(\gamma, h') \geq i = right(\delta', h)$ by Property 1.2. Thus, (iii) holds.

Hence, $S'(n)$ holds.

Induction step: the case of $weight(\Gamma_\gamma) \gg_s \phi$.

Let $\gamma_1 = \bar{\gamma}/u_1 : L_1 \leftrightarrow\!\!\!\!\to \cdots \leftrightarrow\!\!\!\!\to L_{n-1} \in \Gamma_\gamma$ where $u_1 \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)$ and $\bar{\gamma} = \gamma[1, n-1] : M_1 \leftrightarrow\!\!\!\!\to M_2 \leftrightarrow\!\!\!\!\to \cdots \leftrightarrow\!\!\!\!\to M_{n-1}$. (Note that $L_i = M_i/u_1, 1 \leq i \leq n-1$.)

By the definition of γ_1, there exists an ε-reduction in γ_1. Since TRS R is non-E-overlapping, there exists a peak subsequence $\delta_1 = \gamma_1[i-1, j] : L_i \xleftarrow{\varepsilon} L_{i+1} (= \sigma''(\alpha')) \leftrightarrow\!\!\!\!\to^* L_j (= \sigma'''(\alpha')) \xrightarrow{\varepsilon} L_{j+1}$ for some i, j where $1 \leq i < j < n-1, \alpha' \to \beta' \in R$ and $\sigma'', \sigma''' : X \to T$. By the induction hypothesis $S'(n')$ for $n' < n$, $\exists \eta_1 : L_i \leftrightarrow\!\!\!\!\to^* L_{j+1}$ such that

$$|\eta_1|_p = |\delta_1|_p, |\eta_1|_{np} \leq |\delta_1|_{np} - 2 \text{ and } H(\eta_1) \leq H(\delta_1) \quad (*s3)$$

$$K_{ldis}(\eta_1) \leqq_s K_{ldis}(\delta_1) \quad (*s4)$$

$$K_{right}(\eta_1) \leqq_s K_{right}(\delta_1) \quad (*s5)$$

Let $\gamma' = \gamma[\eta_1/\delta_1]$. Then, by (*s3), we have

$$|\gamma'|_p = |\gamma|_p = n, |\gamma'|_{np} \leq |\gamma|_{np}, \text{ and } H(\gamma') \leq H(\gamma) \quad (*s6)$$

and, by (*s3), (*s4), (*s5) and Property 3, we have

$$K_{ldis}(\gamma') \leqq_s K_{ldis}(\gamma) \quad (*s7)$$

$$K_{right}(\gamma') \leqq_s K_{right}(\gamma) \quad (*s8)$$

Obviously $weight(\Gamma_{\gamma'}) \ll_s weight(\Gamma_\gamma)$ holds since for $\gamma_1' = \gamma_1[\eta_1/\delta_1]$, $|\gamma_1'|_{np} < |\gamma_1|_{np}$ and $H(\gamma_1') \leq H(\gamma_1)$ hold by (*s3). Thus, by the induction hypothesis, $S(n)$ holds for γ', so that $\exists \delta' : \sigma(\beta) \leftrightarrow\!\!\!\!\to^* \sigma'(\beta)$ such that

(i) $|\delta'|_p \leq n - 2$

(ii) If β is a variable, then $H(\delta') < H(\gamma')$. Otherwise, δ' is ε-invariant and $H(\delta') \leq H(\gamma')$.

(iii) $K_{ldis}(\delta') \ll_s K_{ldis}(\gamma')$.

By (ii) and (*s6), if β is a variable, then $H(\delta') < H(\gamma)$. Otherwise, δ' is ε-invariant and $H(\delta') \leq H(\gamma)$. By (iii) and (*s7), $K_{ldis}(\delta') \ll_s K_{ldis}(\gamma') \leqq_s K_{ldis}(\gamma)$. Thus, $S(n)$ holds.

Similarly, by the induction hypothesis, $S'(n)$ holds for γ', so that $\exists \delta' : \sigma(\beta) \leftrightarrow\!\!\!\!\to^* \sigma'(\beta)$ such that

(i) $|\delta'|_p = |\gamma'|_p, |\delta'|_{np} \leq |\gamma'|_{np} - 2$

(ii) If β is a variable, then $H(\delta') < H(\gamma')$. Otherwise, δ' is ε-invariant and $H(\delta') \leq H(\gamma')$.

(iii) $K_{ldis}(\delta') \leqq_s K_{ldis}(\gamma')$ and $K_{right}(\delta') \leqq_s K_{right}(\gamma')$.

By (i), (ii) and (*s6), $|\delta'|_p = |\gamma|_p$ and $|\delta'|_{np} \leq |\gamma|_{np} - 2$ hold. And if β is a variable, then $H(\delta') < H(\gamma)$. Otherwise,δ' is ε-invariant and $H(\delta') \leq H(\gamma)$. By (iii), (*s7) and (*s8), $K_{ldis}(\delta') \leqq_s K_{ldis}(\gamma)$ and $K_{right}(\delta') \leqq_s K_{right}(\gamma)$. Thus, $S'(n)$ holds. □

Proof of $P(k) \Rightarrow P'(k)$

We prove $P'(k)$ by induction on the number $l \geq 1$ of ε-reductions appearing in γ where $\gamma : M_0 \leftrightarrow\!\!\!\!\to^* M_{i_1} \xrightarrow{\varepsilon} M_{i_1+1} \leftrightarrow\!\!\!\!\to^* M_{i_2} \xrightarrow{\varepsilon} M_{i_2+1} \cdots \leftrightarrow\!\!\!\!\to^* M_{i_l} \xrightarrow{\varepsilon} M_{i_l+1} \leftrightarrow\!\!\!\!\to^* M_n$ and $H(\gamma) \leq k$.

Basis: Obvious.

Induction step: the case of $l > 1$.

We first apply $P(k)$ to $\gamma[0, i_1+1]$, so that there exists $\delta : M_0 \leftrightarrow\!\!\!\!\to^* N \leftrightarrow\!\!\!\!\to^* M_{i_1+1}$ for some N such that

(i) $H(\delta) \leq H(\gamma[0, i_1+1]) \leq k$ (*p'1)

(ii) $M_0 \to^* N$ (*p'2)

(iii) for the subsequence $\delta' : N \leftrightarrow\!\!\!\!\to^* M_{i_1+1}$ of δ, either $H(\delta') < H(\gamma[0, i_1+1])$ or δ' is ε-invariant.

If $H(\delta') < H(\gamma[0, i_1+1])$ holds, then let $j = 1$, i.e., $i_j = i_1$, so that the above conditions (i)–(iii) ensure $P'(k)$.

The case in which δ' is ε-invariant remains. Let $\gamma' = (\delta'; \gamma\ [i_1+1, n]) : N \leftrightarrow\!\!\!\!\to^* M_{i_1+1} \leftrightarrow\!\!\!\!\to^* M_n$. Note that $H(\gamma') \leq k$ by (*p'1). Since δ' is ε-invariant, γ' contains $(l-1)$ ε-reductions: $M_{i_2} \xrightarrow{\varepsilon} M_{i_2+1}, \cdots, M_{i_l} \xrightarrow{\varepsilon}$

M_{i_l+1}, so that the induction hypothesis ensures that there exist i_j $(2 \leq j \leq l)$ and $\eta : N \leftrightarrow\!\!\!\rightarrow^* N' \leftrightarrow\!\!\!\rightarrow^* M_{i_j+1}$ for some N' such that the following conditions (i)–(iii) hold:

(i) $H(\eta) \leq H(\gamma'')$ where $\gamma'' = (\delta'; \gamma[i_1+1, i_j+1]) : N \leftrightarrow\!\!\!\rightarrow^* M_{i_j+1}$ (*p'3)

(ii) $N \rightarrow^* N'$ (*p'4)

(iii) for the subsequen $\eta' : N' \leftrightarrow\!\!\!\rightarrow^* M_{i_j+1}$ of η, either $H(\eta') < H(\gamma'')$ holds or $i_j = i_l$ and η' is ε-invariant. (*p'5)

Let $\xi = (\delta_0; \eta) : M_0 \leftrightarrow\!\!\!\rightarrow^* N \leftrightarrow\!\!\!\rightarrow^* N' \leftrightarrow\!\!\!\rightarrow^* M_{i_j+1}$ where $\delta_0 : M_0 \leftrightarrow\!\!\!\rightarrow^* N$ is the subsequence of δ such that $\delta = (\delta_0; \delta')$. Then, we can show that ξ satisfies the following required conditions:

(i) $H(\xi) \leq H(\gamma[0, i_j+1])$ (*p'6)

(ii) $M_0 \rightarrow^* N'$ (*p'7)

(iii) for the subsequence $\eta' : N' \leftrightarrow\!\!\!\rightarrow^* M_{i_j+1}$ of ξ, either $H(\eta') < H(\gamma[0, i_j+1])$ holds or $i_j = i_l$ and η' is ε-invariant. (*p'8)

Note that $H(\delta_0) \leq H(\delta) \leq H(\gamma[0, i_1+1])$, and $H(\eta) \leq H(\gamma'') = \max(H(\delta'), H(\gamma[i_1+1, i_j+1])) \leq H(\gamma[0, i_j+1])$ hold by (*p'1) and (*p'3), so that we have $H(\xi) \leq H(\gamma[0, i_j+1])$ by $\xi = (\delta_0, \eta)$. Thus, (*p'6) holds. By (*p'2) and (*p'4), we have $M_0 \rightarrow^* N'$, so that (*p'7) holds. And (*p'8) holds by (*p'5) and $H(\gamma'') \leq H(\gamma[0, i_j+1])$.

Hence, $P'(k)$ holds. □

Proof of $Q(k) \Rightarrow Q'(k)$

Let $\gamma_i : M \leftrightarrow\!\!\!\rightarrow^* M_i$ where $H(\gamma_i) \leq k, 1 \leq i \leq n$. We prove that $\exists \delta : M \leftrightarrow\!\!\!\rightarrow^* N$ for some N such that $H(\delta) \leq k$ and $\forall i \ (1 \leq i \leq n) \ M_i \rightarrow^* N$ by induction on $n \geq 2$.

Basis: the case of $n = 2$.

By $\gamma_1 : M \leftrightarrow\!\!\!\rightarrow^* M_1$ and $\gamma_2 : M \leftrightarrow\!\!\!\rightarrow^* M_2$ where $H(\gamma_1) \leq k, H(\gamma_2) \leq k$, there exists $\gamma_1' : M_1 \leftrightarrow\!\!\!\rightarrow^* M_2$ such that $H(\gamma_1') \leq k$. By $Q(k)$, then there exists $\delta_2 : M_1 \leftrightarrow\!\!\!\rightarrow^* N_2 \leftrightarrow\!\!\!\rightarrow^* M_2$ for some N_2 such that $H(\delta_2) \leq k$ and $M_1 \rightarrow^* N_2 \bigwedge M_2 \rightarrow^* N_2$. By concatenating γ_1 and the subsequence $M_1 \leftrightarrow\!\!\!\rightarrow^* N_2$ of δ_2, we have $\delta_2' : M \leftrightarrow\!\!\!\rightarrow^* M_1 \leftrightarrow\!\!\!\rightarrow^* N_2$. Then, $H(\delta_2') \leq k$. Thus, δ_2' satisfies the required condition.

Induction step: the case of $n > 2$.

By the induction hypothesis, $Q'(k)$ holds for $\gamma_i : M \leftrightarrow\!\!\!\rightarrow^* M_i$ where $1 \leq i \leq n-1$, so that there exists

$\delta : M \leftrightarrow\!\!\!\rightarrow^* N_{n-1}$ for some N_{n-1} such that $H(\delta) \leq k \wedge \forall i \ (1 \leq i \leq n-1) \ M_i \rightarrow^* N_{n-1}$. (*q'1)

By applying $Q(k)$ to $(\delta^R; \gamma_n) : N_{n-1} \leftrightarrow\!\!\!\rightarrow^* M \leftrightarrow\!\!\!\rightarrow^* M_n$ where δ^R is the reverse of δ, there exists

$\delta' : N_{n-1} \leftrightarrow\!\!\!\rightarrow^* L \leftrightarrow\!\!\!\rightarrow^* M_n$ for some L such that $H(\delta') \leq k$, $N_{n-1} \rightarrow^* L$ and $M_n \rightarrow^* L$ (*q'2)

Let $\xi = (\delta; \delta'') : M \leftrightarrow\!\!\!\rightarrow^* N_{n-1} \leftrightarrow\!\!\!\rightarrow^* L$ for the subsequence $\delta'' : N_{n-1} \leftrightarrow\!\!\!\rightarrow^* L$ of δ'. Then, $H(\xi) \leq k$ and $M_i \rightarrow^* L$ for all i $(1 \leq i \leq n)$ by (*q'1) and (*q'2). Hence, $Q'(k)$ holds. □

Proof of $P(k) \bigwedge Q(k)$

We prove $P(k) \bigwedge Q(k)$ by induction on $k \geq 0$. We first prove $P(k)$, then $Q(k)$.

Basis: the case of k=0.

Proof of $P(0)$

In this case, for $\gamma : M \leftrightarrow\!\!\!\rightarrow^* \sigma(\alpha) \rightarrow \sigma(\beta)$, note that $H(\gamma) = 0$ implies that β is not a variable, since $\alpha \in F$ (i.e., α is a function symbol) and $\sigma(\alpha) = M$ since $\bar{\gamma} : M \leftrightarrow\!\!\!\rightarrow^* \sigma(\alpha)$ is ε-invariant. Let $N = \sigma(\beta)$ and $\delta : \sigma(\alpha) = M \rightarrow \sigma(\beta) = N$. Then, $H(\delta) = 0, M \rightarrow^* N$ and $\delta' : \sigma(\beta) = N$ is ε-invariant, as claimed.

Proof of $Q(0)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_n$ where $n \geq 0, H(\gamma) = 0, M_0 = M$ and $M_n = N$. We prove $Q(0)$ by induction on $n \geq 0$. In the case in which $n = 0$ or $n = 1$, the proof is obvious. Note that $M_0 \leftrightarrow\!\!\!\rightarrow M_1$ implies that $M_0 \xrightarrow{\varepsilon} M_1$ or $M_1 \xrightarrow{\varepsilon} M_0$ holds by $H(\gamma) = 0$. Consider the case of $n > 1$. If $M_0 \rightarrow M_1$ holds, then by applying the induction hypothesis to $\gamma' : M_1 \leftrightarrow\!\!\!\rightarrow M_2 \cdots \leftrightarrow\!\!\!\rightarrow M_n$, $Q(0)$ holds for γ. The case of $M_1 \xrightarrow{\varepsilon} M_0$ remains. In this case, M_1 is a function symbol and $M_1 \rightarrow M_0$ is a rule in R by $H(\gamma) = 0$. Similarly, for $M_i \leftrightarrow\!\!\!\rightarrow M_{i+1}$ where $1 \leq i < n$, $M_i \xrightarrow{\varepsilon} M_{i+1}$ or $M_{i+1} \xrightarrow{\varepsilon} M_i$ holds. If $M_{i+1} \xrightarrow{\varepsilon} M_i$ for all i $(1 \leq i < n)$, then $\gamma : M_0 \leftarrow M_1 \cdots \leftarrow M_n$, so that obviously $Q(0)$ holds. Otherwise, i.e., if $M_i \xrightarrow{\varepsilon} M_{i+1}$ for some i $(1 \leq i < n)$, then $M_{i-1} \xleftarrow{\varepsilon} M_i \xrightarrow{\varepsilon} M_{i+1}$ for some i $(1 \leq i < n)$. Since TRS R is non-E-overlapping and M_i is a function symbol, $M_{i-1} = M_{i+1}$ holds. Hence, by applying the induction hypothesis to $\gamma' : M_0 \leftarrow M_1 \cdots M_{i-1} \leftrightarrow\!\!\!\rightarrow M_{i+2} \cdots \leftrightarrow\!\!\!\rightarrow M_n$ where $|\gamma'|_p = n-2$, $Q(0)$ holds.

Induction step: the case of $k > 0$.

Proof of $P(k)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_{n-1} \leftrightarrow\!\!\!\rightarrow M_n$

where $H(\gamma) = k$, $M_0 = M$, $M_{n-1} = \sigma(\alpha)$ and $M_n = \sigma(\beta)$.
Let $\bar{\gamma} = \gamma[0, n-1]$ and $\Gamma_\gamma = \{\gamma_i \mid \gamma_i = \bar{\gamma}/u_i$ for some $u_i \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)\}$.

We define the weight of $\bar{\gamma}$ as follows:

$$weight_2(\bar{\gamma}) = \bigsqcup_{\gamma_i \in \Gamma_\gamma} K_{ldis}(net(\gamma_i))$$

where $\bigsqcup$ denotes the union of multisets. We use $\ll_w$ as the ordering of $weight_2(\bar{\gamma})$'s.

Basis: the case of $weight_2(\bar{\gamma}) = \phi$, i.e., $\Gamma_\gamma = \phi$ (In this case, $\bar{\gamma}$ is α-keeping).

For any $x \in V(\alpha)$, let $O_x(\alpha) = \{u_{x_1}, u_{x_2}, \cdots, u_{x_{l_x}}\}$. Then, $|u_{x_1}| = |u_{x_2}| = \cdots = |u_{x_{l_x}}|$ holds by the strongly depth-preserving property.

Since the reductions of γ occur only in the variable parts $\sigma(x)$'s, $x \in V(\alpha)$, we have

$$\zeta_i : \sigma(x)(= M_{n-1}/u_{x_i}) \leftrightarrow\!\!\!\leftrightarrow M_{n-2}/u_{x_i} \cdots \leftrightarrow\!\!\!\leftrightarrow M_0/u_{x_i}(= M/u_{x_i})$$

for all $u_{x_i} \in O_x(\alpha), 1 \le i \le l_x$ and $x \in V(\alpha)$.
Note that $\zeta_i = (\bar{\gamma}/u_{x_i})^R$ and

$$H(\zeta_i) \le H(\gamma) - |u_{x_i}| = k - |u_{x_i}| < k.$$

Let $h_x = Max\{H(\zeta_i) \mid 1 \le i \le l_x\}$. Then,

$$h_x + |u_{x_i}| \le k \text{ and } h_x < k \qquad (*\text{p0})$$

Hence, we can apply Assertion $Q'(h_x)$ to $\zeta_1, \cdots, \zeta_{l_x}$ by the induction hypothesis. Thus, there exists $\xi_x : \sigma(x) \leftrightarrow\!\!\!\leftrightarrow^* N_x$ for some N_x such that

$$H(\xi_x) \le h_x \text{ and } \forall i\ (1 \le i \le l_x)\ M/u_{x_i} \to^* N_x \qquad (*\text{p1})$$

Let $\sigma'(x) = N_x$ for all $x \in V(\alpha)$. Then,

$$\xi_x : \sigma(x) \leftrightarrow\!\!\!\leftrightarrow^* \sigma'(x) \text{ and } H(\xi_x) + |u_{x_i}| \le k \qquad (*\text{p2})$$

Let $\delta_{x_i} = (\zeta_i^R; \xi_x) : M/u_{x_i} \leftrightarrow\!\!\!\leftrightarrow^* \sigma(x) \leftrightarrow\!\!\!\leftrightarrow^* \sigma'(x)$. Then, $H(\delta_{x_i}) \le h_x$ holds by (*p1). Since $M = \alpha[u_{x_i} \leftarrow M/u_{x_i} \mid u_{x_i} \in O_x(\alpha), x \in V(\alpha)]$, we have $\eta : M \leftrightarrow\!\!\!\leftrightarrow^* \sigma'(\alpha)$ where $\eta/u_{x_i} = \delta_{x_i}$ for all $u_{x_i} \in O_x(\alpha)$, $x \in V(\alpha)$.
Note that for any term M' occurring in η, $H(M') \le Max\{|u_{x_i}| + h_x \mid u_{x_i} \in O_x(\alpha), x \in V(\alpha)\}$ or $H(M') \le H(\alpha)$. By (*p0) and $H(\alpha) \le k$, we have $H(M') \le k$, so that we have

$$H(\eta) \le k. \qquad (*\text{p3})$$

(To obtain this (*p3), we need the strongly depth-preserving condition, which is used only to establish (*p3) in this paper.)

Since rule $\alpha \to \beta$ is depth-preserving, $H(\sigma'(\beta)) \le Max\{H(\beta), H(\sigma'(\alpha))\} \le k$. Hence, $\eta' = (\eta; \sigma'(\alpha) \leftrightarrow\!\!\!\leftrightarrow \sigma'(\beta)) : M \leftrightarrow\!\!\!\leftrightarrow^* \sigma'(\alpha) \leftrightarrow\!\!\!\leftrightarrow \sigma'(\beta)$ satisfies that

$$H(\eta') \le k \text{ and } M \to^* \sigma'(\beta) \qquad (*\text{p4})$$

by (*p1). (See **Fig. 8**.)

Using $\xi_x^R : \sigma'(x) \leftrightarrow\!\!\!\leftrightarrow^* \sigma(x)$, let

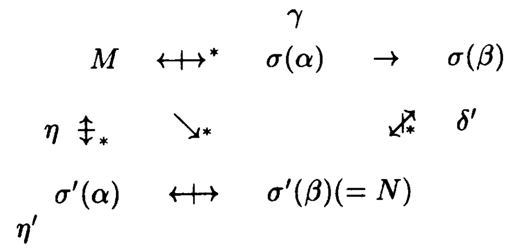


Fig. 8 Parallel reduction sequences in the proof of $P(k)$.

$\delta' : \sigma'(\beta) \leftrightarrow\!\!\!\leftrightarrow^* \sigma(\beta)$.

Then, the strongly depth-preserving property ensures that for all $x \in V(\beta)$ and $v_x \in O_x(\beta)$, $|v_x| \le |u_{x_i}|$ for $u_{x_i} \in O_x(\alpha)$, so that by (*p2) we have

$$H(\delta') \le k. \qquad (*\text{p5})$$

Now, let $N = \sigma'(\beta)$ and $\delta = (\eta'; \delta') : M \leftrightarrow\!\!\!\leftrightarrow^* \sigma'(\beta) = N \leftrightarrow\!\!\!\leftrightarrow^* \sigma(\beta)$. Then, we can prove that the conditions (i)–(iii) of $P(k)$ hold for δ. Since $H(\delta) \le k = H(\gamma)$ and $M \to^* N$ hold by (*p4) and (*p5), the conditions (i) and (ii) hold.

If β is a variable, then for $\delta' : \sigma'(\beta) \leftrightarrow\!\!\!\leftrightarrow^* \sigma(\beta)$ obviously $H(\delta') < k$ holds. Otherwise, δ' is ε-invariant. Thus, the condition (iii) holds. Hence, $P(k)$ holds.

Induction step: the case of $weight_2(\bar{\gamma}) \gg_w \phi$, i.e, $\bar{\gamma}$ is not α-keeping.

(Here, $\gamma : M_0(= M) \leftrightarrow\!\!\!\leftrightarrow M_1 \cdots \leftrightarrow\!\!\!\leftrightarrow M_{n-1}(= \sigma(\alpha)) \to M_n(= \sigma(\beta))$ and $\bar{\gamma} = \gamma[0, n-1]$.)

Let $\gamma_1 = \bar{\gamma}/u_1$ where $u_1 \in MR(\bar{\gamma}) \cap \bar{O}(\alpha)$. Let $\delta = net(\gamma_1) : L_0 \leftrightarrow\!\!\!\leftrightarrow L_1 \cdots \leftrightarrow\!\!\!\leftrightarrow L_m$ where $m \le n-1$, $L_0 = M_0/u_1$, $L_m = M_{n-1}/u_1$.
Note that there exists an ε-reduction in γ_1.

There are two cases depending on whether there exists $L_i \xleftarrow{\varepsilon} L_{i+1}$ for some i $(0 \le i < m)$ or not.

Case 1. The case in which $L_i \xleftarrow{\varepsilon} L_{i+1}$ for some i $(0 \le i < m)$.

In this case, the non-E-overlapping property ensures that there exists a peak subsequence $\xi = \delta[i, j+1] : L_i(= \sigma'(\beta')) \xleftarrow{\varepsilon} L_{i+1}(= \sigma'(\alpha')) \leftrightarrow\!\!\!\leftrightarrow^* L_j(= \sigma''(\alpha')) \xrightarrow{\varepsilon} L_{j+1}(= \sigma''(\beta'))$ of δ for some i, j where $0 \le i < j < m, \alpha' \to \beta' \in R$ and $\sigma', \sigma'' : X \to T$.

By $S(|\xi|_p)$, there exists $\xi' : L_i \leftrightarrow\!\!\!\leftrightarrow^* L_{j+1}$ satisfying the conditions:

(i) $|\xi'|_p \le |\xi|_p - 2$

(ii) If β' is a variable, then $H(\xi') < H(\xi)$. Otherwise, ξ' is ε-invariant and $H(\xi') \le H(\xi)$.

(iii) $K_{ldis}(\xi') \ll_s K_{ldis}(\xi)$.

Let $\delta' = \delta[\xi'/\xi]$. By the conditions (i) and (ii), we have

$$|\delta'|_p < |\delta|_p \quad (*p6)$$
$$H(\delta') \leq H(\delta) \quad (*p7)$$

By (i) and (iii), Property 3'.1 ensures that $K_{ldis}(\delta') \ll_s K_{ldis}(\delta)$. Let $\delta'' = net(\delta')$. Then, $K_{ldis}(\delta'') \leqq_s K_{ldis}(\delta')$ holds by Property 4. Hence, we have

$$K_{ldis}(\delta'') \ll_s K_{ldis}(\delta). \quad (*p8)$$

Let γ_1' be a sequence satisfying that $net(\gamma_1') = \delta''$ and $|\gamma_1'|_p = |\gamma_1|_p$. Let $\bar{\gamma}' = \bar{\gamma}[\gamma_1'/\gamma_1]$. Note that $net(\gamma_1) = \delta$. Thus, $weight_2(\bar{\gamma}') \ll_w weight_2(\bar{\gamma})$ holds by (*p8), and $H(\bar{\gamma}) \geq H(\bar{\gamma}')$ holds by (*p7). (Note that $A \ll_s B$ implies that $A \ll_w B$.) Hence, by the induction hypothesis, $P(k)$ holds for $\gamma' = (\bar{\gamma}'; M_{n-1} \leftrightarrow\!\!\!\leftrightarrow M_n)$, so that $P(k)$ holds for γ.

Case 2. The case in which there exist no reductions $L_i \stackrel{\varepsilon}{\longleftarrow} L_{i+1}$, $0 \leq i < m$.

In this case, if $L_i \leftrightarrow\!\!\!\leftrightarrow L_{i+1}$ is an ε-reduction, then $L_i \stackrel{\varepsilon}{\longrightarrow} L_{i+1}$ holds, where $0 \leq i < m$.

Assume that $\delta : L_0 \leftrightarrow\!\!\!\leftrightarrow^* L_m$ has l (> 0) ε-reductions, i.e., $L_{i_1} \stackrel{\varepsilon}{\longrightarrow} L_{i_1+1}, \cdots, L_{i_l} \stackrel{\varepsilon}{\longrightarrow} L_{i_l+1}$ are the ε-reductions where $0 \leq i_1 < \cdots < i_l < m$. By the induction hypothesis $P(k')$ and $P'(k'), k' < k$ (since $P(k') \Rightarrow P'(k')$), for $\delta : L_0 \leftrightarrow\!\!\!\leftrightarrow^* L_m$, there exists i_j ($1 \leq j \leq l$) and $\eta : L_0 \leftrightarrow\!\!\!\leftrightarrow^* N \leftrightarrow\!\!\!\leftrightarrow^* L_{i_j+1}$ for some N such that

(i) $H(\eta) \leq H(\delta[0, i_j + 1])$ (*p9)

(ii) $L_0 \rightarrow^* N$ (*p10)

(iii) for the subsequence $\eta' : N \leftrightarrow\!\!\!\leftrightarrow^* L_{i_j+1}$ of η, either $H(\eta') < H(\delta[0, i_j + 1])$ holds or $i_j = i_l$ and η' is ε-invariant. (*p11)

Using this η', let $\bar{\delta} = (\eta'; \delta[i_j + 1; m]) : N \leftrightarrow\!\!\!\leftrightarrow^* L_{i_j+1} \leftrightarrow\!\!\!\leftrightarrow^* L_m$. Then, by (*p9) and (*p11), we have

$$H(\bar{\delta}) \leq H(\delta), \quad (*p12)$$

either $\bar{\delta}$ is ε -invariant or

$$K_{ldis}(\bar{\delta}) \ll_w K_{ldis}(\delta), \quad (*p13)$$

since $H(\eta') < H(\delta[0, i_j + 1])$ implies that $K_{ldis}(\bar{\delta}) \ll_w K_{ldis}(\delta)$ by Property 5.

Let $\bar{\gamma}' = \bar{\gamma}[(\eta_0 : L_0 \leftrightarrow\!\!\!\leftrightarrow^* N; \bar{\delta})/\gamma_1]$ where $\eta = (\eta_0; \eta')$. Note that $net(\gamma_1) = \delta$. Let $M' = M[u_1 \leftarrow N]$. Then, $\bar{\gamma}' : M_0 \leftrightarrow\!\!\!\leftrightarrow^* M_{n-1}$ is decomposed into two subsequences $\bar{\gamma}_1 : M_0 \leftrightarrow\!\!\!\leftrightarrow^* M'$ and $\bar{\gamma}_2 : M' \leftrightarrow\!\!\!\leftrightarrow^* M_{n-1}$. Since $H(\bar{\gamma}_1) \leq H(\gamma) = k$ by (*p9) and $M_0 \rightarrow^* M'$ by (*p10), it is sufficient to show that $P(k)$ holds for $(\bar{\gamma}_2; (M_{n-1} \leftrightarrow\!\!\!\leftrightarrow M_n)) : M' \leftrightarrow\!\!\!\leftrightarrow^* M_n$, instead of the original γ.

We can show that $weight_2(\bar{\gamma}_2) \ll_w weight_2(\bar{\gamma})$. Since $weight_2(\bar{\gamma}_2)$ is obtained from $weight_2(\bar{\gamma})$ by replacing $K_{ldis}(net(\gamma_1)) = K_{ldis}(\delta)$ by $\bigsqcup_{v \in MR(\bar{\delta}) \text{where } u_1 v \in \bar{O}(\alpha)} K_{ldis}(net(\bar{\delta}/v))$, if $\bar{\delta}$ is ε-invariant, then obviously $weight_2(\bar{\gamma}_2) \ll_w weight_2(\bar{\gamma})$ holds by (*p12). Otherwise, $K_{ldis}(net(\bar{\delta})) \ll_w K_{ldis}(\delta)$ holds by (*p13) and Property 4, so that $weight_2(\bar{\gamma}_2) \ll_w weight_2(\bar{\gamma})$ holds. Hence, the induction hypothesis ensures that $P(k)$ holds for $(\bar{\gamma}_2; (M_{n-1} \leftrightarrow\!\!\!\leftrightarrow M_n))$. It follows that $P(k)$ holds for γ as already explained. □

Proof of $Q(k)$

Let $\gamma : M_0 \leftrightarrow\!\!\!\leftrightarrow M_1 \leftrightarrow\!\!\!\leftrightarrow M_2 \cdots \leftrightarrow\!\!\!\leftrightarrow M_n$ where $H(\gamma) \leq k$, $M_0 = M$ and $M_n = N$. We prove $Q(k)$ by induction on $weight_3(\gamma)$ which is defined as follows:

$$weight_3(\gamma) = (H(\gamma), K_{width}(\gamma), \varepsilon(\gamma))$$

where $\varepsilon(\gamma)$ is the number of ε-reductions in γ. We use the lexicographic ordering $<$ as the ordering of $weight_3(\gamma)$'s and $\ll_w$ as the ordering of $K_{width}(\gamma)$'s. If $H(\gamma) \leq k - 1$ holds or γ is ε-invariant, then the proof can reduce to that of $Q(k-1)$. So, assume that $H(\gamma) = k$ and γ has ε-reductions. There are two cases depending on whether or not there exists a peak.

Case 1. The case in which there exists a peak.

Let a peak be $\eta : M_i \stackrel{\varepsilon}{\longleftarrow} M_{i+1} \leftrightarrow\!\!\!\leftrightarrow^* M_j \stackrel{\varepsilon}{\longrightarrow} M_{j+1}$ such that $0 \leq i < j < n$ and $M_{i+1} \leftrightarrow\!\!\!\leftrightarrow^* M_j$ is ε-invariant.

There are two cases depending on whether (1-1) $i \leq left(\gamma, H(\eta)) \leq j + 1$ or $i \leq right(\gamma, H(\eta)) \leq j + 1$ or (1-2) not.

Case 1-1. The case in which $i \leq left(\gamma, H(\eta)) \leq j + 1$ or $i \leq right(\gamma, H(\eta)) \leq j + 1$.

By Assertion $S'(|\eta|_p)$, there exists $\xi : M_i \leftrightarrow\!\!\!\leftrightarrow^* M_{j+1}$ such that

(i) $|\xi|_p = |\eta|_p$, $|\xi|_{np} \leq |\eta|_{np} - 2$

(ii) either $H(\xi) < H(\eta)$ or $H(\xi) = H(\eta)$ and ξ is ε-invariant.

(iii) $K_{ldis}(\xi) \leqq_s K_{ldis}(\eta)$ and $K_{right}(\xi) \leqq_s K_{right}(\eta)$.

Let $\gamma' = \gamma[\xi/\eta]$. Then, $H(\gamma') \leq H(\gamma)$ holds by (ii). By (i)–(iii) and Property 3.3,

$$K_{width}(\gamma') \leqq_s K_{width}(\gamma) \quad (*q0)$$

holds. If $H(\gamma') < H(\gamma)$ or $K_{width}(\gamma') \ll_w K_{width}(\gamma)$, then the induction hypothesis for γ' ensures that $Q(k)$ holds for γ.

The case in which $H(\gamma') = H(\gamma)$ and $K_{width}(\gamma') = K_{width}(\gamma)$ remains. If ξ is ε-invariant, i.e., $\varepsilon(\xi) = 0$, then $\varepsilon(\gamma') < \varepsilon(\gamma)$ holds by $\varepsilon(\eta) = 2$, so that $weight_3(\gamma') < weight_3(\gamma)$. Hence, the induction hypothesis for γ' ensures that $Q(k)$ holds for γ.

The case of $H(\xi) < H(\eta)$ remains. Note that $width(\gamma, H(\eta)) \downarrow$ in this case (1-1). So, $width(\gamma', H(\eta)) \downarrow$ and $width(\gamma', H(\eta)) = width(\gamma, H(\eta))$ hold by $K_{width}(\gamma') = K_{width}(\gamma)$.

We first assume that $i \leq left(\gamma, H(\eta)) \leq j+1$ holds. Then, if $right(\gamma, H(\eta)) \downarrow$, then $right(\gamma, H(\eta)) > j+1$ since $width(\gamma', H(\eta)) \downarrow$ and $H(\xi) < H(\eta)$. Thus, $Min\{h'' \mid h'' \geq H(\eta) \bigwedge right(\gamma, h'') \downarrow\} = Min\{h'' \mid h'' \geq H(\eta) \bigwedge right(\gamma', h'') \downarrow\}$ $(= h'')$. Let $h' = Min\{h' \mid h' \geq H(\eta) \bigwedge left(\gamma', h') \downarrow\}$. Then, $left(\gamma', h') > j+1$ holds by $H(\xi) < H(\eta)$. Hence, $width(\gamma, H(\eta)) = right(\gamma, h'') - left(\gamma, H(\eta)) = right(\gamma', h'') - left(\gamma, H(\eta)) > right(\gamma', h'') - left(\gamma, h') = width(\gamma', H(\eta))$. This is a contradiction. Next, we assume that $i \leq right(\gamma, H(\eta)) \leq j+1$. By a similar argument, we have a contradiction. Hence, $H(\xi) = H(\eta)$ must hold. Thus, $Q(k)$ holds for γ.

Case 1-2. The case in which if $left(\gamma, H(\eta)) \downarrow$, then $left(\gamma, H(\eta)) < i$ and if $right(\gamma, H(\eta)) \downarrow$, then $right(\gamma, H(\eta)) > j+1$.

By Assertion $S(|\eta|_p)$, there exists $\xi : M_i \leftrightarrow\!\!\!\rightarrow^* M_{j+1}$ such that

(i) $|\xi|_p \leq |\eta|_p - 2$

(ii) either $H(\xi) < H(\eta)$ or $H(\xi) = H(\eta)$ and ξ is ε-invariant.

(iii) $K_{ldis}(\xi) \ll_s K_{ldis}(\eta)$

Let $\gamma' = \gamma[\xi/\eta]$. Then, $H(\gamma') \leq H(\gamma)$ holds by (ii).

Let $h_1 = Min\{h_1 \mid h_1 \geq H(\eta), left(\gamma, h_1) \downarrow\}$ and $h_2 = Min\{h_2 \mid h_2 \geq H(\eta), right(\gamma, h_1) \downarrow\}$. Then, $left(\gamma, h_1) < i$ and $right(\gamma, h_2) > j+1$ hold in this case (1-2). Let $h = Min(h_1, h_2)$. Then, $width(\gamma, h) = right(\gamma, h_2) - left(\gamma, h_1)$ and $width(\gamma', h) = right(\gamma', h_2) - left(\gamma', h_1)$. Since (i) holds, i.e., $|\xi|_p \leq |\eta|_p - 2$, obviously $width(\gamma, h) > width(\gamma', h)$. And, for any $h' > h$, if $width(\gamma, h) \downarrow$ or $width(\gamma', h) \downarrow$, then the both are defined and $width(\gamma, h) \geq width(\gamma', h)$ by $|\xi|_p \leq |\eta|_p - 2$. Hence, $K_{width}(\gamma') \ll_w K_{width}(\gamma)$ holds. Thus, $weight_3(\gamma') < weight_3(\gamma)$. The induction hypothesis for γ' ensures that $Q(k)$ holds for γ.

Case 2. The case in which there exist no peaks.

In this case, there exists l $(0 \leq l \leq n)$ such that for any ε-reduction $M_i \leftrightarrow\!\!\!\rightarrow M_{i+1}$ $(0 \leq i < n)$ either $i < l$ and $M_i \xrightarrow{\varepsilon} M_{i+1}$ or $i \geq l$ and $M_i \xleftarrow{\varepsilon} M_{i+1}$ holds.

So, let $\gamma_1 = \gamma[0, l] : M_0 \leftrightarrow\!\!\!\rightarrow M_1 \cdots \leftrightarrow\!\!\!\rightarrow M_l$ and $\gamma_2 = \gamma[l, n] : M_l \leftrightarrow\!\!\!\rightarrow M_{l+1} \cdots \leftrightarrow\!\!\!\rightarrow M_n$ where $\gamma = (\gamma_1; \gamma_2)$.

If γ_1 has ε-reductions, then by $P'(k)$ there exists i $(0 \leq i \leq l)$ and $\eta : M_0 \leftrightarrow\!\!\!\rightarrow^* N' \leftrightarrow\!\!\!\rightarrow^* M_i$ for some N' such that $H(\eta) \leq H(\gamma[0, i])$, $M_0 \rightarrow^* N'$, and for the subsequence $\eta' : N' \leftrightarrow\!\!\!\rightarrow^* M_i$ of η either $H(\eta') < H(\gamma[0, i])$ or $i = l$ and η' is ε-invariant.

We first consider the case of $H(\eta') < H(\gamma[0, i])$. Let $\gamma' = (\eta'; \gamma[i, n]) : N' \leftrightarrow\!\!\!\rightarrow^* M_i \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* M_n$.

By $H(\eta') < H(\gamma[0, i])$ and Property 5,

$$K_{width}(\gamma') \ll_w K_{width}(\gamma)$$

holds. Thus, the induction hypothesis $Q(k)$ for γ' ensures that $Q(k)$ holds for γ by $M_0 \leftrightarrow\!\!\!\rightarrow^* N'$ and $M_0 \rightarrow^* N'$.

The case in which $i = l$ and $\eta' : N' \leftrightarrow\!\!\!\rightarrow^* M_l$ is ε-invariant remains.

We apply the same argument for γ_2^R. (Note that $K_{width}(\gamma) = K_{width}(\gamma^R)$.) There exist j $(l \leq j \leq n)$ and $\bar{\eta} : M_n \leftrightarrow\!\!\!\rightarrow^* N'' \leftrightarrow\!\!\!\rightarrow^* M_j$ for some N'' such that $H(\bar{\eta}) \leq H(M_n \leftrightarrow\!\!\!\rightarrow M_{n-1} \cdots \leftrightarrow\!\!\!\rightarrow M_j)$, $M_n \rightarrow^* N''$ and either $j = l$ and the subsequence $\bar{\eta}' : N'' \leftrightarrow\!\!\!\rightarrow M_j$ of $\bar{\eta}$ is ε-invariant or $H(\bar{\eta}') < H(M_n \leftrightarrow\!\!\!\rightarrow M_{n-1} \cdots \leftrightarrow\!\!\!\rightarrow M_j)$ holds.

Similarly, the case in which $j = l$ and $\bar{\eta}' : N'' \leftrightarrow\!\!\!\rightarrow^* M_l$ is ε-invariant remains. In this case, note that

$\exists \zeta : M_0 \leftrightarrow\!\!\!\rightarrow^* N' \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* N'' \leftrightarrow\!\!\!\rightarrow^* M_n$ where $H(\zeta) \leq k$, $M_0 \rightarrow^* N'$, $\zeta' : N' \leftrightarrow\!\!\!\rightarrow^* M_l \leftrightarrow\!\!\!\rightarrow^* N''$ is ε-invariant and $M_n \rightarrow^* N''$.

Thus, the proof can reduce to the proof that $Q(k)$ holds for ζ'. Since ζ' is ε-invariant, the induction hypothesis $Q(k-1)$ ensures that $Q(k)$ holds for ζ'. Hence, $Q(k)$ holds for γ. □

6. Conclusion

In the previous section, we have proven that $Q(k)$ holds for all $k \geq 0$, so that we have the following main result.

Theorem 1

All the non-E-overlapping and strongly depth-preserving TRS's are CR. □

Example. If the third rule of R_2 in Section 1 is replaced by $h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))$, then we obtain a new TRS

$$R_2' = \{f(x, x) \rightarrow a,\ c \rightarrow h(c, g(c)),\ h(g(x), g(x)) \rightarrow f(x, h(x, g(c)))\}$$

which is non-E-overlapping and strongly depth-preserving, so that Theorem 1 ensures that R_2' is CR, though R_2 is not CR.

A TRS is said to be *non-ω-overlapping* if it is non-overlapping even if we use mappings from variables to infinite terms. Matsuura, et al.[3)] showed that if a TRS R is non-ω-overlapping and depth-preserving, then R is non-E-overlapping, so that we have the following corollary.

Corollary

All the non-ω-overlapping and strongly depth-preserving TRS's are CR. □

Note

Whether TRS R is non-ω-overlapping can be checked efficiently[3)].

We make a comment on the strongly depth-preserving property. This property is defined by the depth-preserving property and the condition that for each rule $\alpha \to \beta$ and for any $x \in V(\alpha)$, all the depths of the x occurrences in α are the same. By replacing the restriction on α by that on β, we can define an analogous property. That is, this new property is defined by the depth-preserving property and the condition that for each rule $\alpha \to \beta$ and for any $x \in V(\beta)$, all the depths of the x occurrences in β are the same. However, this new property and non-E-overlapping do not necessarily ensure CR.

Example.

$$R_6 = \{f(g(x),x) \to a,\ c \to h(c,g(c)),\ h(x,g(x)) \to f(g(x),h(x,g(c)))\}.$$

TRS R_6 is non-E-overlapping and satisfies this new condition, but R_6 is not CR.

Finally, we make a comment on an extension of the notion of depth-preserving. Note that the depth of an occurrence of a variable x is defined by the length of the path from the root to this x occurrence, which is equal to the number of function symbols appearing in this path. By assigning a positive integer (we call *weight* of the function symbol) to each function symbol, the notion of depth is naturally extended to that of weight: The weight of the x occurrence is the sum of weights of function symbols appearing in this path. Using the notion of weight, we can define strongly weight-preserving TRS's. We can easily show that non-E-overlapping and strongly weight-preserving TRS's are CR.

It will be a next step following the work of this paper to study the CR property of E-overlapping and strongly depth-preserving TRS's. That is, to find restriction conditions that E-critical pairs must satisfy for ensuring the CR property of strongly depth-preserving TRS's.

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Appendix

The proof of Property 1 is obvious by the definitions.

Proof of Property 2

We only prove the case of $Y = ldis$, since the proofs of the other cases are similar.

($\Leftarrow$) Let $(h_1,l_1),(h_2,l_2) \in K_{ldis}(\delta)$ where $h_1 > h_2$. Then, $l_2 > l_1$ holds, so that $(h_2,l_2)\not\leq_s(h_1,l_1)$. For $(h',l') \in K_{ldis}(\gamma)$, if $(h_1,l_1) \leq_s (h',l')$ and $(h_2,l_2) \leq_s (h',l')$, then $h' \geq h_1 > h_2$ and $l' \geq l_2 > l_1$, so that $(h_1,l_1) <_s (h',l')$ and $(h_2,l_2) <_s (h',l')$.

Let $B = K_{ldis}(\gamma) \bigcap K_{ldis}(\delta)$. Note that $K_{ldis}(\gamma)$ (and also $K_{ldis}(\delta)$) does not contain the same value more than once. Then, $\forall(h,l) \in K_{ldis}(\delta) - B\ \exists(h',l') \in K_{ldis}(\gamma) - B\quad (h,l) <_s (h',l')$. Because, if this is not true, then either $\exists(h'',l'') \in K_{ldis}(\gamma) - B\ \ (h'',l'') = (h,l)$ or $\exists(h'',l'') \in B\quad (h,l) <_s (h'',l'')$ and

$(h,l) \neq (h'',l'')$. The former case contradicts to the choice of B and in the latter case $(h'',l''),(h,l) \in K_{ldis}(\delta)$, $(h'',l'') \in K_{ldis}(\gamma)$ and $(h,l) <_s (h'',l'')$, so that $(h'',l'') <_s (h'',l'')$ and $(h,l) <_s (h'',l'')$ by the above arguments, which is a contradiction.

Hence, $K_{ldis}(\delta) \mathrel{\underline{\ll}}_s K_{ldis}(\gamma)$.

($\Rightarrow$) If $K_{ldis}(\delta) = K_{ldis}(\gamma)$, then the proof is obvious. Assume that $K_{ldis}(\delta) \ll_s K_{ldis}(\gamma)$. Then, $\exists S,T$ such that $S \subseteq K_{ldis}(\gamma)$, $K_{ldis}(\delta) = (K_{ldis}(\gamma)-S) \bigcup T$ and $\forall (h,l) \in T \;\; \exists (h',l') \in S \;\; (h,l) <_s (h',l')$.

For $\forall (h,l) \in K_{ldis}(\delta)$, if $(h,l) \in T$, then $\exists (h',l') \in K_{ldis}(\gamma)$ such that $(h,l) <_s (h',l')$. If $(h,l) \in (K_{ldis}(\gamma)-S)$, then $(h,l) \in K_{ldis}(\gamma)$. Thus, the condition (p.2) holds. □

Proof of Property 3

Proof of the condition 1.

Assume that $(h, ldis(\gamma',h)) \in K_{ldis}(\gamma')$ for $h \leq H(\gamma')$. Then, $left(\gamma',h)\downarrow$. So let $left(\gamma',h) = l$ where $0 \leq l \leq n$. If $l \geq j$ or $l \leq i$, then $H(M_l) = h$ holds, so that $\exists h' \geq h$ such that $left(\gamma,h') \leq left(\gamma',h) = l$ by Property 1.1.

If $i < l < j$, then $h = H(M_l[u \leftarrow L_l])$ holds. If $|u| + H(L_l) < h$, then $H(M_l) = h$, so that $\exists h' \geq h$ such that $left(\gamma,h') \leq left(\gamma',h) = l$ by Property 1.1. If $|u| + H(L_l) = h$, then $left(\delta, H(L_l))\downarrow$ and $left(\delta,H(L_l)) = l - i$. So, by $K_{ldis}(\delta) \mathrel{\underline{\ll}}_s K_{ldis}(\bar{\gamma})$ and Property 2.1, $\exists h' \geq H(L_l)$ such that $left(\bar{\gamma},h') \leq l-i$, i.e., $\exists l' \leq l$ such that $H(M_{l'}/u) \geq H(L_l)$. Thus, $H(M_{l'}) \geq |u| + H(M_{l'}/u) \geq h$, so that $\exists h' \geq h$ such that $left(\gamma,h') \leq l' \leq l = left(\gamma',h)$.

In either case, $\exists h' \geq h$ such that $ldis(\gamma,h') \geq ldis(\gamma',h)$. Hence, $\exists (h', ldis(\gamma,h')) \in K_{ldis}(\gamma)$ such that $(h, ldis(\gamma',h)) \leq_s (h', ldis(\gamma,h'))$. By Property 2, this condition 1 holds. □

The proof of condition 2 is similar to that of condition 1.

Proof of the condition 3.

By the conditions 1 and 2 of Property 3, we have

$$\begin{aligned} &K_{ldis}(\gamma') \mathrel{\underline{\ll}}_s K_{ldis}(\gamma) \text{ and} \\ &K_{right}(\gamma') \mathrel{\underline{\ll}}_s K_{right}(\gamma) \end{aligned} \tag{3.1}$$

Assume that $(h, width(\gamma',h)) \in K_{width}(\gamma')$ for $h \leq H(\gamma')$. Then, either $left(\gamma',h)\downarrow$ or $right(\gamma',h)\downarrow$. Let

$$h' = Min\{h' \mid h' \geq h, \;\; left(\gamma',h')\downarrow\} \text{ and}$$
$$h'' = Min\{h'' \mid h'' \geq h, \;\; right(\gamma',h'')\downarrow\}.$$

Then, $width(\gamma',h) = right(\gamma',h'') - left(\gamma',h')$ holds. Since $K_{right}(\gamma') \mathrel{\underline{\ll}}_s K_{right}(\gamma)$ by (3.1), $\exists \bar{h}'' \geq h''$ such that

$$right(\gamma',h'') \leq right(\gamma,\bar{h}'').$$

Since $K_{ldis}(\gamma') \mathrel{\underline{\ll}}_s K_{ldis}(\gamma)$ by (3.1), $\exists \bar{h}' \geq h'$ such that

$$left(\gamma,\bar{h}') \leq left(\gamma',h').$$

Let $s = Min(\bar{h}',\bar{h}'')$, $s' = Min\{s' \mid s' \geq s, left(\gamma,s')\downarrow\}$ and $s'' = Min\{s'' \mid s'' \geq s, right(\gamma,s')\downarrow\}$. Then, $width\ (\gamma,s) = right\ (\gamma,s'') - left(\gamma,s')$ and $s \geq h$ hold. By $\bar{h}' \geq s'$, we have $left\ (\gamma,s') \leq left\ (\gamma,\bar{h}')$ and by $\bar{h}'' \geq s''$, $right\ (\gamma,\bar{h}'') \leq right\ (\gamma,s'')$. Since $width\ (\gamma,s) = right\ (\gamma,s'') - left\ (\gamma,s') \geq right\ (\gamma,\bar{h}'') - left\ (\gamma,\bar{h}') \geq right\ (\gamma',h'') - left\ (\gamma',h') = width(\gamma',h)$, we have $(h, width(\gamma',h)) \leq_s (s, width(\gamma,s))$, so that $K_{width}(\gamma') \mathrel{\underline{\ll}}_w K_{width}(\gamma)$ holds by Property 2. □

Proof of Property 3'

Proof of the condition 1.

Note that $left(\gamma',h)\downarrow$ implies that $H(M_0) \leq h \leq H(\gamma')$. And $ldis(\gamma',H(M_0)) = |\gamma'|_p < ldis(\gamma,H(M_0)) = |\gamma|_p$ holds by $|\delta|_p < |\bar{\gamma}|_p$. Hence, $(H(M_0),|\gamma'|_p) \in K_{ldis}(\gamma')$, $(H(M_0),|\gamma|_p) \in K_{ldis}(\gamma)$ and $(H(M_0),|\gamma'|_p) <_s (H(M_0),|\gamma|_p)$.

Next, assume that $(h, ldis(\gamma',h)) \in K_{ldis}(\gamma')$ for $h > H(M_0)$. Then, $left(\gamma',h)\downarrow$. So, let $left(\gamma',h) = l$. If $l \geq j$ or $l \leq i$, then $H(M_l) = h$ holds, so that $\exists h' \geq h$ such that $left(\gamma,h') \leq left(\gamma',h) = l$ by Property 1.1. Thus, $ldis(\gamma,h') \geq ldis(\gamma',h)$.

If $i < l < j$, then $h = H(L_l)$ holds. Note that $H(M_{i'}) < h$ for all $i' \in \{0,\cdots,i\}$ and $H(L_{i''}) < h$ for all $i'' \in \{i,\cdots,l-1\}$. And $ldis(\gamma',h) = n - j + ldis(\delta,h)$, $left(\delta,h)\downarrow$, $left(\delta,h) = l-i$ and $ldis(\delta,h) = |\delta|_p - (l-i)$. So, by $K_{ldis}(\delta) \mathrel{\underline{\ll}}_s K_{ldis}(\bar{\gamma})$, $\exists (h', ldis(\bar{\gamma},h')) \in K_{ldis}(\bar{\gamma})$ such that $(h', ldis(\bar{\gamma},h')) \geq_s (h, ldis(\delta,h))$, i.e., $h' \geq h$ and $ldis(\bar{\gamma},h') \geq ldis(\delta,h)$. Note that $H(M_k) = h'$ holds for $k = j - ldis(\bar{\gamma},h')$. Hence, $left(\gamma,h') = k$ and $ldis(\gamma,h') = n - k = (n-j) - (j-k) = (n-j) + ldis(\bar{\gamma},h') \geq (n-j) + ldis(\delta,h) = ldis(\gamma',h)$. Thus, $(h', ldis(\gamma,h')) \geq_s (h, ldis(\gamma',h))$.

Hence, $K_{ldis}(\gamma') \mathrel{\underline{\ll}}_s K_{ldis}(\gamma)$ by Property 2 and $K_{ldis}(\gamma') \neq K_{ldis}(\gamma)$, so that $K_{ldis}(\gamma') \ll_s K_{ldis}(\gamma)$ holds. □

The proof of the condition 2 is similar to that of condition 1. □

Property 4 is a direct consequence of Property 3'.

Proof of Property 5

Let $h_1 = H(\gamma[0,i])$. Then, $left(\gamma,h_1)\downarrow$ and $0 \leq left(\gamma,h_1) \leq i$, so that $ldis(\gamma,h_1) \geq n - i$

holds. If $left(\gamma', h_1) \downarrow$, then $ldis(\gamma', h_1) < n - i$ holds, since $H(\delta) < h_1$. Thus, $left(\gamma', h_1) \uparrow$ or $left(\gamma', h_1) \downarrow$ and $ldis(\gamma', h_1) < ldis(\gamma, h_1)$. And for any $h > h_1$, if $left(\gamma, h) \downarrow$ or $left(\gamma', h) \downarrow$, then $left(\gamma, h) \downarrow, left(\gamma', h) \downarrow$ and $ldis(\gamma, h) = ldis(\gamma', h)$ hold, since $H(\delta) < h_1$. Hence, $K_{ldis}(\gamma') \ll_w K_{ldis}(\gamma)$. (Note that $\ll_w$ is used in Property 5, while $\ll_s$ is used in the other properties.)

Similarly, for any $h > h_1$, if $width(\gamma, h) \downarrow$ or $width(\gamma', h) \downarrow$, then $width(\gamma, h) \downarrow, width(\gamma', h) \downarrow$ and $width(\gamma, h) = width(\gamma', h)$ hold, since $H(\delta) < h_1$.

Let $h_2 = Min\{h_2 \mid h_2 \geq h_1 \bigwedge right(\gamma, h_2) \downarrow \}$. Note that $width(\gamma, h_1) \downarrow$ and $width(\gamma, h_1) = right(\gamma, h_2) - left(\gamma, h_1)$. If $width(\gamma', h_1) \uparrow$, then $K_{width}(\gamma') \ll_w K_{width}(\gamma)$ holds. So, consider the case of $width(\gamma', h_1) \downarrow$. Let $width(\gamma', h_1) = right(\gamma', h_1'') - left(\gamma', h_1')$ where $h_1' = Min\{h_1' \mid h_1' \geq h_1 \bigwedge left(\gamma', h_1') \downarrow\}$, $h_1'' = Min\{h_1'' \mid h_1'' \geq h_1 \bigwedge right(\gamma', h_1'') \downarrow\}$, and $h_1 = Min(h_1', h_1'')$. Then, $left(\gamma', h_1') > |\delta|_p$ and $right(\gamma', h_1'') > |\delta|_p$, since $H(\delta) < h_1$. Thus, $right(\gamma, h_1'') \downarrow$, so that $h_2 \leq h_1''$ holds by the definition of h_2. If $h_2 < h_1''$, then $right(\gamma, h_1'') < right(\gamma, h_2)$ would hold, so that $right(\gamma', h_2) \downarrow$ which contradicts to the definition of h_1'' by $h_1'' > h_2 \geq h_1$. Hence, $h_2 = h_1''$ holds. Since $ldis(\gamma, h_1) \geq n - i > ldis(\gamma', h_1')$, it follows that $width(\gamma', h_1) = right(\gamma', h_1'') - left(\gamma', h_1') = ldis(\gamma', h_1') - (|\gamma'|_p - right(\gamma', h_1'')) = ldis(\gamma', h_1') - (|\gamma|_p - right(\gamma, h_2)) < ldis(\gamma, h_1) - (|\gamma|_p - right(\gamma, h_2)) = right(\gamma, h_2) - left(\gamma, h_1) = width(\gamma, h_1)$

Hence, $K_{width}(\gamma') \ll_w K_{width}(\gamma)$ holds. □

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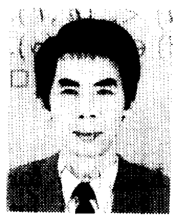

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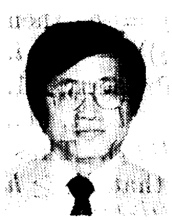

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