# Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid

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**Abstract:** Common unfolding of a regular tetrahedron and a Johnson-Zalgaller solid is investigated. More precisely, we investigate the sets of all edge unfoldings of Johnson-Zalgaller solids. Among 92 Johnson-Zalgaller solids, some of edge unfolding of J17 and J84 admit to fold into a regular tetrahedron. On the other hand, there are no edge unfolding of the other Johnson-Zalgaller solids that admit to fold into a regular tetrahedron.

## 1. Introduction

In 1525 the painter and printmaker Albrecht Dürer published a book, translated as "The Painter's Manual," in which he explained the methods of perspective [13]. In the book, he includes a description of many polyhedra, which he presented as surface unfoldings, are now called "nets." An *edge unfolding* is defined by a development of the surface of a polyhedron to a plane, such that the surface becomes a flat polygon bounded by segments that derive from edges of the polyhedron. We would like an unfolding to possess three characteristics. (1) The unfolding is a single, simply connected piece. (2) The boundary of the unfolding is composed of (whole) edges of the polyhedron, that is, the unfolding is a union of polyhedron faces. (3) The unfolding does not self-overlap, that is, it is a simple polygon. We call a simple polygon that satisfies these conditions a *net* for the polyhedron.

Since then, nets for polyhedra have been widely investigated (rich background can be found in [5], and recent results can be found in [10]). For example, Alexandrov's theorem states that every metric with the global topology and local geometry required of a convex polyhedron is in fact the intrinsic metric of some convex polyhedron. Thus, if $P$ is a net of a convex polyhedron $Q$, then the shape (as a convex polyhedron) is uniquely determined. Alexandrov's theorem was stated in 1942, and a constructive proof was given by Bobenko and Izmestiev in 2008 [4]. A pseudo-polynomial algorithm for Alexandrov's theorem, given by Kane et al. in 2009, runs in $O(n^{456.5}r^{1891}/\epsilon^{121})$ time, where $r$ is the ratio of the largest and smallest distances between vertices, and $\epsilon$ is the coordinate relative accuracy [9]. The exponents in the time bound of the result are remarkably huge.

Therefore, we have to restrict ourselves to smaller classes of polyhedra to investigate from the viewpoint of efficient algorithms. In this paper, we consider some classes of polyhedra that have common nets. In general, a polygon can be a net of two or more convex polyhedra. Such a polygon is called a *common net* of the polyhedra[*1]. Recently, several polygons folding into two different polyhedra have been investigated (see [12] for comprehensive list). In this context, it is natural to ask whether there is a common net of two (or more) different Platonic solids. This question has arisen several times independently, and it is still open (see [5], Section 25.8.3). In general nets, there is a polygon that can folds into a cube and an almost regular tetrahedron with small error $\epsilon < 2.89200 \times 10^{-1796}$ [12]. On the other hand, when we restrict ourselves to deal with only edge unfoldings, there are no edge unfolding of the Platonic solids except a regular tetrahedron that can fold into a regular tetrahedron [8]. This result is not trivial since a regular icosahedron and a regular dodecahedron have 43,380 edge unfoldings. In fact, it is confirmed that all the edge unfolding are nets (i.e., without self-overlapping) recently [6]).

In this paper, we broaden the target of research from the set of five Platonic solids to the set of 92 Johnson-Zalgaller solids (JZ solids for short). A JZ solid is a strictly convex polyhedron, each face of which is a regular polygon, but which is not uniform, i.e., not a Platonic solid, Archimedean solid, prism, or antiprism (see, e.g., `http://mathworld.wolfram.com/JohnsonSolid.html`). Recently, the number of edge unfoldings of the JZ solids are counted [7], however, it has not been investigated how many nets (without self-overlapping) are there. On the other hand, the tilings of edge unfoldings of JZ solids are classified [2]. That is, they classified the class of the JZ solids whose edge unfoldings form tilings. Some tilings are well investigated in the context of nets; a polygon is a net of a regular tetrahedron if and only if it belongs to a special class of tilings [1].

In this paper, we concentrate on common nets of a regular tetrahedron and the JZ solids. More precisely, we classify the set of

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*1 Note that an edge of an unfolding can passes through a flat face of the polyhedra.

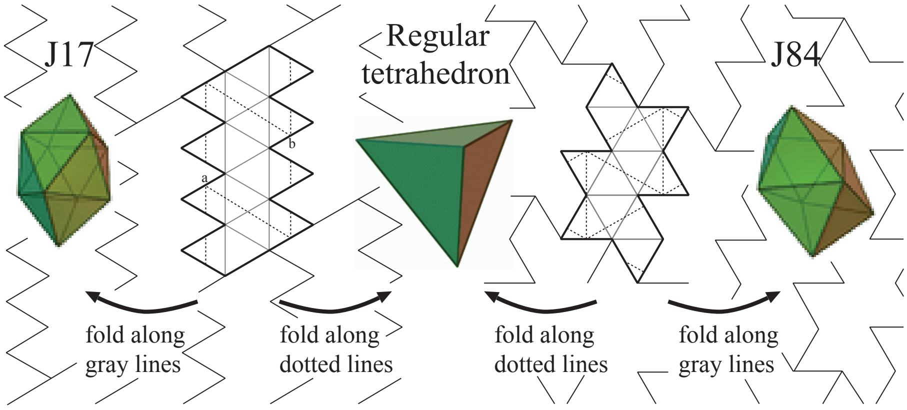



**Fig. 1** (Left) an edge unfolding of the JZ solid J17, and (right) an edge unfolding of the JZ solid J84, which are also nets of a regular tetrahedron, respectively. These polygons are also p2 tilings.

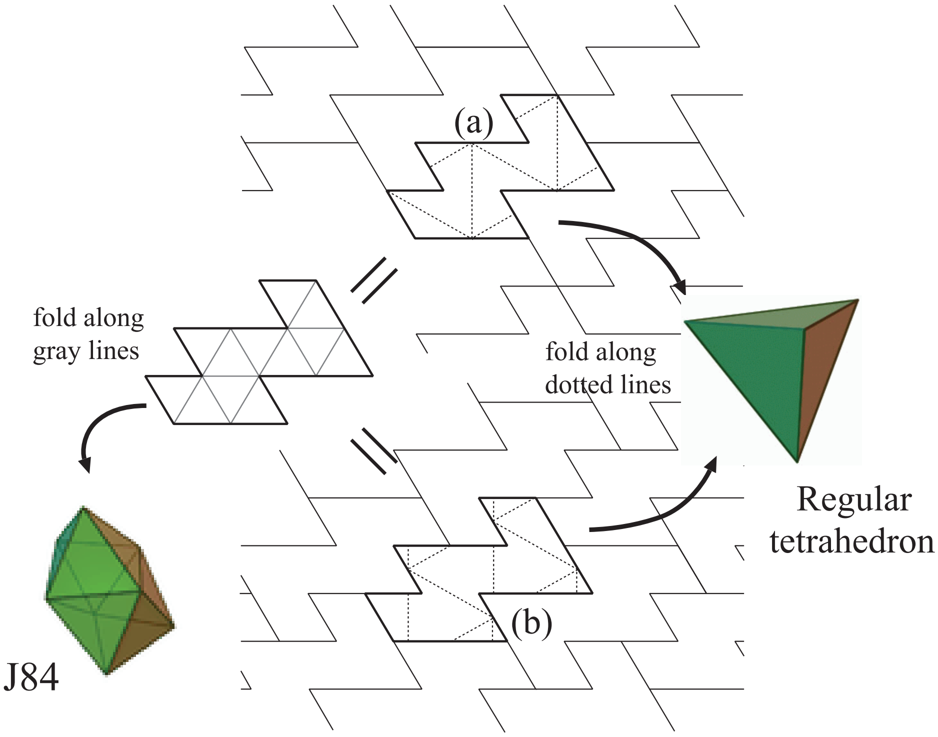



**Fig. 2** An edge unfolding of the JZ solid J84. It has two different types of p2 tilings, and hence there are two different ways to fold into a regular tetrahedron.

edge unfoldings of the JZ solids such that each of them is also folded into a regular tetrahedron. We first show that there exists edge unfoldings of some JZ solids that are also nets of a regular tetrahedra:

**Theorem 1** An edge unfolding of the JZ solid J17 and an edge unfolding of the JZ solid J84 fold into a regular tetrahedron.

We will show that Fig. 1 certainly proves Theorem 1. Next we also compute all common nets that fold into both of a JZ solid and a regular tetrahedra:[*2]

**Theorem 2** (1) Among 13,014 edge unfoldings of the JZ solid J17 [7], there are 87 nets that fold into a regular tetrahedron, which consist of 78 nets that have one way of folding into a regular tetrahedron, 8 nets that have two ways of folding into a regular tetrahedron, and 1 net that has three ways of folding into a regular tetrahedron. (2) Among 1,109 edge unfoldings of the JZ solid J84 [7], there are 37 nets that fold into a regular tetrahedron, which consist of 32 nets that have one way of folding into a regular tetrahedron, and 5 nets that have two ways of folding into a regular tetrahedron.

We note that some nets allow to fold into a regular tetrahedron in two or more different ways of folding. A typical example that has two ways of folding is shown in Fig. 2. We can tile the net of the JZ solid J84 in two different ways, hence we can fold a regular tetrahedron in two different ways according to the tilings. The unique net that has three ways of folding is shown in Fig. 3.

Among 92 JZ solids, Akiyama et al. found that 18 JZ solids have edge unfoldings that are also tilings [2]. We will show that all of them are also p2 tiling, which imply that they can be folded into tetramonohedra. As shown in Theorem 1, two of them can be folded into regular tetrahedra. On the other hand, the other 16

*2 These numbers are counted on the "unlabeled" solids, and congruent unfoldings are not reduced. See [7] for further details.

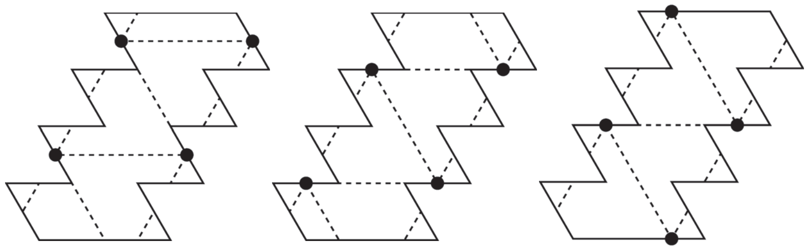
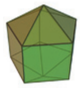
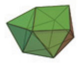
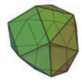


**Fig. 3** An edge unfolding of the JZ solid J17 that can be folded into a regular tetrahedron in three different ways.

**Table 1** The JZ solids whose some edge unfoldings are nets of tetramonohedra.

| Name | J1 | J8 | J10 | J12 | J13 | J14 | J15 | J16 | J17 |
|---|---|---|---|---|---|---|---|---|---|
| Image |  |  |  |  |  |  |  |  |  |
| # of □s | 1 | 5 | 1 | 0 | 0 | 3 | 4 | 5 | 0 |
| # of △s | 4 | 4 | 12 | 6 | 10 | 6 | 8 | 10 | 16 |
| $L_{J_i}$ | $\sqrt{\frac{1}{\sqrt{3}}+1}$ $= 1.255\cdots$ | $\sqrt{\frac{5}{\sqrt{3}}+1}$ $= 1.971\cdots$ | $\sqrt{\frac{1}{\sqrt{3}}+3}$ $= 1.891\cdots$ | $\sqrt{1.5}$ $= 1.224\cdots$ | $\sqrt{2.5}$ $= 1.581\cdots$ | $\sqrt{\frac{3}{\sqrt{3}}+\frac{3}{2}}$ $= 1.797\cdots$ | $\sqrt{\frac{4}{\sqrt{3}}+2}$ $= 2.075\cdots$ | $\sqrt{\frac{5}{\sqrt{3}}+\frac{5}{2}}$ $= 2.320\cdots$ | 2 |
| Name | J49 | J50 | J51 | J84 | J86 | J87 | J88 | J89 | J90 |
| Image |  |  |  |  |  |  |  |  |  |
| # of □s | 2 | 1 | 0 | 0 | 2 | 1 | 2 | 3 | 4 |
| # of △s | 6 | 10 | 14 | 12 | 12 | 16 | 16 | 18 | 20 |
| $L_{J_i}$ | $\sqrt{\frac{2}{\sqrt{3}}+\frac{3}{2}}$ $= 1.629\cdots$ | $\sqrt{\frac{1}{\sqrt{3}}+\frac{5}{2}}$ $= 1.754\cdots$ | $\sqrt{3.5}$ $= 1.870\cdots$ | $\sqrt{3}$ $= 1.732\cdots$ | $\sqrt{\frac{2}{\sqrt{3}}+3}$ $= 2.038\cdots$ | $\sqrt{\frac{1}{\sqrt{3}}+4}$ $= 2.139\cdots$ | $\sqrt{\frac{2}{\sqrt{3}}+4}$ $= 2.270\cdots$ | $\sqrt{\frac{3}{\sqrt{3}}+\frac{9}{2}}$ $= 2.496\cdots$ | $\sqrt{\frac{4}{\sqrt{3}}+5}$ $= 2.703\cdots$ |

JZ solids do not have such edge unfoldings:

**Theorem 3** Except J17 and J84, there is no other JZ solid such that its edge unfolding is a net of a regular tetrahedron.

Therefore, we classify the set of edge unfoldings of the JZ solids by the foldability of a regular tetrahedron.

## 2. Preliminaries

We first show some basic results about unfolding of a polyhedron.

**Lemma 4 ([5], Sec. 22.1.3)** All vertices of a polyhedron $X$ are on the boundary of any unfolding of $X$.

Let $P$ be a polygon on the plane, and $R$ be a set of four points (called *rotation centers*) on the boundary of $P$. Then $P$ has a *tiling* called symmetry group *p2* (*p2 tiling*, for short) if $P$ fills the plane by the repetition of 2-fold rotations around the points in $R$. The filling should contain no gaps nor overlaps. The rotation defines an equivalence relation on the points in the plane. Two points $p_1$ and $p_2$ are mutually equivalent if $p_1$ can be moved to $p_2$ by the 2-fold rotations. More details of p2 tiling can be found, e.g., in [11]. Based on the notion of p2 tiling, any unfolding of a tetramonohedron[*3] can be characterized as follows:

**Theorem 5 ([1], [3])** $P$ is an unfolding of a tetramonohedron if and only if (1) $P$ has a p2 tiling, (2) four of the rotation centers consist in the triangular lattice formed by the triangular faces of the tetramonohedron, (3) the four rotation centers are the lattice points, and (4) no two of the four rotation centers belong to the same equivalent class on the tiling.

*3 A *tetramonohedron* is a tetrahedron that consists of four congruent triangular faces.

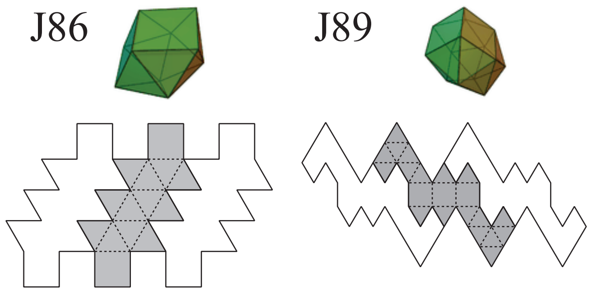



**Fig. 4** p2 tilings by (left) an edge unfolding of JZ solid J86, and (right) an edge unfolding of JZ solid J89.

We can obtain the characterization of the unfolding of a regular tetrahedron if each triangular face in Theorem 5 is a regular triangle. By Theorem 5, Theorem 1 is directly proved by Fig. 1. (Of course it is not difficult to check these nets in Fig. 1 by cutting and folding directly.)

In the classification in [2], they show only p1 tilings for the JZ solids J84, J86 and J89. However, they also have edge unfoldings that form p2 tilings as shown in Fig. 1 (J84) and Fig. 4 (J86 and J89), and hence they can fold into tetramonohedra.

Let $L_{J_i}$ be the length of an edge of a regular tetrahedron $T_{J_i}$ that has the same surface area of the JZ solid $J_i$. We assume that each face of $J_i$ is a regular polygon that consists of edges of unit length. Thus it is easy to compute $L_{J_i}$ from its surface area of $J_i$ as shown in Table 1. If an edge unfolding $P_{J_i}$ of the JZ solid $J_i$ can be folded into a regular tetrahedron, the tetrahedron is congruent to $T_{J_i}$ since they have the same surface area. Moreover, by Theorem 5, $P_{J_i}$ is a p2 tiling, and its four of the rotation centers form the regular triangular lattice filled by regular triangles of edge length $L_{J_i}$. Let $c_1$ and $c_2$ be any pair of the rotation centers of distance $L_{J_i}$. Then, by Lemma 4, $c_1$ and $c_2$ are on boundary of $P_{J_i}$ and $P'_{J_i}$ for some polygons $P_{J_i}$ and $P'_{J_i}$, respectively. By the same extension of Theorem 25.3.1 in [5] used in [8], Lemma 8, we can assume that $c_1$ and $c_2$ are on the corners or the middlepoints on some edges of regular faces of JZ solids $J_i$ without loss of generality. Summarizing them, we obtain the following lemma:

**Lemma 6** Assume that a polygon $P_{J_i}$ is obtained by an edge unfolding of a JZ solid $J_i$. If $P_{J_i}$ can be folded into a regular tetrahedron $T_{J_i}$, $P_{J_i}$ forms a p2 tiling $\mathcal{T}$. Let $c_1$ and $c_2$ be any two rotation centers on $\mathcal{T}$ such that the distance between $c_1$ and $c_2$ is $L_{J_i}$, equal to the length of an edge of $T_{J_i}$. Then, the vertices $c_1$ and $c_2$ are on the corners or the middlepoints on edges of unit length in $\mathcal{T}$.

## 3. The JZ solids J17 and J84

In this section, we describe an algorithm to obtain Theorem 2. By applying the technique in [6], we can enumerate a set of spanning trees of any polyhedron, where a spanning tree is obtained as a set of edges. By traversing each spanning tree, we can obtain its corresponding unfolding $P_{J_i}$. Since all edges of a JZ solid have the same length, $P_{J_i}$ can be represented by a cyclic list $C_{J_i}$ of its interior angles $a_j$, where vertices $v_j$ of $P_{J_i}$ correspond to the corners or the middlepoints on some edges of the original JZ solid. Since a spanning tree has $n-1$ edges, each edge appears twice as the boundary of $P_{J_i}$, and each edge is broken into two halves, $P_{J_i}$ has $4(n-1)$ vertices. Fig. 5 illustrates (a) a spanning tree of the JZ solid J17, and (b) its corresponding unfolding, which can be represented by $C_{J_i} = \{60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180\}$.

Now, we use Theorem 5 and check if each edge unfolding is a p2 tiling or not. We can use the similar idea with the algorithm for gluing borders of a polyhedron (see [5], Chap. 25.2): around each rotation center, check if the corresponding points make together 360°. If not, we dismiss this case, and otherwise, we obtain a gluing to form a regular tetrahedron.

We first consider the JZ solid J17. In this case, we can determine the length of each edge of the triangular lattice equals to 2, since each face of the (potential) regular tetrahedron consists of four unit tiles. We can check if each unfolding of the JZ solid J17 can be folded into a regular tetrahedron as follows:

(1) For each pair of $v_{j_1}$ and $v_{j_2}$, suppose they are rotation centers, and check if the distance between them is 2.
(2) Obtain a path $v_{j'_1}$–$v_{j_1}$–$v_{j''_1}$ which is glued to $v_{j''_1}$–$v_{j_1}$–$v_{j'_1}$ by a 2-fold rotation around $v_{j_1}$. So do a path $v_{j'_2}$–$v_{j_2}$–$v_{j''_2}$ for $v_{j_2}$.
(3) Replace interior angles of $a_{j'_1}, \ldots, a_{j_1}, \ldots, a_{j''_1}$ in $C_{J_i}$ with angle $a'_{j_1}$, where $a'_{j_1} = a_{j'_1} + a_{j''_1}$ if $j'_1 \neq j''_1$ and $a'_{j_1} = a_{j_1} + 180$ if $j'_1 = j''_1 = j_1$. So do $a_{j'_2}, \ldots, a_{j''_2}$. Let $C'_{J_i}$ be the resulting cyclic list.
(4) For each pair of $v_{j_3}$ and $v_{j_4}$, suppose they are rotation centers, and check if a path $v_{j_3}$–$v_{j_4}$ in $C'_{J_i}$ is glued to the remaining path $v_{j_4}$–$v_{j_3}$.
(5) Check if $v_{j_3}$ and $v_{j_4}$ are the lattice points of the regular triangular lattice defined by $v_{j_1}$, $v_{j_2}$, and check if no two of $v_{j_1}$, $v_{j_2}$, $v_{j_3}$ and $v_{j_4}$ belong to the same equivalent class.

In Step 1, since every face of the JZ solid J17 is a triangle, $a_j$ is always a multiple of 60. The relative position of $v_j$ from $v_0$ can be represented as a linear combination of two unit vectors $\vec{u}$ and $\vec{v}$ that make a 60° angle. Thus, we check if vector $\overrightarrow{v_{j_1}v_{j_2}}$ is one of $\pm 2\vec{u}$, $\pm 2\vec{v}$, $\pm 2(\vec{u}-\vec{v})$ in this step.

In Step 2, two vertices $v_{j'_1}$ and $v_{j''_1}$ are obtained as $v_{j_1-k}$ and $v_{j_1+k}$ with an integer $k$ satisfying $a_{j_1-k} + a_{j_1+k} < 360$ and $a_{j_1-k'} + a_{j_1+k'} = 360$ for all $0 \le k' < k$. In Fig. 5, $v_1$ and $v_7$ are supposed to be rotation centers, and paths $v_0$–$v_1$–$v_2$ and $v_4$–$v_7$–$v_{10}$ are glued to $v_2$–$v_1$–$v_0$ and $v_{10}$–$v_7$–$v_4$, respectively. By rotating $P_{J_i}$ around $v_{j_1}$ and $v_{j_2}$ repeatedly, we obtain a horizontally infinite sequence of $P_{J_i}$ as shown in Fig. 5(c), whose upper and lower borders are the repetition of the path denoted in double line. The list of the interior angles along the double line is obtained as $C'_{J_i}$ in Step 3. In Fig. 5(c), $C'_{J_i}$ is $\{180, 180, 240, 180, 300, 180, 60, 180, 300, 180, 60, 180, 120, 180, 180, 180, 60, 180, 300, 180, 60, 180, 300, 180, 60, 180, 300, 180\}$.

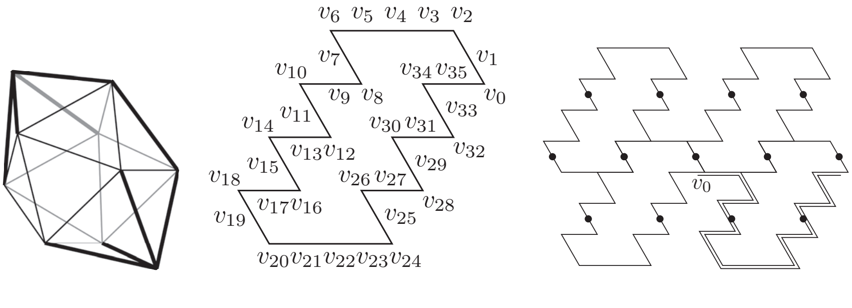
In Step 4, we check if $a_{j_3} = a_{j_4} = 180$ holds and $a_{j_3-k} + a_{j_3+k} = 360$ for other gluing of vertices $v_{j_3-k}$ and $v_{j_3+k}$ in $C'_{J_i}$. If $P_{J_i}$ passes all checks in Steps 1–4, $P_{J_i}$ has a p2 tiling with rotation centers $v_{j_1}$, $v_{j_2}$, $v_{j_3}$ and $v_{j_4}$. In Step 5, we check if the four points meet Theorem 5(2)–(4) and if each triangular face is a regular triangle. As in Step 1, this check can be done from the positions of vertices represented as a linear combination of $\vec{u}$ and $\vec{v}$.

For the JZ solid J84, we can check in the same way by letting the length of the triangular lattice equal to $\sqrt{3}$, and thus, in Step 1, we check if vector $\overrightarrow{v_{j_1}v_{j_2}}$ is one of $\pm(\vec{u}+\vec{v})$, $\pm(2\vec{u}-\vec{v})$, $\pm(2\vec{v}-\vec{u})$. The complete catalogue of 87 and 37 nets of the JZ solids J17 and J84, respectively, that fold into a regular tetrahedron is given in `http://www.al.ics.saitama-u.ac.jp/horiyama/research/unfolding/common/`.

## 4. The other JZ solids

In this section, we prove Theorem 3. Combining the results in [2] and the tilings in Fig. 1 and Fig. 4, the set $\mathcal{J}$ of JZ solids whose edge unfoldings can be p2 tiling is $\mathcal{J}$ = {J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90}. In other words, some edge unfoldings of the JZ solids in $\mathcal{J}$ can be folded into tetramonohedra. Among them, J17 and J84 allow to fold into regular tetrahedra from their edge unfoldings as shown in Fig. 1. We will show that the other JZ solids do not. Hereafter, we only consider the JZ solids in $\mathcal{J}$. Then each face is either a unit square or a unit triangle. We call each of them a *unit tile* to simplify. We consider the rotation centers form the regular triangular lattice of size $L_{J_i}$. Let $c_1$ and $c_2$ be any pair of the rotation centers of distance $L_{J_i}$. We use the fact that the distance between $c_1$ and $c_2$ is equal to $L_{J_i}$, and show that any combination of unit tiles cannot achieve the length.

Intuitively, two points $c_1$ and $c_2$ are joined by a sequence of edges of unit length that are supported by unit tiles in $\mathcal{T}$. Thus, by Lemma 6, we can observe that there exists a linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$ such that (1) $c_1$ is on either $p_0$ or $q_1$, (2) $c_2$ is on either $q_k$ or $p_k$, (3) the length of $p_ip_{i+1}$ is 1, (4) the length of $p_{i-1}q_i$ and $q_ip_i$ is 1/2 (in other




(a) (b) (c)

**Fig. 5** (a) a spanning tree of the JZ solid J17, (b) its corresponding unfolding, and (c) a p2 tiling around the rotation centers.

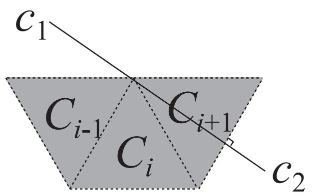



**Fig. 6** The shortest way to pierce three consecutive unit tiles.

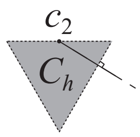



**Fig. 7** The shortest intersection with the last unit tile $C_h$.

words, $q_i$ is the center point of $p_{i-1}p_i$), (5) each angle at $q_i$ ($1 \leq i \leq k$) is 180°, (6) each angle at $p_i$ ($1 \leq i \leq k-1$) is in $\{60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°\}$, and (7) the linkage is not self-crossing. (See [5] for the definition of the notion of *linkage*.) Without loss of generality, we suppose that $L_{J_i}$ has the minimum length among the linkages satisfying the conditions from (1) to (7). By the minimality, we also assume that (8) $p_i \neq p_j$ for each $i \neq j$, and (9) if $|i-j| > 1$, the distance between $p_i$ and $p_j$ is not 1 (otherwise, we obtain a shorter linkage).

Therefore, by Theorem 5, for sufficiently large $k$, if all possible pairs $c_1$ and $c_2$ on the linkages satisfying the conditions from (1) to (9) do not achieve the required distance $L_{J_i}$, any edge unfolding of the JZ solid $J_i$ cannot be folded into a corresponding regular tetrahedron $T_{J_i}$. We show an upper bound of $k$:

**Theorem 7** Let $\mathcal{J}$ be the set {J1, J8, J10, J12, J13, J14, J15, J16, J17, J49, J50, J51, J84, J86, J87, J88, J89, J90} of the JZ solids that have some edge unfoldings which are also p2 tilings. For some $J_i \in \mathcal{J}$, suppose that the linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$ defined above exist. Then $k \leq 10$.

**Proof.** By simple calculation, $L_{J_i}$ takes the maximum value $\sqrt{(4+5\sqrt{3})/\sqrt{3}} = 2.703\cdots$ for $J90$ in $\mathcal{J}$. Thus the length of the line segment $c_1c_2$ is at most $2.703\cdots$.

Now we assume that the line segment $c_1c_2$ passes through a sequence $C_1C_2\cdots C_h$ of unit tiles in this order. That is, the line segment $c_1c_2$ has nonempty intersection with each of $C_i$ in this order. If $c_1c_2$ passes an edge shared by two unit tiles, we take arbitrary one of two in the sequence. We consider the minimum length of the part of $c_1c_2$ that intersects three consecutive unit tiles $C_{i-1}C_iC_{i+1}$ in the sequence. Since they are unit triangles and/or squares, three unit tiles make greater than or equal to 180° at a vertex. Therefore the minimum length is achieved by the three consecutive triangles arranged in Fig. 6, and in this case, the length is greater than or equal to $\sqrt{3}/2 = 0.866\cdots$. Thus, if $c_1c_2$ passes through nine unit tiles, the intersection has length at least $3\sqrt{3}/2 = 2.598\cdots$. On the other hand, the last point $c_2$ is on the vertex or a midpoint of an edge of the last unit tile $C_h$. Then the intersection of $c_1c_2$ and $C_h$ has at least $\sqrt{3}/4 = 0.433\cdots$ (Fig. 7). Since $3\sqrt{3}/2 + \sqrt{3}/4 = 3.03\cdots > 2.703$, $c_1c_2$ passes through at most 9 unit tiles.

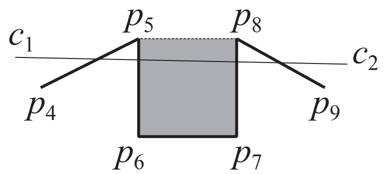



**Fig. 8** Unit square can contribute three edges to the linkage.

Now we turn to the linkage $L_{J_i} = (p_0, q_1, p_1, q_2, p_2, \ldots, q_k, p_k)$ supported by the unit tiles $C_1C_2\cdots C_h$ with $h \leq 9$. We consider the number of edges of a unit tile that contributes to $L_{J_i}$. Locally, the worst case is that a unit square that contributes three edges to $L_{J_i}$ (Fig. 8). However, in this case, the length of the intersection of the square and $c_1c_2$ has length at least 1. Therefore, further analysis for the remaining length at most $2.703\cdots - 1 = 1.703\cdots$, and we can confirm that this case does not give the worst value of $k$. In the same reason, if $c_1c_2$ passes through an entire edge of length 1, it does not give the worst value of $k$. Next considerable case is that a unit tile $C_i$ contributes two edges to $L_{J_i}$ independent from $C_{i-1}$ and $C_{i+1}$. That is, $c_1c_2$ passes through two vertices of $C_i$. Then $C_i$ is not a unit triangle since we can replace two edges by the third edge and obtain a shorter linkage. Thus $C_i$ is a unit square, and $c_1c_2$ passes through the diagonal of $C_i$ since two edges are not shared by $C_{i-1}$ and $C_{i+1}$. Then the intersection of $c_1c_2$ and $C_i$ has length $\sqrt{2} = 1.414\cdots$, and hence this case does not give the worst value of $k$ again. Therefore, in the worst case, each unit tile contributes exactly two edges to $L_{J_i}$, and each edge is shared by two consecutive unit tiles in the sequence $C_1C_2\cdots C_h$, where $h \leq 9$. Therefore, the linkage consists of at most 10 unit length edges, that is, $k \leq 10$. □

Now, for $k \leq 10$, if all possible pairs $c_1$ and $c_2$ on the linkages satisfying the conditions from (1) to (9) do not realize any distance $L_{J_i}$ in Table 1, any edge unfolding of the JZ solid $J_i$ cannot be folded into a corresponding regular tetrahedron $T_{J_i}$. However, the number of possible configurations of the linkage is still huge. When we regard each edge in the linkage as a unit vector, we can swap two edges without changing the coordinate of $c_2$ (one can find the same idea in, e.g., [5], Section 5.1.1). Therefore, to check only the possible distances between $c_1$ and $c_2$, it is sufficient to consider the standardized linkage that satisfies the conditions (1)-(5), (8) and (9) with the following conditions (6'-0) and (6'-1). For each $j = 0, 1, \ldots, 11$, let set $S_j$ be the set of unit vectors $\vec{v_i}$ from $(0, 0)$ to $(\cos i\pi/6, \sin i\pi/6)$ for each $j \leq i < 12$. Then (6'-0) the vector $\overrightarrow{p_0 p_1} = (1, 0)$, and (6'-1) for $i > 1$, each vector $\vec{v_i} = \overrightarrow{p_{i-1} p_i}$ is in $S_j$, where $j$ is determined by $\overrightarrow{v_{i-1}} = \overrightarrow{p_{i-2} p_{i-1}}$ such that if $\overrightarrow{v_{i-1}} \in S_{j'}$, $j$ is any number with $j' \leq j < 12$. Intuitively, the linkage is admitted to turn only to the left up to 360° in total, and each turn angle should be a multiple of 30°.

Now we prove the main theorem in this section:

**Proof.** (of Theorem 3) We check the all possible linkages up to $k = 10$ satisfying conditions from (1)-(5), (6'-0), (6'-1), (8), and (9) by exhaustive search. Each output is produced when the resulting distance is in [1.20..2.80]. The program runs in a second, and we obtain 11226 distinct arrangements of linkages, and 457 distinct distances between $(p_0, p_k)$, $(q_1, p_k)$, and $(q_1, q_k)$. For $\mathcal{J}$, only J17 and J84 have feasible solutions $L_{J_{17}} = 2$ and $L_{J_{84}} = \sqrt{3}$ in the distances.[*4] □

## 5. Convex Polyhedra with Regular Polygonal Faces

According to the classification in [2], there are 23 polyhedra with regular polygonal faces whose edge unfoldings allow tilings. Among them, 18 JZ solids have been discussed in Section 4, and four Platonic solids were discussed in [8]. The remaining one is hexagonal antiprism that consists of two regular hexagons and 12 unit triangles. By splitting each regular hexagon into six unit triangles, which is called *coplanar deltahedron*, we can show the following theorem using the same argument above:

**Theorem 8** The hexagonal antiprism has no edge unfolding that can fold into a regular tetrahedron.

Thus we can conclude as follows:

**Corollary 9** Among convex polyhedra with regular polygonal faces, including the Platonic solids, the Archimedean solids, and the JZ solids, regular prisms, and regular anti-prisms, only the JZ solids J17 and J84 (and regular tetrahedron) admit to fold into regular tetrahedra from their edge unfoldings.

*4 We here summarize the resulting data as: $J1(1.25423 < 1.2559 < 1.25945)$, $J8(1.97078 < 1.9715 < 1.97655)$, $J10\ (1.89119 < 1.8914 < 1.89466)$, $J12(1.21053 < 1.2247 < 1.22527)$, $J13(1.5785 < 1.5811 < 1.58155)$, $J14(1.79449 < 1.79779 < 1.79913)$, $J15(2.07341 < 2.0759 < 2.07805)$, $J16\ (2.32034 < 2.3209 < 2.32241)$, $J49(1.62847 < 1.629325 < 1.63397)$, $J50(1.75302 < 1.7542 < 1.75777)$, $J51(1.8686 < 1.8708 < 1.87211)$, $J86(2.03676 < 2.0383 < 2.03999)$, $J87\ (2.13818 < 2.1395 < 2.14237)$, $J88(2.26978 < 2.2704 < 2.2719)$, $J89(2.4921 < 2.49640758 < 2.49667)$, and $J90(2.70345 < 2.70359 < 2.70831)$. For example, $L_{J_{90}} = 2.70359$, while the closest linkages achieved the lengths 2.70345 and 2.70831 from above and below. (The program correctly found for $J17$ and $J84$, and hence omitted.)

## 6. Concluding Remarks

In this paper, we show that the JZ solids J17 and J84 are exceptionally in the sense that their edge unfoldings admit to fold into regular tetrahedra. Especially, some edge unfoldings can fold into a regular tetrahedron in two or three different ways. In this research, the characterization of nets by tiling (Theorem 5) plays an important role. In general, even the decision problem that asks if a polyhedron can be folded from a given polygon is quite difficult problem [5], Chapter 25. More general framework to solve the problem is future work.

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