Union-Connections and
A Simple Readable Winning Way in $7 \times 7$ Hex

Kohei Noshita
Department of Computer Science
The University of Electro-Communications
Chofu, Tokyo 182, Japan

Summary
A notion of union-connection in the game of Hex is introduced as a generalization of Anshelevich's virtual connection. This is applied to obtain a simple easy-to-verify winning way in $7 \times 7$ Hex. A certain extension of our notion leads to a new proof for $8 \times 8$ Hex.

1. Introduction

In the game of Hex the first player Black always wins. This is proved by the well-known strategy-stealing argument [4]. On the $7 \times 7$ board J. Yang et al. first published an explicit winning way by exhaustive searching of the whole game-tree [5]. In their homepages, B. Enderton reports about his computer program for obtaining it [2], and Yang reports about $8 \times 8$ and $9 \times 9$ cases [6]. R. Hayward et al. have obtained a winning way for each of forty-nine opening positions on the $7 \times 7$ board [3].

This paper presents a simple readable complete winning way for $7 \times 7$ which is described as a small game-tree. The main idea is a generalization of the concept of virtual connections investigated by Anshelevich [1]. We introduce a union of connections for handling a wider class of connecting patterns, in that Anshelevich's AND-OR connections generate only series-parallel-graph patterns.

Our union-connection has been applied to $7 \times 7$ Hex for reducing the case-analyses of tree searching considerably, i.e., to the final eleven patterns. For each of them, Black winning move is given along with the correctness proof. Thanks to union-connection lemmas, the correctness can be checked in a direct and concise manner without the aid of a computer.

Yang's winning way for $7 \times 7$ consists of as many as forty patterns, each of which requires detailed case-analyses by further searching [5]. Hayward's winning way for $7 \times 7$ is generated by searching all the game-tree by their computer program [3]. Apparently they are too complicated for a human reader to read the correctness of their winning ways. For $8 \times 8$, no complete winning ways have been formally published.

We have applied our method to $8 \times 8$ Hex and obtained a reasonably simple description of a complete winning way. For this purpose a certain extension of our union-connection is needed.

2. Definitions and Basic Lemmas

Let $S[ab]$ be a virtual connection (VC in short) in $S$ between two places (hexagons) $a$ and $b$, where $S$ is a set of empty places [1]. Unless otherwise stated, connections are assumed to be for Black, and thus a player's name will be omitted in our notation.

By definition, $S[ab]$ holds if and only if, for any White move-sequence, Black connects $a$ and $b$ where both players make moves only at places in $S$. 

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We define a union-connection
\[ S[ a_1 b_1 | a_2 b_2 | ... | a_k b_k ] \]
if, for any White move-sequence, there exists \( i \) \( (1 \leq i \leq k) \) such that Black connects \( a_i \) and \( b_i \) in \( S \). If \( k = 1 \), union-connection is identical to VC. In this definition, the order of pairs as well as the order of two places in a pair is arbitrary (interchangeable).

We denote \( a - b \) for \( S[ab) \) when \( S \) is understood in the context. Similar notations like \( a - b Jc - d \) will be used for union-connections.

For any Black place \( x \), we simply write \( x \) to mean the set of Black places which are directly connected to \( x \) (by transitive adjacency) including itself.

Let \( t \) and \( b \) represent the top-border and the bottom-border, respectively, as shown in Figure 1. Thus \( t - b \) implies Black winning connection.

The following properties will be used for deducing various connections. The proof is straightforward.

Let \( \sigma \) denote a form of \( \alpha_1 b_1 | \alpha_2 b_2 | ... | \alpha_k b_k \) \( (k \geq 0) \).
1. \( S[ab \sigma] \) implies \( S[ab \sigma] \).
2. If \( S_1[ab \sigma] \) and \( S_2[bc] \) and \( S_1 \cap S_2 = \phi \), we have \( S[ac \sigma] \) where \( S = S_1 \cup S_2 \).

In a similar way other types of deduction rules can be introduced when necessary. Two basic lemmas are shown in Figure 1. They are also well-known properties of VC. We call a type of the upper-left connection Skip Lemma (S-lemma in short), in which \( x - y \) (and \( t - z \)) holds. We call a type of the lower-right connection Trapezoid Lemma (T-lemma in short), in which \( u - b \) holds. These lemmas will be freely used without mentioning it explicitly.

The following Union Lemma (U-lemma) is the key property of our winning way.

**Lemma (U1-lemma)**

In Figure 2 we have \( S[ tx | tb ] \), where \( S \) is the set of empty places as shown there.

Proof: Let \( ij \) denote the place at column \( i \) and row \( j \). This position in the figure forces White to make the next move only at the five places, i.e., 51, 61, 52, 43 and 53. Otherwise Black will make a move at 52, leading to \( t - x \). In case of White \( W61, W52 \) or \( W53 \), Black immediately leads to \( t - x \) by \( T \)-lemma and \( S \)-lemma. In case of \( W51 \), move-sequences beginning with \( B62 \) (e.g., \( B62 W53 B43 W52 B32 \)) lead to \( t - x \). In case of \( W43 \), the hardest move-sequence leading to \( t - b \) is

\[ W43 B53 W61 B52 W51 B32 W42 B24 W34 B25 W35 B17. \]

By combining this with other forcing sequences, we have \( t - x \) or \( t - b \).

As a corollary of this proof, we have \( U_{1+}\)-lemma in Figure 3. In a similar way, we can prove several useful variants of \( U \)-lemma as shown in Figures 4 to 7.

### 3. Our Game-Tree for the 7 × 7 Board

The complete game-tree is shown in Figure A. A node (written in the form of a move) represents a position obtained by the corresponding move. For example, B44 at the root represents the position obtained by the first Black move at 44 in the initial position.

As seen in the tree, there are twenty-two nodes for Black turn, i.e., \( a \) to \( f \), \( f.p \) to \( f.w \) and \( g.p \) to \( g.v \), among which twenty nodes are terminal leaf positions. Those twenty positions are classified into the final eleven patterns, for each of which Black winning move is shown in the tree.

In Figures 8 to 28, those winning moves along with the correctness proof and the preceding analyses are described in the order of depth-first traversal of the tree. The nodes in the tree with their corresponding Figure numbers are listed in Table B.
For a given node of White turn, the following naive reasoning determines the set of nontrivial next moves. In the *analysis* step, we preclude apparently ineffective White moves. We choose an empty place $z$ for making a fictitious Black move at $z$ which attains $t-b$. Let $S_z$ be a minimal set of empty places such that $S_z[tb]$. Note that there may be other choices for such $z$ and that, for any fixed $z$, this minimal set is not always unique. We take the intersection of all the sets $S_z \cup \{z\}$. All empty places that are not in the intersection can be precluded from White next moves. (The proof is obvious.) If the intersection is empty, this position can be regarded as a terminal leaf node.

By examining our tree more carefully along this reasoning, we can further reduce the size of the tree (e.g., from twenty-five nodes to five), though the analysis at each node becomes more complicated.

### 4. A Winning Way in $8 \times 8$ Hex

Our method has been applied to the $8 \times 8$ board to make a complete winning way. It is reasonably simple (for the complexity of $8 \times 8$), though the whole description is still too long to include here. We shall sketch the key idea of our new proof.

The total number of nodes in our game-tree is exactly 200, 37 of which are terminal leaf nodes. No deeper searches are necessary for the leaf nodes. The tree includes some loop-like paths (having similar positions), unlike our $7 \times 7$ case. The maximum length of paths from the root to leaves is eighteen. Twenty leaf nodes of 37 represent positions in which all White
next moves are precluded by the analyses. Seventeen other leaf nodes represent final-stage positions in which Black has the straightforward forcing winning move-sequences (i.e., all White moves are forced).

The initial analysis at the root for White second moves (depth 1) leaves eighteen nontrivial cases as shown in Figure 29. The first Black move is B54. For each White move, Black next move (depth 2) is:


The hardest two cases in terms of the number of nodes are h (W28) and q (W53). The easiest nine cases are c, g, i, k, l, m, n, o and p in that Black next move immediately attains $t - b$.

The most important idea is an extension of our union-connection, which is best explained by means of examples in Figure 30. The top-left T-lemma will be given an additional property named $AB$-property. In the figure, we can prove that not only $x - t$ but also the following property holds: At any time of White turn, if White occupies $A$, then Black occupies $B$.

Note that, at any time, when White occupies both $A$ and $B$ ($A$ after $B$), Black has already secured $x - t$ and is able to make a move outside the $T$ area. The bottom-right shows another example of useful lemmas having this $AB$-property. Slightly more complicated patterns of $AB$-property are also needed.

The $AB$-property serves as a connecting interface between two adjacent areas whose sets of empty places are mutually disjoint.

We have made a working library of about sixty lemmas, most of which can be proved straightforwardly, while several lemmas are relatively hard. For example, Figure 31 shows a huge lemma in the library. Proving $x - b|y - b|t - b$ may be fun.

As another example, Figure 32 shows an interesting move-sequence which appears in case W53(q). At White tenth move (W35), four other nontrivial White moves must be considered, which can be dealt with in a similar way. All other White moves from 2 to 24 are forced for preventing Black from attaining $t - b$ immediately. Note that, for any White move in the lower area before W34 in the figure, Black has an easier (or immediate) way to attain $t - b$. For example, if W28 instead of 2 (W61), then B47. It is easy to see that, by applying $AB$-property in Figure 30 to this position, we have Black forcing move-sequence for $t - b$.

References


Let $x$ be $B_{44}$. We have $t - x \mid t - y \mid t - b$. The proof is similar to $U_1$-lemma. If $W_{43}$, then $B_{53}$. For this position, we have $U_{24}$-lemma.

Figure 5 $U_2$-lemma

Let $x$ be $B_{44}$. We have $t - x \mid t - b$. The proof is similar to $U_3$-lemma. If $W_{43}$, then $B_{53}$. For this position, we have $U_{34}$-lemma.

Figure 6 $U_3$-lemma

Let $x$ be $B_{44}$. We have $t - x \mid t - b$. This position appears in the proof of $U_1$-lemma.

Figure 3 $U_{14}$-lemma

Let $x$ be $B_{44}$. Let $y$ be $B_{26}$. We have $t - x \mid t - y$.

Figure 7 $U_4$-lemma

Let $x$ be $B_{44}$. We have $t - x \mid t - b$. White blocking of $B_{71}$ leads to a similar position in the proof of $U_{14}$-lemma.

Figure 4 $U_{14}$-lemma

Let $x$ be $B_{44}$. We have $x - b$. The proof is easy. Note that $W_{45}$ $B_{36}$ $W_{35}$ $B_{55}$ leads to $x - b$.

Figure 8 $M$-lemma
For White second move, the seven moves \((a \text{ to } f)\) are to be considered. All other moves are precluded by \(M\)-lemma along with its left-shift \((\to \ast)\) and top-bottom symmetry. For example, if \(W33, B54\) makes \(t - b\) by two applications of \(M\)-lemma.

Figure 9  White second moves

If \(a \) (W27) or \(b \) (W36), then B55. Let \(x\) be B44. Let \(y\) be B55. We have \(t - x \mid t - b\) by \(U_1\)-lemma. We have \(x - y\) by \(S\)-lemma. We have \(y - b\) by \(T\)-lemma. Hence we have \(t - b\).

Figure 10  Case a and b

If c (W47) or d (W46), then B36. Let \(x\) be B44. Let \(y\) be B36. We have \(t - x \mid t - y \mid t - b\) by \(U_3\)-lemma. We have \(x - y\) by \(S\)-lemma. We have \(y - b\) by \(S'\)-lemma. Hence we have \(t - b\).

Figure 11  Cases c and d

If \(e \) (W37), then B26. Let \(x\) be B44. Let \(y\) be B26. We have \(t - x \mid t - y\) by \(U_4\)-lemma. We have \(y - b\) by \(S'\)-lemma. We have \(x - y \mid x - b\) in the asterisk set. (The proof is immediate, since W35 B45 W36 B56.) Hence we have \(t - b\).

Figure 12  Case e

If \(f \) (W35), then B45. Let \(x\) be B44. If White next move is not in the dotted set, Black moves at B52. This leads to \(t - b\), since \(t - x\) and \(x - b\) by \(T\)-lemma. Hence all the places without dots are precluded.

Figure 13  Case f (Analysis 1)

If White next move is in the minus (-) set, Black moves at B36. This leads to \(x - b\). We also have \(t - x \mid t - b\) by \(U_3\)-lemma. Hence we have \(t - b\). All the places with minuses are also precluded.

Figure 14  Case f (Analysis 2)

For White fourth move, the eight moves \((p \text{ to } w)\) are to be considered. All other moves are precluded by the above analyses in Figures 13 and 14. (We shall consider only the places in the intersection of sets which are not precluded in Analyses 1 and 2.)

Figure 15  Case f (White fourth moves)

If \(f.p \) (W37), then B56. Let \(x\) be B44. We have \(t - x \mid t - b\) by \(U_3\)-lemma. Let \(y\) be B56. We have \(x - y \mid y - b\). Hence we have \(t - b\).

Figure 16  Case f.p
If \( f.q \) (\( W36 \)) or \( f.r \) (\( W27 \)), then \( B53 \). Let \( x \) be \( B44 \). We have \( t - x \) (by \( T \)-lemma). We also have \( x - b \) in the dotted set. (The proof is easy. If \( W46 \) or \( W37, B65 \) suffices. If \( W47, B46 W37 B66 W56 B65 \) suffices.) Hence we have \( t - b \).

Figure 17 Cases \( f.q \) and \( f.r \)

If \( f.s \) (\( W43 \)), then \( B53 \). Let \( x \) be \( B44 \). We have \( t - x \) \( t - b \) by \( U_3 \)-lemma in the top-left area. We also have \( x - b \) as seen in Figure 17. Hence we have \( t - b \).

Figure 18 Case \( f.s \)

If \( f.t \) (\( W53 \)), \( f.u \) (\( W52 \)) or \( f.v \) (\( W61 \)), then \( B33 \). Let \( x \) be \( B44 \). Let \( y \) be \( B33 \). We have \( t - y \) and \( y - x \) and \( x - b \) by \( T \)-, \( S \)- and \( T \)-lemmas. Hence we have \( t - b \).

Figure 19 Cases \( f.t \), \( f.u \) and \( f.v \)

If \( f.w \) (\( W51 \)), then \( B62 \). Let \( x \) be \( B44 \). We have \( t - x \), since \( W53 B43 W52 B32 \) makes it. We also have \( x - b \) by \( T \)-lemma. Hence we have \( t - b \). Note that \( B62 \) can be replaced by \( B43 \).

Figure 20 Case \( f.w \)

If \( g \) (\( W45 \)), then \( B35 \). Let \( x \) be \( B44 \). We have \( t - x \). The proof is similar to \( U_4 \)-lemma. If White next move is not at the dotted places, Black makes \( x - b \). This implies \( t - b \). Hence all the places without dots are precluded. Note that \( 54 \) is precluded.

Figure 21 Case \( g \) (Analysis 1)

If White next move is not at the places with dots or pluses (+), \( B54 \) makes \( t - x \) (in dots by \( M \)-lemma) and \( x - b \) (in pluses by \( B26 \) or \( B46 \)). Hence 21, 22, 23, 24, 25, 31, 32, 33 and 34 are also precluded.

Figure 22 Case \( g \) (Analysis 2)

If White next move is not at the places with dots, \( B25 \) makes \( t - x \) (in dots). The proof is similar to \( U_1 \)-lemma. Note that (a) \( W43 \) (\( W52, W61 \) or \( W41 \)) \( B33 \), (b) \( W42 \) (or \( W51 \)) \( B23 \), (c) \( W33 \) \( B43 \), and (d) \( W32 \) (or \( W31 \)) \( B42 \). \( B25 \) also makes \( x - b \) (in pluses). Hence 71, 62 and 53 are also precluded.

Figure 23 Case \( g \) (Analysis 3)

For White fourth move, the seven places (\( p \) to \( v \)) are to be considered. All other places are precluded through Analyses 1 to 3.

Figure 24 Case \( g \) (White fourth moves)
If \( g.p \) (W61) or \( g.q \) (W52), then B33. Let \( x \) be B44. Let \( y \) be B33. We have \( t - y \) by \( T \)-lemma and \( x - y \) by \( S \)-lemma. We also have \( x - b \mid t - b \) by \( U_{14} \)-lemma. Hence we have \( t - b \).
Figure 25 Cases \( g.p \) and \( g.q \)

If \( g.r \) (W41), \( g.s \) (W42) or \( g.t \) (W51), then B52. Let \( x \) be B44. Let \( y \) be B52. We have \( x - y \). We also have \( t - y \). (Cases \( g.r \) and \( g.s \) are obvious. For \( g.t \), the hardest sequence B52 W61 B32 W42 B24 suffices to see this.) We have \( x - b \mid y - b \mid t - b \) by \( U_{24} \)-lemma in the bottom-right area. Hence we have \( t - b \). Note that \( g.t \) can also be done by B62.
Figure 26 Cases \( g.r \), \( g.s \) and \( g.t \)

If \( g.u \) (W43), then B53. Let \( x \) be B44. We have \( x - b \). (The proof is similar to \( U_{14} \)-lemma.) By the top-bottom symmetry, We also have \( t - x \). Hence we have \( t - b \).
Figure 27 Case \( g.u \)

If \( g.v \) (W27), then B25. Let \( x \) be B44. We have \( t - x \). (See Figure 23.) We also have \( x - b \mid t - b \) by \( U_{14} \)-lemma. Hence we have \( t - b \).
Figure 28 Case \( g.v \)

In the bottom-right, let \( y \) and \( z \) be B54 and B47, respectively. We have \( y - z \mid y - b \) and AB-property.
Figure 30 Two examples of AB-property

Let \( x \) and \( y \) be B54 and B43, respectively. We have \( x - b \mid y - b \mid t - b \).
Figure 31 A huge lemma

Figure 29 The initial analysis for White second moves in 8 \( \times \) 8 Hex

Figure 32 An example of Black winning move-sequence