局所完全ダイグラフの独立双方向支配集合問題について

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概要：ダイグラフ D に対して，頂点の部分集合 S が，任意の頂点 $u \notin S$ に対して，vu, uw が D の弧であるような $v, w \in S$ が存在するという条件を満たすとき，S を D の双方向支配集合という．また双方向支配集合 S のどの 2 頂点も隣接しないとき，S は独立であるという．すべてのダイグラフが独立双方向支配集合を持つとは限らない．本報告では，与えられたダイグラフに独立双方向支配集合が存在するかどうかを判定する問題が NP 完全であることを示す．また，ダイグラフを局所完全ダイグラフと呼ばれるクラスに限定すると，最小の独立双方向支配集合が多項式時間で計算できることを示す．

キーワード：独立双方向支配集合，完全ダイグラフ，局所完全ダイグラフ，ラウンドダイグラフ，ラウンド分解．

Independent Twin Domination in Locally Semicomplete Digraphs

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Abstract: For a digraph D, a twin dominating set of D is a set S of vertices if, for every vertex $u \notin S$, there are vertices v, w in S such that vu and uw are arcs of D. A twin dominating set S is independent if any pair of vertices in S are not adjacent. Every digraph does not have independent twin dominating set. We show that deciding whether a given digraph has a independent twin dominating set is NP-complete. Then, we also show that there is a polynomial time algorithm for computing a minimum independent dominating set for digraphs in the class of locally semicomplete digraphs.

Keywords: Independent twin domination, semicomplete digraph, locally semicomplete digraph, round digraph, round decomposition.

1. Introduction

Throughout this paper, all digraphs (directed graphs) are finite without loops and parallel arcs. For a digraph D, we denote $V(D)$ and $A(D)$ the vertex set and the arc set of D, respectively. The subdigraph induced by a subset $S \subseteq V(D)$ is denoted by $D[S]$. For a vertex $v \in V(D)$, the *open out-neighborhood* $N^+(u)$ and the *open in-neighborhood* $N^-(u)$ is defined by $N^+(u) = \{x \mid ux \in A(D)\}$ and $N^-(u) = \{y \mid yu \in A(D)\}$. The *closed out-neighborhood* and the *closed in-neighborhood* of u are $N^+[u] = N^+(u) \cup \{u\}$ and $N^-[u] = N^-(u) \cup \{u\}$, respectively. Similarly, for a subset S of vertices, $N^+(S) = \cup_{u \in S} N^+(u)$, $N^-(S) = \cup_{u \in S} N^-(u)$, $N^+[S] = N^+(S) \cup S$ and $N^-[S] = N^-(S) \cup S$. The *outdegree* $\mathrm{od}\, u$ and *indegree* $\mathrm{id}\, u$ of a vertex u is the cardinality of $N^+(u)$ and $N^-(u)$, respectively. A digraph is *strongly connected* (or just *strong*) if there exists a path from a vertex u to a vertex v for any distinct vertices u and v. A digraph is *acyclic* if it has is no directed cycle.

A subset $S \subseteq V(D)$ is a *twin dominating set* of D if $N^+[S] = V(D)$ and $N^-[S] = V(D)$. In other words, S is a twin dominating set of D if, for any vertex $u \notin S$, there are vertices $x, y \in S$ such that $ux, yu \in A(D)$. The minimum cardinality of all twin dominating set of D is called the *twin domination number* of D and denoted by

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$\gamma^*(D)$. A twin dominating set S of D is *independent* if any pair of vertices in S are non-adjacent. Note that not every digraph has an independent twin dominating set. For example, the directed cycle of odd length has no independent twin dominating set. Independent dominating sets of digraphs is closely related to a *kernel* of digraphs, for example, see [4, 5, 7, 8].

Domination and its variants in undirected graphs have been extensively studied [10, 11]. Compared with undirected graphs, there are smaller number of works for dominating set of digraphs. The concept of twin domination was proposed by Chartrand, Denkelmann, Shults and Swart [6]. In [6], sharp upper bounds of the twin domination numbers for digraphs were given, and also a Nordhaus-Gaddum type inequality for the twin domination number were presented. The author considered the related problem in de Bruijn digraphs and obtained the twin domination number of these digraphs [1, 2]. Shan et al. [14] studied the problem for generalized de Bruijn digraphs. Hasunuma [9] generalized the results in [2] by investigated the connected twin domination problem for iterated line digraphs. Given a undirected graph G and an integer k, the problem of determining whether G has a (ordinary) dominating set whose cardinality is less than k is NP-complete. Let D be a digraph obtained by replacing edges in an undirected graph G with bidirectional arcs. Then S is a twin dominating set of D if and only if S is a dominating set of G. Hence determining whether a digraph has the twin dominating set whose cardinality if less than k is also NP-complete.

In this paper, we focus on *locally semicomplete digraphs* which is a particular class of digraphs (definition of locally semicomplete digraphs are given in Section 2). We give a polynomial time algorithm for finding the minimum independent twin dominating set of a locally semicomplete digraphs.

2. Locally semicomplete digraphs

In this section, we give a definition of a locally semicomplete digraph and its properties.

A digraph is *semicomplete* if there is at least one arc between any pair of vertices. If there is exactly one arc between any pair of vertices in a semicomplete digraph, then it is called a *tournament*. A digraph is *locally semicomplete* if, for any vertex v, both of $N^+(v)$ and $N^-(v)$ induce semicomplete digraphs.

A digraph D of n vertices is *round* if the vertices can be labeled as $v_0, v_1, \ldots, v_{n-1}$ such that, for each ver-

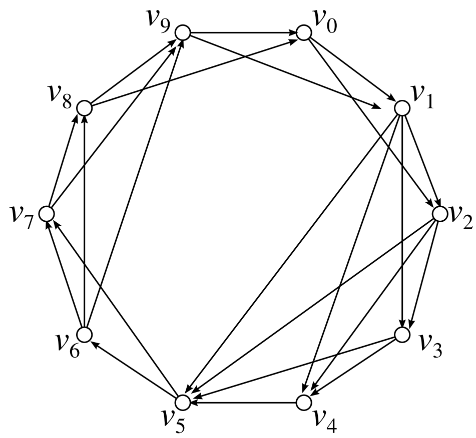

図 1 An example of a round digraph of 10 vertices.

tex v_i, $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+s}\}$, $s = \mathrm{od}\, v_i$, and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-t}\}$, $t = \mathrm{id}\, v_i$, (subscripts are taken modulo n). We refer the ordering $v_0, v_1, \ldots, v_{n-1}$ as a *round labeling* of D. Observe that every round digraph is either strong or acyclic. A round digraph is strong if and only if $\mathrm{od}\, v_i > 0$ for every v_i. Figure 1 represents an example of a strong round digraph.

There is a polynomial time algorithm to decide whether a given digraph is round and find a round labeling if it is round [12]. Structure of round digraphs were discussed in several articles, for example, [3, 5, 12]. The following theorem is shown by Huang [12].

Theorem 2.1. *Every round digraph is locally semicomplete.*

Let R be a digraph with the vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$, and let $H_0, H_1, \ldots, H_{n-1}$ be a collection of digraphs. Then $R[H_0, H_1, \ldots, H_{n-1}]$ is the new digraph obtained from R by replacing v_i with H_i and adding an arc from every vertex of H_i to every vertex of H_j if and only if $v_i v_j$ is an arc of R. If $D = R[H_0, H_1, \ldots, H_{n-1}]$, then $R, H_0, \ldots, H_{n-1}$ are subdigraphs of D. A locally semicomplete digraph D is *round decomposable* if there exists a round digraph R on n vertices such that $D = R[H_0, H_1, \ldots, H_{n-1}]$, where each H_i is a strong semicomplete digraph. $D = R[H_0, H_1, \ldots, H_{n-1}]$ is called a *round decomposition* of D.

In [4], it was proved that there exists a polynomial time algorithm to decide if a given locally semicomplete digraph has a round decomposition and to find this decomposition if it exists. It was also shown that if a locally semicomplete digraph is round decomposable, then it has a unique round decomposition.

The following property is very important for our algorithm.

Lemma 2.2 (Bang-Jensen et al. [4]). *Let D be a locally semicomplete digraph. If D is not round decomposable, the independence number of $U(D)$ is at most two.*

3. NP-completeness

As we mentioned in Section 1, not every digraph has an independent twin dominating set. In this section, we show that the problem to decide if a given digraph has an independent twin dominating set is NP-complete.

Theorem 3.1. *The problem to decide if a given digraph has an independent twin dominating set is NP-complete.*

Proof. The problem is clearly in NP.

Next we construct a reduction from a known NP-complete problem 3-SAT to the independent twin domination problem. Let $P = (X, C)$ be an instance of 3-SAT with variables $X = \{x_1, x_2, \ldots, x_n\}$ and clauses $C = \{C_1, C_2, \ldots, C_m\}$ such that each clause consists of exactly three literals. From the instance C, we construct a digraph $D(P)$ as follows. The vertex set of $D(P)$ is $\{s\} \cup \{x_1, x_2, \ldots, x_n\} \cup \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\} \cup C$. The arc set of $D(P)$ is $\{sC_j \mid 1 \leq j \leq m\} \cup \{C_j v \mid v \in C_j, 1 \leq j \leq m\} \cup \{x_i\overline{x}_i, \overline{x}_i x_i \mid 1 \leq i \leq n\}$. We show that C is satisfiable if and only if $D(P)$ has an independent twin dominating set.

Assume that C has a satisfying truth assignment. Then we create a set $S = \{s\} \cup \{x_i \mid x_i \text{ is true}\} \cup \{\overline{x}_i \mid x_i \text{ is false}\}$. It easy to see that S is an independent twin dominating set of $D(P)$.

Conversely, we suppose that $D(P)$ has an independent twin dominating set S. Since indegree of s is zero, the vertex s must be in S. Hence S does not contain vertices C_j, $1 \leq j \leq m$. Thus exactly one of x_i or $\overline{x}_i$ is in S for any $1 \leq i \leq n$. Notice that, for any $1 \leq j \leq m$, a vertex C_j is adjacent to some x_i or $\overline{x}_i$ in S. We can therefore make a satisfying truth assignment for C as follows: for each variable x_i, the value is true if $x_i \in S$, otherwise assign x_i the value false. □

Let $P = (X, C)$ be an instance of 3-SAT. Define a graph $G(P) = (V, E)$ with vertex set $V = X \cup C$ and edges $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \{x_i x_{i+1} \mid i \leq n\}$, $E_2 = \{x_i\overline{x}_i \mid 1 \leq i \leq n\}$, and $E_3 = \{x_i C_j \mid C_j \text{ contains } x_i \text{ or } \overline{x}_i\}$. A 3-SAT formula P is *planar* if the corresponding graph $G(P)$ is planar. It is known that the planar 3-SAT is NP-complete [13].

We will show that the independent twin domination problem if NP-complete even for planar digraphs. Let $P = (X, C)$ be an instance of planar 3-SAT and $G(P) = (V, E)$ be the corresponding planar graph. We construct a digraph $DG(P)$ from $G(P)$ as follows. The vertex set of $DG(P)$ is $S \cup V$, where $S = \{s_1, s_2, \ldots, s_m\}$. The arc set of $DG(P)$ is $A_0 \cup A_1 \cup A_2 \cup A_3$, where $A_0 = \{(s_j, C_j) \mid 1 \leq j \leq m\}$, and $A_1 = \{(x_i, x_{i+1}) \mid x_i x_{i+1} \in E_1\}$, $A_2 = \{(x_i, \overline{x}_i), (\overline{x}_i, x_i) \mid 1 \leq i \leq n\}$, and $A_3 = \{(C_j, x_i) \mid x_i \in C_j\}$. Figure 2 shows that an example of the planar digraph for an instance $C_1 = x_1 \vee x_2 \vee \overline{x}_3$, $C_2 = x_1 \vee \overline{x}_3 \vee x_5$, $C_3 = \overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_5$, $C_4 = \overline{x}_2 \vee x_3 \vee \overline{x}_4$, and $C_5 = x_3 \vee x_4 \vee x_5$.

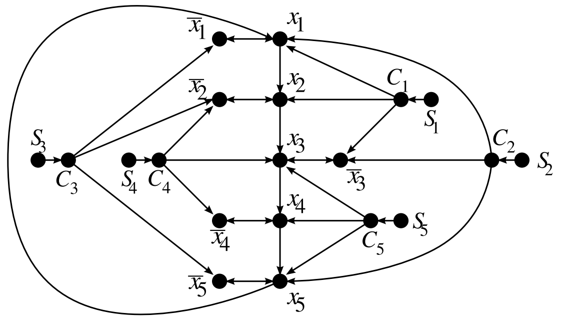

図 2 An example of the planar digraph $DG(P)$.

By a similar argument to the proof of Theorem 3.1, we can show that the 3-SAT P is satisfiable if and only if the corresponding planar digraph $DG(P)$ has an independent twin dominating set.

Theorem 3.2. *The problem to decide if a given planar digraph has an independent twin dominating set is NP-complete.*

4. Polynomial-time Algorithm

In this section, we show that there is a polynomial-time algorithm to decide if a given locally semicomplete digraph has an independent twin dominating set.

First we consider an algorithm for computing the minimum independent twin dominating set of round digraphs. If a given digraph D is acyclic, Algorithm 1 computes whether the acyclic digraph has an independent twin dominating set, and output $\gamma_i^*(D)$ if it exists.

Lemma 4.1. *Algorithm 1 computes correctly if a given acyclic digraph has an independent twin dominating set.*

Hence, we can compute $\gamma_i^*(R)$ if a round digraph R is acyclic.

Next we consider a round digraph is strongly connected. For a strong round digraph R, let R_i be the acyclic round digraph constructed as follows. The vertex set $V(R_i) = (V \setminus \{v_i\}) \cup \{v_i^0, v_i^1\}$. The arc set of R_i is obtained by removing arcs from and to v_i and then adding arcs so that $N^+(v_i^0) = N^+(v_i)$ and $N^-(v_i^1) = N^-(v_i)$.

Algorithm 1 Decide if an acyclic digraph D has an independent twin dominating set.

```
i ← 0
U ← V(D)
S ← ∅
while U is not empty do
   S_i ← the set of vertices v ∈ U such that id v = 0
   T ← N^+(S_i) \ S_i
   if T contains a vertex that has no outgoing arc then
      return "D has no independent twin dominating set"
   end if
   U ← U \ (S_i ∪ T)
   S ← S ∪ S_i
   i ← i + 1
end while
return |S|
```

The following lemma states the relationship between twin dominating sets of R and R_i, which can be proved easily and hence we omit it.

Lemma 4.2. *Let S be an independent twin dominating set of a strong round digraph R such that $v_i \in S$. Then $(S \setminus \{v_i\}) \cup \{v_i^0, v_i^1\}$ is an independent twin dominating set of R_i. Conversely, if $S \cup \{v_i^0, v_i^1\}$ is an independent twin dominating set of R_i, then $(S \setminus \{v_i^0, v_i^1\}) \cup \{v_i\}$ is an independent twin dominating set of R. Hence*

$$\gamma_i^*(R) = \min_{0 \le i \le n-1} \{\gamma_i^*(R_i) - 1\}.$$

By Lemma 4.1 and 4.2, we obtain a desired algorithm for round digraphs. If a given round digraph R is acyclic, the existence of an independent twin dominating set is computed by Algorithm 1. If R is strong, we construct R_i for $i = 0, 1, \ldots, n-1$ and computes $\gamma_i^*(R_i)$. If $\gamma_i^*(R_i)$ are undefined for all $i = 0, 1, \ldots, n-1$, then R has no independent twin dominating set. If $\gamma_i^*(R_i)$ are defined for some $i = 0, 1, \ldots, n-1$, then we obtain $\gamma_i^*(R)$ by Lemma 4.2.

Theorem 4.3. *There is a polynomial-time algorithm to decide if a given round digraph has an independent twin dominating set, and also computes the cardinality of the minimum independent twin dominating set. It runs in $O(n^2)$-time.*

Then we assume that a locally semicomplete digraph D is round decomposable.

Theorem 4.4. *If a locally semicomplete digraph D is round decomposable, there is a polynomial-time algorithm to decide if D has an independent twin dominating set, and also computes the cardinality of the minimum independent twin dominating set. It runs in $O(n^2)$-time if the round decomposition of D is given.*

Proof. Let $D = R[D_0, D_1, \ldots, D_{r-1}]$ be the round decomposition of D, where D_i are strong semicomplete digraphs.

If $|V(D_i)| = 1$ for all $i = 0, 1, \ldots, r-1$, then D is a round digraph and hence the problem can be solved in polynomial-time by Theorem 4.3. We assume that some D_i has at least two vertices. It should be noted that, if D_i has no independent twin dominating set, no independent twin dominating set of D can contain a vertex of D_i. Since D_i is semicomplete, it is easy to check if D_i has an independent twin dominating set.

Let $V(R) = \{v_0, v_1, \ldots, v_{r-1}\}$ be the vertex set of R. The digraph D has an independent twin dominating set if and only if R has an independent twin dominating set S such that $v_i \in S$ only when D_i has only one vertex or has at least two vertex and it has a universal vertex. It is checked by Theorem 4.3. The complexity of the corresponding algorithm is at most $O(n^2)$. □

Theorem 4.5. *There exists a polynomial-time algorithm to decide if a given locally semicomplete digraph has an independent twin dominating set.*

Proof. Let D be a locally semicomplete digraph. If D is semicomplete, we check if there is a universal vertex. So we assume that D is not semicomplete. If $U(D)$ has independence number two, then we can simply check that, for each pair of nonadjacent vertices, it form an independent twin dominating set. Assume that $U(D)$ has the independence number at least three. In this case D is round decomposable by Lemma 2.2, By Theorem 4.4, the minimum independent twin dominating set of D can be computed in $O(n^2)$ time if it exist. The complexity of overall algorithm is $O(n^3)$. □

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参考文献

[1] T. Araki, The k-tuple twin domination in de Bruijn and Kautz digraphs, Discrete Mathematics 308 (2008) 6406–6413.

[2] T. Araki, Connected twin domination in de Bruijn and Kautz digraphs, Discrete Mathematics 309 (2009) 6229–6234.

[3] J. Bang-Jensen, Locally semicomplete digraphs: a generalization of tournaments, Journal of Graph Theory 14 (3) (1990) 371–390.

[4] J. Bang-Jensen, Y. Guo, G. Gutin, L. Volkmann, A classification of locally semicomplete digraphs, Discrete Mathematics 167/168 (1997) 101–114.

[5] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd Edition, Springer-Verlag London,

2009.
[6] G. Chartrand, P. Dankelmann, M. Schultz, H. C. Swart, Twin domination in digraphs, Ars Combinatoria 67 (2003) 105–114.
[7] A. S. Fraenkel, Planar kernel and grundy with $d \leq 3$, $d^+ \leq 2$, $d^- \leq 2$ are NP-complete, Discrete Applied Mathematics 3 (1981) 257–262.
[8] G. Gutin, T. Kloks, C. M. Lee, A. Yeo, Kernels in planar digraphs, Journal of Computer and System Sciences 71 (2005) 174–184.
[9] T. Hasunuma, M. Otani, On the (h, k)-domination numbers of iterated line digraphs, Discrete Applied Mathematics 160 (12) (2012) 1859–1863.
[10] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[11] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
[12] J. Huang, Which digraphs are round?, Australasian Journal of Combinatorics 19 (1999) 203–208.
[13] D. Lichtenstein, Planar formulae and their uses, SIAM Journal on Computing 11 (1981) 329–343.
[14] E. Shan, Y. Dong, Y. Cheng, The twin domination number in generalized de Bruijn digraphs, Information Processing Letters 109 (2009) 856–860.