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不動点定理によるドロネー性の確認
Characterizing Delaunay Graphs via Fixed Point Theorem

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Abstract: This paper discusses a problem for determining whether a given plane graph is a Delaunay graph, i.e., whether it is topologically equivalent to a Delaunay triangulation. There exists a theorem which characterizes Delaunay graphs and yields a polynomial time algorithm for the problem only by solving a certain linear inequality system. The theorem was proved by Rivin based on arguments of hyperbolic geometry. Independently, Hiroshima, Miyamoto and Sugihara gave another proof of the theorem based on primitive arguments on Euclidean geometry. Unfortunately, the existing proofs of the theorem are rather difficult or long. In this paper, we give a simple proof of the theorem characterizing Delaunay graphs, which is based on the fixed point theorem.

1. Introduction

The two-dimensional Delaunay triangulation and its dual, the Voronoi diagram, are fundamental concepts in computational geometry, and have many practical applications such as interpolation and mesh generation [1], [3], [8]. It is also important to recognize Delaunay triangulations. The recognition problem can be divided into two types: geometric and combinatorial. The geometric problem is to judge whether a given drawing is a Delaunay triangulation. The combinatorial problem, which is discussed in this paper, determines whether a given embedded graph is topologically equivalent to a Delaunay triangulation. The combinatorial problem is important not only theoretically but also practically because it is closely related to the design of a topologically consistent algorithm for constructing the Delaunay/Voronoi diagram in finite-precision arithmetic [7], [11], [12].

Hodgson et al. [6] characterized the convex polyhedra that can be inscribed in a sphere, and constructed a polynomial time algorithm for judging whether a given graph is realizable as a convex polyhedron with all the vertices on a common sphere. On the basis of this characterization, Rivin [9], [10] reduced the recognition problem on the Delaunay graph to a certain linear programming problem, and thus gave a polynomial time algorithm. His proof was based on sophisticated arguments about hyperbolic geometry, and hence is not easy to understand. Almost the same time Hiroshima et al. independently found the same algorithm [5]. Their proof is simple in the sense that it is based on primitive arguments on Euclidean geometry, but the proof is long and intricate.

In this paper, we give a simple short proof of the theorem characterizing Delaunay graphs by employing the fixed point theorem. After making preparations in Section 2, we give our main result (a simple proof) in Section 3.

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2. Preliminaries

2.1 Delaunay Graph

First, we briefly review the notion of a Delaunay triangulation. Given a set of mutually distinct points $P \subseteq \mathbb{R}^2$, a *Delaunay triangulation* of P is commonly defined as a triangulation of P satisfying the property that the circumcircle of each inner cell (triangle) contains no point of P in its interior. A Delaunay triangulation of P is also known as the planar dual of the *Voronoi diagram* of P. A Delaunay triangulation is called *non-degenerate* if and only if it satisfies the conditions that no three vertices on the outermost cell are collinear, and no four vertices lie on a common circle that circumscribes an inner cell.

Next, we give a definition of *combinatorial triangulation*. Let G be an undirected graph with vertex set V and edge set E. We assume that G is connected and plane graph (planar graph embedded in the 2-dimensional plane) without selfloops and parallel edges. The outermost cell is unbounded while the other cells, called *inner cells*, are bounded. We also assume that all the inner cells are bounded by exactly three edges. For each inner cell, we associate a directed 3-cycle which is obtained from an undirected 3-cycle of G forming the boundary of the cell by directing edges counterclockwise. Let C be the set of all the directed 3-cycles corresponding to all the inner cells of G. A combinatorial triangulation is defined by a triplet (V, E, C). In the rest of this paper, we write $G = (V, E, C)$ and concentrate our attention on the topological structure of G; we do not care about the actual positions at which the vertices are placed. When a given undirected graph, which is defined by vertex set V and edge set E, is 2-connected, we say that a combinatorial triangulation (V, E, C) is *2-connected*.

In the following, we introduce some notions related to a problem for judging whether a given combinatorial triangulation is obtained from a Delaunay triangulation. Given a combinatorial triangulation $G = (V, E, C)$, we seek an injection $\psi : V \to \mathbb{R}^2$ satisfying that the set of points $\{\psi(v) \in \mathbb{R}^2 \mid v \in V\}$ and the set of line segments between pairs in $\{\{\psi(u), \psi(v)\} \mid \{u, v\} \in E\}$ define a Delaunay triangulation. A map ψ satisfying the above conditions is called a *Delaunay realization*, if it exists. When a combinatorial triangulation G has a Delaunay realization, we say that G is a *Delaunay graph*. In particular, if a corresponding Delaunay triangulation is non-degenerate, then G is called a *non-degenerate Delaunay graph*.

2.2 Characterizing Delaunay Graphs

In this subsection, we briefly review an inequality system which characterizes (non-degenerate) Delaunay graphs. Given a combinatorial triangulation $G = (V, E, C)$, we denote the elements of C by $c_0, c_1, \ldots, c_{|C|-1}$. For each cycle $c_i \in C$, we introduce three variables $x_{3i+1}, x_{3i+2}, x_{3i+3}$ assigned to three vertices in c_i. In the rest of this paper, we interpret these variables as angles in degrees at the corresponding corner of a triangle defined by cycle c_i. So, let us call these variables *angle variables*. There are $3|C|$ angle variables. We denotes the index set of angle variables by $J := \{1, 2, \ldots, 3|C|\}$. For example, a combinatorial triangulation shown in Figure 1 has nine angle variables $x_1, x_2, \ldots, x_9$.

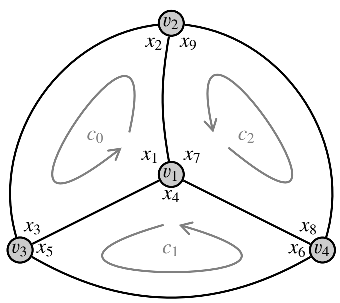

Fig. 1 Combinatorial triangulation and angle variables

A vertex of a combinatorial triangulation G is called an *outer vertex* if it is on the boundary of the outermost cell, and an *inner vertex* otherwise. Similarly, an edge of G is called an *outer edge* if it is on the boundary of the outermost cell, and an *inner edge* otherwise. In the rest of this paper, we denote a set of outer vertices and a set of inner vertices by V^{outer} and V^{inner} respectively.

If a given combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph, a corresponding vector of angle variables, defined by a Delaunay realization, satisfies the following conditions.

C1 For each cycle in C, the sum of the associated three angle variables is equal to 180.

C2 For each inner vertex, the sum of all the associated angle variables is equal to 360.

C3 For each outer vertex, the sum of all the associated angle variables is at most 180.

C4 For each inner edge, the sum of the associated pair of the facing angle variables (i.e., the angle variables corresponding to the vertices that are on the same cycle as, but are not incident to, the inner edge) is at most 180.

C5 Each angle variable is positive.

For example, if we consider the combinatorial triangulation in Figure 1, the above conditions give the following linear inequality system;

$$
\begin{array}{ll}
\text{(C1) defined by } c_0: & x_1 + x_2 + x_3 = 180, \\
\text{(C1) defined by } c_1: & x_4 + x_5 + x_6 = 180, \\
\text{(C1) defined by } c_2: & x_7 + x_8 + x_9 = 180, \\
\text{(C2) defined by } v_1: & x_1 + x_4 + x_7 = 360, \\
\text{(C3) defined by } v_2: & x_2 + x_9 \le 180, \\
\text{(C3) defined by } v_3: & x_3 + x_5 \le 180, \\
\text{(C3) defined by } v_4: & x_6 + x_8 \le 180, \\
\text{(C4) defined by } \{v_1, v_2\}: & x_3 + x_8 \le 180, \\
\text{(C4) defined by } \{v_1, v_3\}: & x_2 + x_6 \le 180, \\
\text{(C4) defined by } \{v_1, v_4\}: & x_5 + x_9 \le 180, \\
\text{(C5)}: & x_1, x_2, \ldots, x_9 > 0.
\end{array}
$$

Figure 2 gives an example of a Delaunay realization of the combinatorial triangulation in Figure 1.

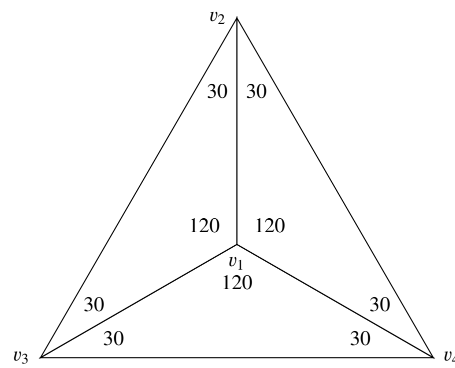

Fig. 2 Realized Delaunay triangulation.

Unfortunately, the values of the angle variables satisfying all the conditions C1–C5 do not necessarily correspond to a Delaunay triangulation. For example, the combinatorial triangulation in Figure 1 has a vector of angle variables defined by

$$
\begin{aligned}
& x_1 = x_4 = x_7 = 120,\ x_2 = x_5 = x_8 = 32, \\
& x_3 = x_6 = x_9 = 28,
\end{aligned}
$$

which satisfies C1–C5, but it does not correspond to any triangulations. If we try to draw the diagram using these angle values, we come across an inconsistency as shown in Figure 3. In order to avoid this inconsistency, we still need other conditions described below.

Let $c \in C$ be an inner cell with three vertices $v_\alpha,\ v_\beta,\ v_\gamma$, and $x_i,\ x_j,\ x_k$ be three angle variables corresponding to the three vertices, respectively. We say that x_j is *cc-facing* (meaning "facing counterclockwise") around v_α and x_k is *c-facing* (meaning "facing clockwise") around v_α. In Figure 4, for example, $x_2,\ x_5,\ x_8$ are cc-facing around v_1 while $x_3,\ x_9,\ x_6$ are c-facing around v_1.

For any inner vertex $v \in V^{\text{inner}}$, let $X_v^{\text{CC}} \subseteq J$ be indices of cc-facing angle variables around v, and $X_v^{\text{C}} \subseteq J$ be indices of c-facing angle variables around v. Furthermore, we introduce a function

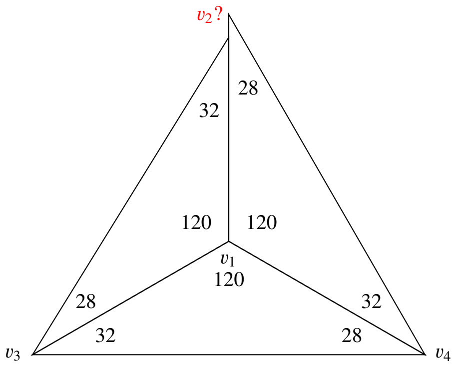

Fig. 3 Angle variables satisfying C1–C5.

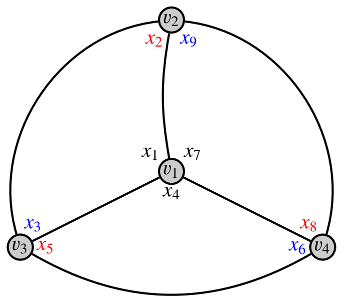

Fig. 4 x_3, x_6, x_9 are c-facing and x_2, x_5, x_8 are cc-facing around v_1.

$$F_v(\boldsymbol{x}) := \frac{\prod_{j \in X_v^{\mathrm{CC}}} \sin x_j}{\prod_{j \in X_v^{\mathrm{C}}} \sin x_j}, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^J$ is a vector of angle variables (in degrees). We only consider angle variables satisfying $0 < x_j < 180$ ($\forall j \in J$), and hence we get $0 < F_v(\boldsymbol{x}) < \infty$.

Now we describe a necessary and sufficient condition that a combinatorial triangulation becomes a Delaunay graph.

Theorem 1 ([5]) A 2-connected combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph if and only if the set of conditions C1–C6 is satisfiable, where

C6 $F_v(\boldsymbol{x}) = 1$ for any inner vertex $v \in V^{\mathrm{inner}}$.

It is not so difficult to prove the above theorem. For example, Hiroshima, Miyamoto and Sugihara gave a short and elementary proof in their paper [5].

If we restrict the Delaunay triangulations to non-degenerate ones, the conditions C3 and C4 are respectively changed in the following way.

C3' For each outer vertex, the sum of all the associated angle variables is *less than* 180.

C4' For each inner edge, the sum of the associated pair of the angle values facing the edge is *less than* 180.

A non-degenerate version of Theorem 1 is as follows.

Theorem 2 ([5]) A 2-connected combinatorial triangulation $G = (V, E, C)$ is a non-degenerate Delaunay graph if and only if the set of conditions C1, C2, C3', C4', C5 and C6 is satisfiable.

Thus, we get a necessary and sufficient condition for a combinatorial triangulation to be a (non-degenerate) Delaunay graph. However, the conditions stated in Theorems 1 and 2 are not useful for the recognition of a Delaunay graph, because we do not know any finite-step algorithm for judging the satisfiability of these conditions.

3. Main Result

Now we describe a theorem which yields an efficient method for recognizing Delaunay graphs. The following theorem says that when we only need to judge whether a given combinatorial triangulation is a Delaunay graph (or not), we can drop condition C6, surprisingly.

Theorem 3 ([5], [9], [10]) A 2-connected combinatorial triangulation $G = (V, E, C)$ is a Delaunay graph if and only if the set of conditions C1–C5 is satisfiable.

We can judge the satisfiability of the set of conditions C1–C5 in finite steps because the conditions C1 through C5 are linear in the variables and the method for checking their satisfiability has been established using linear programming (see [5] for detail). Especially, the obtained linear programming problem satisfies that all the non-zero coefficients are $+1$ or -1, and thus it is solvable in strongly polynomial time [4].

The following theorem deals with the non-degenerate case.

Theorem 4 ([5], [9], [10]) A 2-connected combinatorial triangulation $G = (V, E, C)$ is a non-degenerate Delaunay graph if and only if the set of conditions C1, C2, C3', C4' and C5 is satisfiable.

We employ the fixed point theorem and give simple proofs of Theorem 3 and Theorem 4.

Theorem 5 (Fixed Point Theorem [2])
Every continuous map $f : B^m \to B^m$ defined on an m-dimensional closed ball B^m has a fixed point (a point $x \in B^m$ with $f(x) = x$).
It is well known that we can extend the above theorem to a continuous map defined on a convex compact set.

Before describing our proof, we give a sketch of an important procedure, which transforms a feasible solution of the linear inequality system defined by C1–C5. Let us recall a vector of angle variables shown in Figure 3, that satisfies conditions C1–C5, but not C6. Now we construct a (new) vector by increasing angle variables c-facing around the inner vertex v_1 by α degree, and decreasing angle variables cc-facing around v_1 by α degree. After this procedure, conditions C1–C4 are preserved. When we set $\alpha = 2$, the obtained vector of angle variables, shown in Figure 2, satisfies conditions C1–C6.

Now we describe the above procedure precisely. Given a non-negative vector $\boldsymbol{x} \geq \boldsymbol{0}$ of angle variables satisfying C1–C4, an inner vertex v and a real number α, we introduce a vector $\boldsymbol{x}(\alpha)$ defined by

$$x(\alpha)_j = \begin{cases} x_j + \alpha, & j \in X_v^{\mathrm{CC}}, \\ x_j - \alpha, & j \in X_v^{\mathrm{C}}, \\ x_j, & \text{otherwise}. \end{cases} \tag{2}$$

The following lemma shows some properties of $\boldsymbol{x}(\alpha)$.

Lemma 1 Let $\boldsymbol{x} \geq \boldsymbol{0}$ be a non-negative vector of angle variables satisfying C1–C4 and $\boldsymbol{x}(\alpha)$ be a vector defined by (2) w.r.t. an inner vertex $v \in V^{\mathrm{inner}}$. For any $\alpha \in \mathbb{R}$, vector $\boldsymbol{x}(\alpha)$ satisfies conditions C1–C4. We define

$$\alpha_{\max} = \max\{\alpha \in \mathbb{R} \mid \boldsymbol{x}(\alpha) \geq \boldsymbol{0}\},$$
$$\alpha_{\min} = \min\{\alpha \in \mathbb{R} \mid \boldsymbol{x}(\alpha) \geq \boldsymbol{0}\}.$$

If $\alpha_{\min} < \alpha_{\max}$, then $F_v(\boldsymbol{x}(\alpha)) : (\alpha_{\min}, \alpha_{\max}) \to \mathbb{R}$ is a continuous monotone increasing function w.r.t. α.

Proof 1 It is easy to show that $\boldsymbol{x}(\alpha)$ satisfies conditions C1–C4. The continuity of $F_v(\boldsymbol{x}(\alpha))$ with respect to α is obvious. Let $C(v) \subseteq C$ be a set of cycles including v. For each cycle $c' \in C(v)$, angle variable $x^{\mathrm{CC}}_{c'}$ ($x^{\mathrm{C}}_{c'}$) denotes associated cc-facing (c-facing) angle variable around v. Condition C1, non-negativity of $\boldsymbol{x}$, and inequality $\alpha_{\min} < \alpha_{\max}$ imply that $0 < x^{\mathrm{CC}}_{c'} + x^{\mathrm{C}}_{c'} \leq 180$ ($\forall c' \in C(v)$). We transform the following differentiation and obtain that

$$\begin{aligned}
&\frac{\mathrm{d}\log F_v(\boldsymbol{x}(\alpha))}{\mathrm{d}\alpha}\\
&= \sum_{j \in X^{\mathrm{CC}}_v} \frac{\mathrm{d}\log \sin(x_j + \alpha)}{\mathrm{d}\alpha} - \sum_{j \in X^{\mathrm{C}}_v} \frac{\mathrm{d}\log \sin(x_j - \alpha)}{\mathrm{d}\alpha}\\
&= \sum_{j \in X^{\mathrm{CC}}_v} \frac{\cos(x_j + \alpha)}{\sin(x_j + \alpha)} + \sum_{j \in X^{\mathrm{C}}_v} \frac{\cos(x_j - \alpha)}{\sin(x_j - \alpha)}\\
&= \sum_{c' \in C(v)} \left(\frac{\cos(x^{\mathrm{CC}}_{c'} + \alpha)}{\sin(x^{\mathrm{CC}}_{c'} + \alpha)} + \frac{\cos(x^{\mathrm{C}}_{c'} - \alpha)}{\sin(x^{\mathrm{C}}_{c'} - \alpha)} \right)\\
&= \sum_{c' \in C(v)} \frac{\sin(x^{\mathrm{CC}}_{c'} + x^{\mathrm{C}}_{c'})}{\sin(x^{\mathrm{CC}}_{c'} + \alpha)\sin(x^{\mathrm{C}}_{c'} - \alpha)} > 0,
\end{aligned}$$

where the last inequality is derived from the facts that (1) $\forall \alpha \in (\alpha_{\min}, \alpha_{\max})$, $\forall c' \in C(v)$, $\sin(x^{\mathrm{CC}}_{c'} + \alpha)\sin(x^{\mathrm{C}}_{c'} - \alpha) > 0$ (2) $\forall c' \in C(v)$, $\sin(x^{\mathrm{CC}}_{c'} + x^{\mathrm{C}}_{c'}) \geq 0$, and (3) $\exists c' \in C(v)$, $\sin(x^{\mathrm{CC}}_{c'} + x^{\mathrm{C}}_{c'}) > 0$ (obtained from C2). Thus, both $\log F_v(\boldsymbol{x}(\alpha))$ and $F_v(\boldsymbol{x}(\alpha))$ are monotonically increasing.

In the following, we show that if there exists a vector of angle variables satisfying C1–C5, then there also exists a vector of angle variables satisfying C1–C6 which is obtained by adopting the above procedure around all inner vertices simultaneously.

Proof 2 (Proof of Theorem 3.) From Theorem 1, we only have to show that we once obtain angle variables satisfying C1–C5 there is a vector of angle variables satisfying C1–C6.
Let $\boldsymbol{b} \in \mathbb{R}^J$ be a vector of angle variables satisfying conditions C1–C5, where $J = \{1, 2, \ldots, 3|C|\}$ is a set of indices of angle variables. We define a matrix M whose rows are indexed by J, columns are indexed by the vertex set V, and each entry m_{iv} is defined as follows:

$$m_{iv} = \begin{cases} 1, & \text{angle variable } x_i \text{ is cc-facing around } v,\\ -1, & \text{angle variable } x_i \text{ is c-facing around } v,\\ 0, & \text{otherwise.} \end{cases}$$

Figure 5 shows a matrix M corresponding to Figure 1.
Let $\widetilde{M}$ be a column submatrix of M corresponding to inner vertices V^{inner}. It is easy to see that the vector of angle variables $\widetilde{M}\boldsymbol{y} + \boldsymbol{b}$ is obtained from $\boldsymbol{b}$ by increasing angle variables c-facing around the inner vertex v by y_v, and decrease angle variables cc-facing around v by y_v, for each inner vertex $v \in V^{\mathrm{inner}}$. Lemma 1 directly implies that for any vector $\boldsymbol{y} \in \mathbb{R}^{V^{\mathrm{inner}}}$, a vector of angle variables $\widetilde{M}\boldsymbol{y} + \boldsymbol{b}$ also satisfies conditions C1–C4.
We introduce a subset $\Omega \subseteq \mathbb{R}^{V^{\mathrm{inner}}}$ defined by

$$\Omega := \left\{ \boldsymbol{y} \in \mathbb{R}^{V^{\mathrm{inner}}} \,\middle|\, \widetilde{M}\boldsymbol{y} + \boldsymbol{b} \geq \boldsymbol{0} \right\}.$$

	v_1	v_2	v_3	v_4
x_1	0	−1	1	0
x_2	1	0	−1	0
x_3	−1	1	0	0
x_4	0	0	−1	1
x_5	1	0	0	−1
x_6	−1	0	1	0
x_7	0	1	0	−1
x_8	1	−1	0	0
x_9	−1	0	0	1

Fig. 5 A matrix M corresponding to Figure 1.

Here, we briefly prove the boundedness of Ω by showing that every vector $\boldsymbol{y} \in \Omega$ satisfies $-180|V|\boldsymbol{1} \leq \boldsymbol{y} \leq 180|V|\boldsymbol{1}$. Let $\{u, v\}$ be an inner edge of G. Since $\{u, v\}$ is an inner edge, there exists an angle b_j (in the vector $\boldsymbol{b}$) which is both c-facing around u and cc-facing around v. There also exists an angle $b_{j'}$ (in vector $\boldsymbol{b}$) which is both cc-facing around u and c-facing around v. When both u and v are inner vertices, every vector $\boldsymbol{y} \in \Omega$ satisfies

$$-180 \leq -b_{j'} \leq y_u - y_v \leq b_j \leq 180. \tag{3}$$

If $(u, v) \in V^{\mathrm{inner}} \times V^{\mathrm{outer}}$, then we have

$$-180 \leq -b_{j'} \leq y_u \leq b_j \leq 180. \tag{4}$$

For any inner vertex u, there exists a minimal path Γ_u on G connecting u and an outer vertex. From the minimality, Γ_u consists of inner edges. The telescoping sum of inequalities (3) and (4) w.r.t. inner edges in Γ_u gives

$$-180|V| \leq y_u \leq 180|V|.$$

From the above, Ω becomes a compact convex set.
For any pair $(\boldsymbol{y}, v) \in \Omega \times V^{\mathrm{inner}}$, we define following two values:

$$\alpha_{\max}(\boldsymbol{y}, v) := \max\{\alpha \in \mathbb{R} \mid \boldsymbol{y} + \alpha \boldsymbol{e}_v \in \Omega\},$$
$$\alpha_{\min}(\boldsymbol{y}, v) := \min\{\alpha \in \mathbb{R} \mid \boldsymbol{y} + \alpha \boldsymbol{e}_v \in \Omega\},$$

where $\boldsymbol{e}_v \in \{0, 1\}^{V^{\mathrm{inner}}}$ is a unit vector whose entry is equal to 1 if and only if the corresponding index is equal to v. (Here we note that both the maximum and the minimum always exist, because Ω is a bounded closed set and is nonempty; clearly $\boldsymbol{b} \in \Omega$.) Since $\boldsymbol{y} \in \Omega$, inequalities $\alpha_{\min}(\boldsymbol{y}, v) \leq 0 \leq \alpha_{\max}(\boldsymbol{y}, v)$ hold. When $\alpha_{\min}(\boldsymbol{y}, v) < \alpha_{\max}(\boldsymbol{y}, v)$, we have that

$$\lim_{\alpha \to \alpha_{\max}(\boldsymbol{y}, v)} F_v(\boldsymbol{y} + \alpha \boldsymbol{e}_v) = +\infty,$$
$$\lim_{\alpha \to \alpha_{\min}(\boldsymbol{y}, v)} F_v(\boldsymbol{y} + \alpha \boldsymbol{e}_v) = +0,$$

and thus Lemma 1 and the intermediate value theorem imply that there exists a unique value α^* in the open interval $(\alpha_{\min}(\boldsymbol{y}, v), \alpha_{\max}(\boldsymbol{y}, v))$ satisfying equality $F_v(\boldsymbol{y} + \alpha^* \boldsymbol{e}_v) = 1$. Now we introduce a map $f_v : \Omega \to \Omega$ for each $v \in V^{\mathrm{inner}}$ defined by

$$f_v(\boldsymbol{y}) = \begin{cases} \boldsymbol{y}, & \text{if } \alpha_{\min}(\boldsymbol{y}, v) = \alpha_{\max}(\boldsymbol{y}, v) = 0,\\ \boldsymbol{y} + \alpha^* \boldsymbol{e}_v, & \text{if } \alpha_{\min}(\boldsymbol{y}, v) < \alpha_{\max}(\boldsymbol{y}, v), \end{cases}$$

where α^* is a unique value satisfying $F_v(\boldsymbol{y} + \alpha^* \boldsymbol{e}_v) = 1$. It is obvious that for each inner vertex v, the corresponding map f_v is continuous.
Lastly, we define a map $f : \Omega \to \Omega$ as:

$$f(\boldsymbol{y}) := \frac{1}{|V^{\text{inner}}|} \sum_{v \in V^{\text{inner}}} f_v(\boldsymbol{y}),$$

where $f(\boldsymbol{y})$ is the gravity center of vectors $\{f_v(\boldsymbol{y}) \mid v \in V^{\text{inner}}\}$. Since f_v is continuous for each inner vertex v, f is also continuous.

Now we apply the fixed point theorem to the continuous map f and obtain a result that there exists a fixed point $\boldsymbol{y}^* \in \Omega$, i.e., $\boldsymbol{y}^*$ satisfies $f(\boldsymbol{y}^*) = \boldsymbol{y}^*$.

Every fixed point $\boldsymbol{y}^*$ satisfies that

$$\begin{aligned} &\forall v \in V^{\text{inner}}, \\ &\alpha_{\min}(\boldsymbol{y}^*, v) = \alpha_{\max}(\boldsymbol{y}^*, v) = 0 \text{ or } F_v(\boldsymbol{y}^*) = 1. \end{aligned} \tag{5}$$

Otherwise, there exists at least one inner vertex v' satisfying $\alpha_{\min}(\boldsymbol{y}^*, v') < \alpha_{\max}(\boldsymbol{y}^*, v')$ and $F_{v'}(\boldsymbol{y}^*) \neq 1$. Then v' also satisfies $f_{v'}(\boldsymbol{y}^*) \neq \boldsymbol{y}^*$, which implies $f(\boldsymbol{y}^*) \neq \boldsymbol{y}^*$. It is a contradiction.

We have shown that there exists a non-negative vector of angle variables satisfying C1–C4. Next we discuss condition C5, which also yields condition C6. In the following, we show that $\widetilde{M}\boldsymbol{y}^* + \boldsymbol{b} > \boldsymbol{0}$ for *any* fixed point $\boldsymbol{y}^*$.

When a vertex v satisfies $F_v(\boldsymbol{y}^*) = 1$, it is obvious that $\alpha_{\min}(\boldsymbol{y}^*, v) < 0 < \alpha_{\max}(\boldsymbol{y}^*, v)$. Since $\boldsymbol{y}^*$ is a fixed point, property (5) implies that for any $v \in V^{\text{inner}}$,

$$\alpha_{\min}(\boldsymbol{y}^*, v) = 0 \text{ if and only if } \alpha_{\max}(\boldsymbol{y}^*, v) = 0.$$

Put $\boldsymbol{x}^* = \widetilde{M}\boldsymbol{y}^* + \boldsymbol{b}$. If an inner vertex v has a cc-facing angle variable x_j satisfying $x_j^* = 0$, then $\alpha_{\min}(\boldsymbol{y}^*, v) = 0$ and thus $\alpha_{\max}(\boldsymbol{y}^*, v) = 0$, which implies that v also has a c-facing angle variable $x_{j'}$ satisfying $x_{j'}^* = 0$. Similarly, when an inner vertex v has a c-facing angle variable x_j satisfying $x_j^* = 0$, then $\alpha_{\max}(\boldsymbol{y}^*, v) = 0$ and thus $\alpha_{\min}(\boldsymbol{y}^*, v) = 0$, which implies that v also has a cc-facing angle variable $x_{j'}$ satisfying $x_{j'}^* = 0$.

Let us consider a directed graph H whose incident matrix is $M^\top$: i.e., a directed graph obtained from G by substituting a pair of parallel arcs with opposite direction for each edge in E (see Figure 6). Digraph H has vertex set V and edge set J, which is an index set of angle variables. Each angle variable x_j corresponds to an arc in H from u to v where x_j is a c-facing variable around u and cc-facing variable around v.

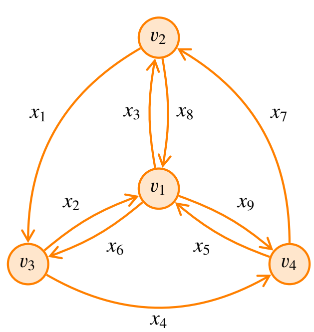

Fig. 6 A directed graph (V, J) corresponding to Figure 1.

If an arc a of H satisfies that the corresponding angle variable, denoted by x_a, satisfies $x_a^* = 0$, we say that a is a *critical* arc. In the directed graph, if an inner vertex v has an incoming critical arc, then v also has at least one outgoing critical arc.

Now we show $\boldsymbol{x}^* > \boldsymbol{0}$. Assume on the contrary that there exists an angle variable x_j satisfying $x_j^* = 0$. Then, there exists a critical arc in H. Let A_0 be a set of critical arcs. From the above discussion, a digraph defined by (V, A_0) has either (Case 1) "a directed elementary path Γ_1 connecting a pair of outer vertices and passing only inner vertices" or (Case 2) "a directed elementary cycle Γ_2 consisting of inner vertices."

Case 1. Let $\chi_1 \in \{0, 1\}^J$ be a characteristic vector of the set of arcs in Γ_1. Since Γ_1 consists of critical edges, $\chi_1^\top \boldsymbol{x}^* = 0$ hold. Every inner vertex v has an incoming arc in Γ_1 if and only if v has an outgoing arc in Γ_1. Accordingly, the equality $\chi_1^\top \widetilde{M} = \boldsymbol{0}$ hold. Thus we have that

$$0 = \chi_1^\top \boldsymbol{x}^* = \chi_1^\top(\widetilde{M}\boldsymbol{y}^* + \boldsymbol{b}) = \chi_1^\top \widetilde{M}\boldsymbol{y}^* + \chi_1^\top \boldsymbol{b} = \chi_1^\top \boldsymbol{b} > 0.$$

Contradiction.

Case 2. Let $\chi_2 \in \{0, 1\}^J$ be a characteristic vector of the set of arcs in Γ_2. Since Γ_2 consists of inner vertices and critical edges, both $\chi_2^\top \widetilde{M} = \boldsymbol{0}$ and $\chi_2^\top \boldsymbol{x}^* = 0$ hold. Thus we have that

$$0 = \chi_2^\top \boldsymbol{x}^* = \chi_2^\top(\widetilde{M}\boldsymbol{y}^* + \boldsymbol{b}) = \chi_2^\top \widetilde{M}\boldsymbol{y}^* + \chi_2^\top \boldsymbol{b} = \chi_2^\top \boldsymbol{b} > 0.$$

Contradiction.

Now we have shown that every fixed point $\boldsymbol{y}^*$ satisfies condition C5 and thus every inner vertex v satisfies $\alpha_{\min}(\boldsymbol{y}^*, v) < 0 < \alpha_{\max}(\boldsymbol{y}^*, v)$. From property (5), every inner vertex v satisfies $F_v(\boldsymbol{y}^*) = 1$. As a consequence, condition C6 is satisfied.

A proof of Theorem 4 is almost the same. Actually, we only have to replace C3 and C4 with C3' and C4' respectively in our proofs of Lemma 1 and Theorem 3.

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