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テクニカルノート

完全独立全域木の十分条件について

荒木 徹[1,a)]

概要：T_1, T_2 をグラフ G の二つの全域木とする．G の任意の 2 頂点 u, v に対して，T_1 上の u-v パスと T_2 上の u-v パスが内素であるとき，T_1 と T_2 は完全独立であるという．本報告では，グラフに二つの完全独立全域木が存在するための必要十分条件を示す．その特徴付けを用いて，さらに二つの十分条件を証明する．まず，n 頂点のグラフの最小次数が $n/2$ 以上であるなら，そのグラフに二つの完全独立全域木が存在することを示す．続いて，G が 2 連結ならその 2 乗 G^2 に二つの完全独立全域木が存在することを示す．これらの条件は，いずれもグラフにハミルトニアンサイクルが存在するための十分条件として知られている条件である．

キーワード：全域木，完全独立全域木，Dirac の定理，Fleischner の定理.

Sufficient Conditions for Completely Independent Spanning Trees

Toru Araki[1,a)]

Abstract: Two spanning trees T_1 and T_2 of a graph G are completely independent if, for any two vertices u and v, the paths from u to v in T_1 and T_2 are internally disjoint. In this report, we show that two sufficient conditions for the existence of completely independent spanning trees. First we show that a graph of n vertices has two completely independent spanning trees if the minimum degree of the graph is at least $n/2$. Then we prove that the square of a 2-connected graph has two completely independent spanning trees. These conditions are known to be sufficient conditions for Hamiltonian graphs.

Keywords: Spanning tree, Completely independent spanning trees, Dirac's Theorem, Fleischner's Theorem.

1. Introduction

Let G be a simple undirected graph. The vertex set and the edge set of G is denoted by $V(G)$ and $E(G)$, respectively. The *degree* of a vertex v is the number of adjacent vertices of v, and is denoted by $\deg_G v$. A *leaf* of G is a vertex of degree 1. The *distance* between u and v is the minimum length of paths from u to v, and is denoted by $dist_G(u, v)$. For a subset $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. For a vertex x, $G - x$ is a graph obtained by removing x and every edges incident with x. For two vertices x, y such that $xy \notin E(G)$, $G + xy$ is a graph obtained by adding an edge xy to G. Let P_1 and P_2 be paths from a vertex x to a vertex y. If P_1 and P_2 have no common vertex except for x and y, then the two paths are *internally disjoint* or *openly disjoint.* A spanning tree of a connected graph G is a tree that is a subgraph of G and contains all vertices of G. If two spanning tree T_1 and T_2 of G is *completely independent* if, for any two distinct vertices u and v of G, two u-v paths on T_1 and T_2 are internally disjoint.

The concept of completely independent spanning trees was proposed by Hasunuma [2]. In [2], a characterization for completely independent spanning trees was shown, and also shown that the existence of k completely independent spanning trees in the underlying graph of k-connected line digraph. In [3], it was shown that there are two completely

1 群馬大学大学院工学研究科情報工学専攻
Department of Computer Science, Gunma University
a) arakit@gunma-u.ac.jp

independent spanning trees in any 4-connected maximal planar graph, and a linear time algorithm for finding such trees was proposed. Recently, in [4], it was shown that there are two completely independent spanning trees in the Cartesian product of two 2-connected graphs. From these results, Hasunuma posed the following conjecture in [3].

Conjecture 1.1. *There are k completely independent spanning trees in any $2k$-connected graph.*

However, recently Péterfalvi [5] proved a negative results against the conjecture as follows.

Theorem 1.2. *For any $k \geq 2$, there exists a k-connected graph that does not contain two completely independent spanning trees.*

By Theorem 1.2, there is no direct relation between the existence of completely independent spanning trees and connectivity. In this paper, we try to provide other sufficient conditions for the existence of completely independent spanning trees.

By simple observations, we can see that a graph has completely independent spanning trees if it has a large number of edges, or the degree of its vertices are large, for example, complete graphs, complete bipartite graphs, and so on. Hence we interested in relations between the existence of completely independent spanning trees and the number of edges or the minimum degree of graphs.

In the history of graph theory, there are similar approach for sufficient conditions of Hamiltonian graphs. One of the simplest condition is shown by Dirac in 1952 which gives a lower bound on the minimum degree.

Theorem 1.3 (Dirac's Theorem)**.** *If G is a graph of $n \geq 3$ vertices such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

One of the purpose of this paper is to show that the degree condition of Dirac's Theorem is also a sufficient condition for the existence of completely independent spanning trees.

Another sufficient condition of Hamiltonian graphs are known. For a graph G and $k \geq 1$, the *k-th power* of G, denoted by G^k, has vertex set $V(G)$ and $uv \in E(G^k)$ if and only if $dist_G(u, v) \leq k$. In particular, G^2 is called the *square* of G. In 1974, Fleischner verified the next theorem. Recent proof of this theorem have been given by [1].

Theorem 1.4 (Fleischner's Theorem)**.** *If G is 2-connected, the square of G is Hamiltonian.*

In Section 4, we show that the condition of Fleischner's Theorem is also a condition for the existence of completely independent spanning trees.

2. Characterization

In this section, we give a characterization for the existence of completely independent spanning trees. A characterization for completely independent spanning trees was shown as follows.

Theorem 2.1 ([2])**.** *Let $T_1, T_2, \dots, T_k$ be spanning trees in G. $T_1, T_2, \dots, T_k$ are pairwise completely independent if and only if they are edge-disjoint, and for any vertex v, there is at most one T_i such that $\deg_{T_i} v > 1$.*

The characterization says that every vertex of G is non-leaf of at most one tree T_i. In other words, vertices of G can be colored with k colors in such a way that if a vertex is a non-leaf of T_i then its color is i. The vertices that are leaves of each tree can be colored arbitrarily. Hence, if G has two completely independent spanning trees T_1 and T_2, we obtain a partition of vertices $V(G) = V_1 \cup V_2$ such that V_i is the set of vertices with color i. Since V_i contains every non-leaf of T_i, the induced subgraph $G[V_i]$ is connected. By Theorem 2.1, vertices in V_1 are leaves of T_2, and vertices in V_2 are leaves of T_1. From this fact, we introduce the following definition.

Let $V(G) = V_1 \cup V_2$ be a partition of the vertex set. For the partition, the bipartite graph $B(V_1, V_2, G)$ has the bipartition $V_1 \cup V_2$ and has the edge set $\{uv \mid uv \in E(G), u \in V_1 \text{ and } v \in V_2\}$. A partition $V(G) = V_1 \cup V_2$ is called a *CIST-partition* of G if it satisfies the following two conditions: (1) for $i = 1, 2$, the induced subgraph $G[V_i]$ is connected, and (2) the bipartite graph $B(V_1, V_2, G)$ has no tree component, that is, every connected component H of $B(V_1, V_2, G)$ satisfies $|E(H)| \geq |V(H)|$.

Theorem 2.2. *A connected graph G has two completely independent spanning trees if and only if G has a CIST-partition.*

Proof. Assume that G has two completely independent spanning trees T_1 and T_2. The vertices in G can be colored with two colors in such a way that if a vertex is an inner vertex of tree T_i then its color is i. The vertices that are leaves of both trees can be colored arbitrary. For $i = 1, 2$, define V_i as the set of vertices with color i. Then, $V_1 \cup V_2$ is a partition of V. Let $B = B(V_1, V_2, G)$. Note that any vertex $x_1 \in V_1$ is adjacent to at least one vertex of V_2 since x_1 is a leaf of T_2. Similarly, $x_2 \in V_2$ is adjacent to at least one vertex of V_1. Hence, for any connected component H of B, the number of edges $|E(H)|$ is at least $|V(H)|$.

Next we assume that G has a CIST-partition $V(G) = V_1 \cup V_2$. Since the induced subgraphs $G[V_i]$ is connected, it has a spanning tree S_i for $i = 1, 2$. We color the edges of S_i with color i. Let H be a connected component of $B = B(V_1, V_2, G)$. By the definition of CIST-partition, we have $|E(H)| \geq |V(H)|$. So, H has a connected spanning subgraph H' such that $|E(H')| = |V(H')|$, that is, H' is a unicyclic graph. Since B (and also H) is bipartite, the unique cycle of H' has even length. We assign two colors to the edges of H' as follows: The edges of the unique cycle are colored with color 1 and 2 alternately. The edges of H' that are not in the cycle induces a forest. We consider a tree component of the forest is rooted at the vertex in the unique cycle. So vertices in the forest has the unique parent. For an edge uv in the forest such that u is the parent of v, it is colored with i if $u \in V_i$. After this edge coloring is finished for every connected component H' of B, each vertex of V_1 is incident with exactly one edge with color 2, and also each vertex of V_2 is incident with exactly one edge with color 1.

Let T_i be the subgraph induced by the edges whose color is i. Every vertex of V_1 is contained in T_1. For a vertex u in V_2, it is adjacent to exactly one vertex of V_1 because u is incident with exactly one edge whose color is 1. Thus T_1 is a spanning tree of G. Similarly T_2 is a spanning tree of G. For each vertex u of G, at most one of $\deg_{T_1} u$ or $\deg_{T_1} u$ is larger than 1. We can see easily that T_1 and T_2 are edge-disjoint. Hence, by Theorem 2.1, T_1 and T_2 are completely independent spanning trees. □

Theorem 2.2 is generalized for the existence of k completely independent spanning trees. The proof of the following theorem is similar to Theorem 2.2, so we omit it.

Theorem 2.3. *A connected graph G has k completely independent spanning trees if and only if there is a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ such that (1) for $i = 1, 2, \ldots, k$, the induced subgraph $G[V_i]$ is connected, and (2) for any $1 \leq i < j \leq k$, the bipartite graph $B(V_i, V_j, G)$ has no tree component.*

3. Degree conditions

In this section, we show that the condition of Dirac's Theorem is also a sufficient condition for the existence of completely independent spanning trees. In order to prove the theorem, we need next lemma.

Let G be a graph of n vertices and $\delta(G) \geq n/2$. Let $V(G) = V_1 \cup V_2$, $|V_1| = m$ for $m = \lceil n/2 \rceil$ be a partition of vertex set. If $V_1 \cup V_2$ is not a CIST-partition, by Theorem 2.2, the bipartite graph $B = B(V_1, V_2, G)$ contains a tree component. Hence B has a leaf x. If $x \in V_1$, it is adjacent to every other vertices in V_1 since $\deg_G x \geq m$. This property of a leaf of B will be used in the proofs of this section. Note that, if n is odd, then V_2 has $(n-1)/2$ vertices, and hence there are no leaf of B in V_2.

Lemma 3.1. *Suppose that $n \geq 8$ and $\delta(G) \geq n/2$ and $V(G) = V_1 \cup V_2$ is a partition such that $|V_1| = m$ for $m = \lceil n/2 \rceil$. Assume that the induced subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ are connected, and V_1 has a leaf x of $B = B(V_1, V_2, G)$ such that it is not a cut-vertex of G_1 and is adjacent to non-leaf $y \in V_2$ of B. Then, there are two completely independent spanning trees in G.*

Proof. If $V(G) = V_1 \cup V_2$ is a CIST-partition, we are done. Assume that it is not a CIST-partition. Let $U_1 = V_1 \setminus \{x\}$ and $U_2 = V_2 \cup \{x\}$. Since x is not a cut-vertex of G_1, the induced subgraph $G[U_1]$ is connected. Also $G[U_2]$ is connected. If $U_1 \cup U_2$ is a CIST-partition, then we are done.

Assume that $U_1 \cup U_2$ is not a CIST-partition, that is, $B_U = B(U_1, U_2, G)$ has a tree component. By assumption, $x \in V_1$ is a leaf of B. Then x is adjacent to every vertex in U_1. Hence B_U is a spanning tree of G. It is possible only when any vertex $z \in V_2 \setminus \{y\}$ is a leaf of B (and thus n is even). Since $|V_1| = |V_2| = m$, V_1 has at most one non-leaf of B, and also V_2 has exactly one non-leaf y of B. This means that the induced subgraphs G_1 and G_2 must be isomorphic to a complete graph K_m. From the above discussion, G has a spanning subgraph that consists of two complete graphs K_m and edges between them. For $m \geq 4$ and two vertices v_1 and v_2 of K_m, we can construct two completely spanning trees T_1 and T_2 in K_m such that $\deg_{T_i} v_i \geq 2$. Let x_1x_2, y_1y_2 be edges between V_1 and V_2 such that $x_i, y_i \in V_i$. We construct completely independent spanning trees T_x, T_y in G_1 such that $\deg_{T_x} x_1 \geq 2$ and $\deg_{T_y} y_1 \geq 2$. Similarly, let T'_x, T'_y by two trees in G_2 such that $\deg_{T'_x} x_2 \geq 2$ and $\deg_{T'_y} y_2 \geq 2$. Finally, let $T_1 = (T_x \cup T'_x) + x_1x_2$ and $T_2 = (T_y \cup T'_y) + y_1y_2$. Then T_1 and T_2 are completely independent spanning trees in G. □

Theorem 3.2. *If $\delta(G) \geq n/2$, then G has two completely independent spanning trees.*

Proof. By Dirac's Theorem, G has a Hamiltonian cycle. Hence there is a partition $V(G) = V_1 \cup V_2$ such that $|V_1| = m$ for $m = \lceil n/2 \rceil$ and the induced subgraphs $G_1 = G[V_1]$ and $G_2 = G[V_2]$ are connected. If the partition $V_1 \cup V_2$ is a CIST-partition, then we are done.

We assume that $V_1 \cup V_2$ is not a CIST-partition. By

Theorem 2.2, the bipartite graph $B = B(V_1, V_2, G)$ has a tree component T. Thus T (and also B) has at least two leaves. If $x \in V_i$ is a leaf of B, then x is adjacent to every other vertices in V_i.

(Case 1) There are two leaves x, y of T in V_1.

Since x and y are adjacent to every other vertices in V_1, each of these vertices is not a cut-vertex of G_1. Since T is a tree and x and y are members of same partite set of T, the vertex x is adjacent to a vertex of degree at least two. Hence, by Lemma 3.1, G has two completely independent spanning trees.

(Case 2) For $i = 1, 2$, there is exactly one leaf x_i of T in V_i, and x_i is adjacent to a vertex of degree at least two of T.

Since V_2 contains a leaf of B, as mentioned earlier, n is even. Since T has exactly two leaves, it is a path of at least four vertices. By Lemma 3.1, we may assume that x_1 and x_2 are cut-vertices of G_1 and G_2, respectively. Let $y_1 \in V_1$ be a vertex adjacent to x_2 and $y_2 \in V_2$ be a vertex adjacent to x_1. Assume that Q is the component of $G_1 - x_1$ that contains y_1. Then

$$\begin{aligned} n/2 \le \deg_G y_1 &= \deg_T y_1 + \deg_{G_1} y_1 \\ &\le 2 + (1 + \deg_Q y_1) \\ &\le |V(Q)| + 2. \end{aligned}$$

(Note that y_1 is adjacent to x_1.) Hence we obtain $|V(Q)| \ge n/2 - 2$. Since $|V_1| = n/2$ and $G_1 - x_1$ has at least two components, we have $|V(Q)| = n/2 - 2$. This means that $G_1 - x_1$ has two components, one is an isolated vertex z_1 and the other component Q_1 has $n/2 - 2$ vertices. Similarly, $G_2 - x_2$ has two components, one is an isolated vertex z_2 and the other component Q_2 has $n/2 - 2$ vertices. Furthermore, z_1 (or z_2) is not adjacent to vertices in Q_1 (or Q_2) and has degree at least $n/2$, it must be adjacent to every vertices in Q_2 (or Q_1). From the above discussion, G has a spanning subgraph H with edge set

$$\begin{aligned} E(H) = \{x_1y_2, x_2y_1\} &\cup \{x_1u \mid u \in V_1, u \ne x_1\} \\ &\cup \{x_2v \mid v \in V_2, v \ne x_2\} \\ &\cup \{y_1u \mid u \in V(Q_1), u \ne y_1\} \\ &\cup \{y_2v \mid v \in V(Q_2), v \ne y_2\} \\ &\cup \{z_1v \mid v \in V(Q_2)\} \cup \{z_2u \mid u \in V(Q_1)\}. \end{aligned}$$

We can construct two completely independent spanning trees T_1 and T_2 in H as follows. We choose $u \in V(Q_1), u \ne y_1$ and $v \in V(Q_2), v \ne y_2$ arbitrary.

$$\begin{aligned} E(T_1) = \{x_1z_1, y_1x_2, y_1z_2, y_2v\} &\cup \{x_1p \mid p \in V(Q_1)\} \\ &\cup \{z_1q \mid q \in V(Q_2), q \ne y_2\}, \\ E(T_2) = \{x_2z_2, y_2x_1, y_2z_1, y_1u\} &\cup \{x_2q \mid q \in V(Q_2)\} \\ &\cup \{z_2p \mid p \in V(Q_1), p \ne y_1\}, \end{aligned}$$

These trees T_1 and T_2 are illustrated in Fig. 1.

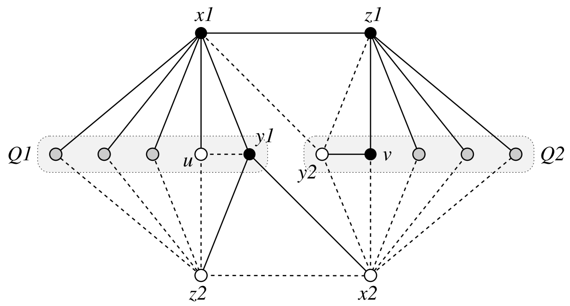

図 1 Case 2: two completely independent spanning trees in G. Edges of T_1 are solid lines, and edges of T_2 are dotted lines.

(Case 3) For $i = 1, 2$, there is exactly one leaf x_i of T in V_i, and x_1 is adjacent to x_2.

In this case, for $i = 1, 2$, x_i is adjacent to every vertex in V_i, and so $G' = G - \{x_1, x_2\}$ satisfies $|V(G')| = n - 2$ and $\delta(G') \ge (n-2)/2$. By Dirac's Theorem, G' has a Hamiltonian cycle C. Let y_1 and y_2 be adjacent vertices in C such that $x_1y_1, x_2y_2 \in E(G)$. Then z_1 and z_2 be vertices such that $y_1z_1, y_2z_2 \in E(C)$. Since z_i is a member of either V_1 or V_2, there are four possibility for z_i's, that is, (1) $z_1, z_2 \in V_1$, (2) $z_1 \in V_1$ and $z_2 \in V_2$, (3) $z_1 \in V_2$ and $z_2 \in V_1$, and (4) $z_1, z_2 \in V_2$. In this proof, we assume that $z_1, z_2 \in V_1$. (The other cases can be proved similarly.) First we construct spanning tree T_1 of G. We color vertices x_1, x_2, z_1 and z_2 with color 1, and color the following edges with color 1:

- $x_1x_2, x_1z_1, x_1z_2, y_1z_1, y_2z_2$,
- x_1v for $v \in V_1$, and x_1w for $w \in V_2$.

Then we construct spanning tree T_2. Assign color 2 into $n - 4$ vertices that are not colored with 1. We color the following edges with color 2: every $n - 4$ edges on C except for y_1z_1 and y_2z_2, and x_1y_1 and x_2y_2. In addition, if either y_1 or y_2 is adjacent to a vertex w with color 2 and $w \notin \{y_1, y_2\}$, then we color edge y_iw with color 2, where $y_iw \in E(G)$. Since x_1 and x_2 has no common adjacent vertex, there is such vertex w if $\deg_G y \ge 5$. Hence we consider the case that $n = 8$ and $\deg_G y_1 = \deg_G y_2 = 4$, and $N_G(y_1) = \{x_1, y_2, z_1, z_2\}$ and $N_G(y_2) = \{x_2, y_1, z_1, z_2\}$.

In this case, we exchange colors of y_1 and z_1, and construct new trees T_1' and T_2' by $T_1' = T_1 - x_1z_1 + x_1y_1$ and $T_2' = T_2 - x_1y_1 + x_1z_1 + z_1y_2$. We can easily see that T_1' and T_2' are completely independent.

From Case 1 to Case 3, we show that G has two completely independent spanning trees. □

The bound described in Theorem 3.2 is tight. That is, there is a graph G such that $\delta(G) = (n-1)/2$ and G does not have two completely independent spanning trees. For example, a graph obtained by identifying a vertex of two complete graphs $K_{(n+1)/2}$. This graph cannot have two completely independent spanning trees because it has a cut-vertex.

4. Power of graphs

One of the purpose of this section is to show that the square of 2-connected graph has two completely independent spanning trees, which is similar to Theorem 1.4. First we consider a condition for the square of trees to have two completely independent spanning trees. Then the square of a 2-connected graph is considered.

The next lemma is simple but useful property of bipartite graphs. It can be proved easily and hence we omit a proof of it.

Lemma 4.1. *Let G be a connected bipartite graph with bipartition $V_1 \cup V_2$. Then, in G^2, the induced subgraphs $G^2[V_1]$ and $G^2[V_2]$ are connected.*

The square of a path of n vertices cannot have two edge-disjoint spanning trees since it has only $2n-3$ edges. Hence there is a tree T such that T^2 does not have two completely independent spanning trees. The following lemma shows more general result. Let $\mathcal{T}$ be the family of trees such that it has at least four vertices and every leaf is adjacent to a vertex of degree 2.

Lemma 4.2. *For a tree $T \in \mathcal{T}$, the square T^2 does not have two completely independent spanning trees.*

Proof. Assume that, for a tree $T \in \mathcal{T}$, there are two completely independent spanning trees in T^2. Let $V_1 \cup V_2$ be a CIST-partition of T^2. Let v be a leaf of T, and $vu, uw \in E(T)$. (Note that the degree of u is 2.) In T^2, the degree of v is 2 and $vu, vw \in E(T^2)$. Hence v is a leaf in each of the two completely independent spanning trees. This means that u and w are in different subsets V_1 or V_2. Furthermore, we may assume that v and w are in the same subset. As a result, for each leaf v and vertices u, w such that $vu, uw \in E(T)$, we have either $v, w \in V_1$ and $u \in V_2$ or $v, w \in V_2$ and $u \in V_1$.

Assume that there is adjacent vertices x and y of T such that $x, y \in V_i$. We may assume that $x, y \in V_1$. Then x (and also y) is not a leaf and is not adjacent to a leaf of T. Let T_x and T_y be two connected components of $T - xy$ such that T_x contains x and T_y contains y. Then T_x and T_y have vertices in V_2. Let $x' \in V(T_x)$ and $y' \in V(T_y)$ be any vertex in V_2. However, $dist(x', y') \geq 3$ in T, and hence $T^2[V_2]$ cannot be connected. This is a contradiction. Therefore, any two vertices in V_i are not adjacent for $i = 1, 2$. This implies that $V_1 \cup V_2$ is equal to the bipartition of T. Hence the bipartite graph $B = B(V_1, V_2, T^2)$ is a tree, and it contradicts the fact that $V_1 \cup V_2$ is a CIST-partition.

Hence there is no CIST-partition in T^2 for $T \in T$. □

We state a characterization for a tree whose square has two completely independent spanning trees.

Theorem 4.3. *Let T be a tree with at least four vertices. The square T^2 has two completely independent spanning trees if and only if T has a leaf that is adjacent to a vertex of degree at least 3, that is, $T \notin \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then T^2 does not have two completely independent spanning trees by Lemma 4.2.

Next we assume that $T \notin \mathcal{T}$. Hence T has a leaf x that is adjacent to y such that $\deg_T y \geq 3$. Let $V_1 \cup V_2$ be the bipartition of T and we assume that $y \in V_1$ and $x \in V_2$. We define $U_1 = V_1 \cup \{x\}$ and $U_2 = V_2 \setminus \{x\}$. We show that $U_1 \cup U_2$ is a CIST-partition of T^2.

First we show that the induced subgraph $T^2[U_i]$ is connected for $i = 1, 2$. For $s, t \in U_2$, there is a s-t path P in T. Since x is a leaf, the path P does not contain x. Hence there is a s-t path in P^2 through vertices of U_2, and thus $T^2[U_2]$ is connected. By similar argument, there is a s-t path on $T^2[U_1]$ if $s \neq x$ and $t \neq x$. For any vertex $s \in V_1$, there is a path from s to y. Since v is adjacent to x, there is a path from s to x in $T^2[U_1]$. Next we show that the bipartite graph $B = B(U_1, U_2, T^2)$ has no tree component. By the definition of U_1 and U_2, the edge set of B is $(E(T) \setminus \{xy\}) \cup \{xw \mid yw \in E(T), w \neq x\}$. Since $\deg_T y \geq 3$, the bipartite graph B is connected and has at least $|V(T)| + 1$ edges. Hence B has no tree component.

Therefore, by Theorem 2.2, T^2 has two completely independent spanning trees. □

Next we focus on the square of 2-connected graphs.

Lemma 4.4. *For a cycle C_n, $n \geq 4$, the square C_n^2 has two completely independent spanning trees.*

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and its edges are $v_i v_{i+1}$ for $0 \leq i \leq n-1$ (additions are taken modulo n). If we take $V_1 = \{v_{2i} \mid 0 \leq i \leq \lceil n/2 \rceil - 1\}$ and $V_2 = \{v_{2i+1} \mid 0 \leq i \leq \lfloor n/2 \rfloor - 1\}$, then we can prove easily that $V_1 \cup V_2$ is a CIST-partition. □

Theorem 4.5. *For a 2-connected graph G of $n \geq 4$ vertices, the square G^2 has two completely independent spanning trees.*

Proof. If G is a cycle, the theorem follows from Lemma 4.4. We assume that G is not a cycle, that is, G has a vertex v of degree three or more. Let w be an adjacent vertex of v. Since G is 2-connected, $G-w$ is connected and it has a spanning tree T such that $\deg_T v \geq 2$. Hence we obtain a spanning tree $T' \notin \mathcal{T}$ of G by adding vertex w and edge vw with T. Hence, by Theorem 4.3, G^2 has two completely independent spanning trees. □

5. Conclusion

In this paper, we consider the problem of completely independent spanning tress. First we provided a characterization of two completely independent spanning trees. This characterization is different from one by Hasunuma given in [2]. Next we show some sufficient conditions for the existence of two completely independent spanning trees. It is interesting that well-known conditions for Hamiltonian graphs are also for completely independent spanning trees (Theorem 3.2 and 4.5). It is known that there are many sufficient conditions for Hamiltonian graphs. These conditions might help the researchers to consider problems of completely independent spanning trees.

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