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Technical Note

Lower Bound of Face Guards of Polyhedral Terrains

Chuzo Iwamoto[1,a)] Junichi Kishi[2] Kenichi Morita[1]

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Abstract: We study the problem of determining the minimum number of face guards which cover the surface of a polyhedral terrain. We show that $\lfloor (2n-5)/7 \rfloor$ face guards are sometimes necessary to guard the surface of an n-vertex triangulated polyhedral terrain.

Keywords: face guards, polyhedral terrains

1. Introduction

The art gallery problem is to determine the minimum number of guards who can observe the interior of a gallery. Chvátal [4] proved that $\lfloor n/3 \rfloor$ guards are the lower and upper bounds for this problem; namely, $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary for observing the interior of an n-vertex simple polygon in the two-dimensional space.

In three dimensions, a similar visibility problem has been considered for n-vertex triangulated polyhedral terrains. It is known that there is a linear-time algorithm for placing $\lfloor n/2 \rfloor$ vertex guards [2]. Here, a *vertex guard* is a guard that is only allowed to be placed at the vertices of a terrain. As for the lower bound, there is a polyhedral terrain for which $\lfloor n/2 \rfloor$ vertex guards are necessary [3]. Thus, $\lfloor n/2 \rfloor$ is the lower and upper bound of the number of vertex guards of a polyhedral terrain.

An *edge guard* is a guard that is only allowed to be placed on the edges of a terrain, and the edge guard can move between the endpoints of the edge. For the edge guarding problem, it was shown that the upper bound is $\lfloor n/3 \rfloor$ [2] and the lower bound is $\lfloor (4n-4)/13 \rfloor$ [3] for n-vertex triangulated polyhedral terrains. Reducing the gap between the upper and lower bounds of edge guards remains an open problem.

Table 1 summarizes the upper and lower bounds of guards of a polyhedral terrain. The paper stating the $\lfloor (4n-4)/13 \rfloor$ lower bound [3] did not present the detailed construction of a polyhedral terrain for which $\lfloor (4n-4)/13 \rfloor$ edge guards are necessary. In 2003, Kaučič et al. [6] presented the detailed construction of a polyhedral terrain for which $\lfloor (2n-4)/7 \rfloor$ edge guards are necessary. In response to this paper, the proof of the $\lfloor (4n-4)/13 \rfloor$ lower bound at a level of detail was presented in 2009 [1].

In Table 1, upper bounds of vertex and edge guards were firstly proved in 1997 [3], [5]. However, these bounds are based on the four color theorem, and for this reason, there seemed to be no practical efficient algorithms achieving these bounds. The first algorithmic upper-bounds were presented in 2003 [2]; it was shown that there are linear-time algorithms for finding $\lfloor n/2 \rfloor$ vertex guards and $\lfloor n/3 \rfloor$ edge guards.

Table 1 Upper and lower bounds of guards of a polyhedral terrain.

	Upper bounds		Lower bounds	
Vertex guards	$\lfloor n/2 \rfloor$	[2], [3]	$\lfloor n/2 \rfloor$	[3]
Edge guards	$\lfloor n/3 \rfloor$	[2], [5]	$\lfloor (4n-4)/13 \rfloor$	[1], [3]
Face guards	$\lfloor n/3 \rfloor$	obvious	$\lfloor (2n-5)/7 \rfloor$	current paper

In the current paper, we study the number of *face guards*, where a face guard is allowed to be placed on the faces of a terrain, and the face guard can walk around only on the allocated face. A face guard can observe the allocated face and its adjacent faces. Here, two faces are said to be adjacent if they share a vertex.

The face guarding problem is motivated by applications in guarding bordering territories. In the real world, a territorial owner keeps watch over neighboring lands not only from an edge (borderline) or a vertex (corner), but also from all his territory.

Given an n-vertex triangulated polyhedral terrain, the face guarding problem is to find the minimum number of face guards which cover the surface of the terrain. In this paper, we show that there is an n-vertex triangulated polyhedral terrain for which $\lfloor (2n-5)/7 \rfloor$ face guards are necessary for every $n \in \{6, 9, 12, \ldots, 3i+6, \ldots\}$. As for the upper bound, it is obvious that $\lfloor n/3 \rfloor$ face guards can be found by a very simple algorithm, which repeatedly removes triangulated faces one by one in some order from the graph. Of course, the upper bound of $\lfloor n/3 \rfloor$ edge guards immediately implies the upper bound of $\lfloor n/3 \rfloor$ face guards.

2. Definitions and Results

The definitions of polyhedral terrains and visibility are mostly from landmark papers on guarding polyhedral terrains [3], [5]. A *polyhedral terrain* is a polyhedral surface in three dimensions such that its intersection with any vertical line is either a point or empty. A polyhedral terrain is *triangulated* if each of its faces is a triangle.

Two points x and y of a terrain are said to be *visible* if the line segment xy does not contain any points below the terrain. A point

1 Hiroshima University, Graduate School of Engineering, Higashi-Hiroshima 739–8527, Japan
2 The City Office of Hiroshima, Naka, Hiroshima 730–8586, Japan
a) chuzo@hiroshima-u.ac.jp

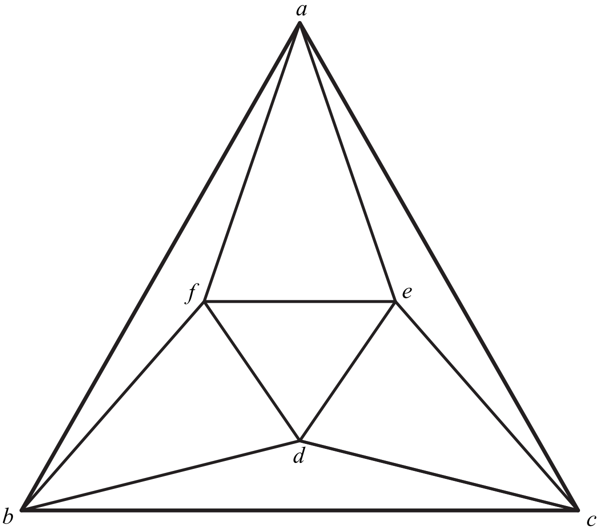

Fig. 1 6-vertex triangulated plane graph G_6, which corresponds to a 6-vertex triangulated convex terrain T_6 from the top view.

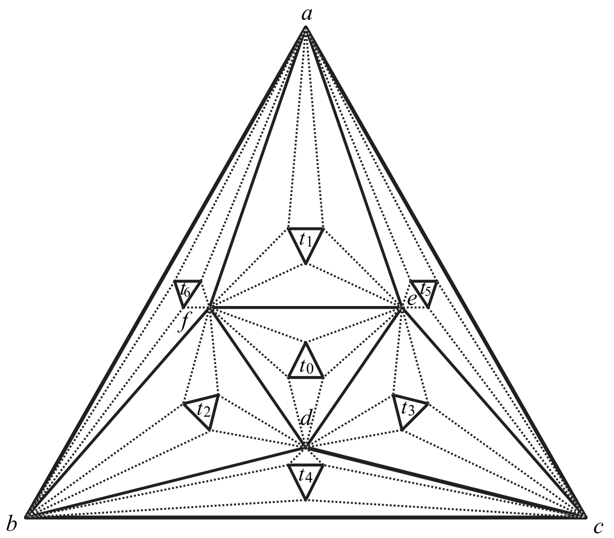

Fig. 2 27-vertex convex terrain T_{27} with 7^2 faces.

x of a terrain is said to be *visible from a face* f if there exists a point y on the face f such that x and y are visible. A set of faces is said to *cover* a terrain if every point of the terrain is visible from one of these faces.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices of a terrain T such that no three vertices of V are collinear. Each vertex v_i is specified by three real numbers (x_i, y_i, z_i) which are its cartesian coordinates and z_i is referred as the height of vertex v_i. In this paper, we assume each z_i is nonnegative.

Let $V' = \{v'_1, v'_2, \ldots, v'_n\}$ denote the orthogonal projections of the points $V = \{v_1, v_2, \ldots, v_n\}$ on the X-Y plane, i.e., v'_i is specified by the two real numbers (x_i, y_i). Let E' denote the orthogonal projections of T's edges on the X-Y plane. The graph $G = (V', E')$ is an n-vertex *triangulated plane graph* corresponding to the terrain T.

Theorem 1 *For every $n \in \{6, 9, 12, \ldots, 3i + 6, \ldots\}$, there exists an n-vertex triangulated polyhedral terrain which needs at least $\lfloor (2n - 5)/7 \rfloor$ face guards.*

3. Proof of Theorem 1

We first construct an n-vertex triangulated plane graph G_n and the corresponding n-vertex triangulated convex terrain T_n. Then, we will prove that $\lfloor (2n - 5)/7 \rfloor$ is necessary for T_n.

Consider the 6-vertex triangulated plane graph G_6 of **Fig. 1**. This figure is also regarded as the top view of a 6-vertex triangulated polyhedral convex terrain T_6. Here, all points on the triangle $\triangle(def)$ of T_6 have the same positive height.

Consider the triangle $\triangle(def)$ in Fig. 1 (see also $\triangle(def)$ in **Fig. 2**) and one more 6-vertex convex terrain T_6. We place T_6 on the face $\triangle(def)$ of Fig. 1 so that the vertices a, b, c of T_6 correspond exactly to the vertices d, e, f of Fig. 1. In Fig. 2, the height of t_0 above $\triangle(def)$ must be a sufficiently small positive number compared to the height of $\triangle(def)$ above $\triangle(abc)$ so that the resulting terrain is convex.

Furthermore, we place T_6 on the face $\triangle(afe)$ so that the vertices a, b, c of T_6 correspond exactly to the vertices a, f, e of Fig. 1. For such a placement, four edges $(a, b), (a, f), (a, e)$ and (a, c) of T_6 should be stretched so that a, b, c of T_6 correspond exactly to a, f, e of Fig. 1 (see also $\triangle(afe)$ in Fig. 2). One can see that, by continuing this construction, we will obtain an n_i-vertex convex terrain T_{n_i} for every $n_i \in \{9, 12, \ldots, 3i + 6, \ldots\}$.

In an analogous way, we place 7^2 six-vertex convex terrains T_6 on 7^2 faces of Fig. 2. Then we obtain a triangulated polyhedral convex terrain, say, T_{174}, with $6 + 3 \cdot 7 + 3 \cdot 7^2 = 174$ vertices.

Lemma 1 For every $n_i \in \{9, 12, \ldots, 3i + 6, \ldots\}$, $\lfloor (2n_i - 5)/7 \rfloor$ face guards are necessary for T_{n_i}.

Proof of Lemma 1. First of all, we analyze the numbers of guards for T_6, T_9, and T_{12}. Consider T_6 of Fig. 1. A guard must be placed on the face $\triangle(def)$ to observe all the seven faces of Fig. 1. (Thus, Theorem 1 holds for $n = 6$.) Consider T_9. A point inside the small triangle t_0 is not visible from any face guard outside $\triangle(def)$, since the height of t_0 above $\triangle(def)$ is nonzero and T_9 is convex. Thus, at least one guard must be placed in $\triangle(def)$ to observe inside t_0. Hence, Lemma 1 holds for $n_1 = 9$.

In the current proof, the lower bound for T_6 is 1, and the lower bound for T_9 is also 1; we need no "new" guard for T_9. (By an exhaustive search, we can prove that two guards are necessary and sufficient for T_9.)

Consider T_{12}. By the same reason as above, at least one "new" guard must be placed in $\triangle(afe)$ to observe inside t_1. The total number of guards is two; one is in $\triangle(afe)$, and the other is in $\triangle(def)$. Hence, Lemma 1 holds for $n_2 = 12$.

In a similar fashion, one can see that at least one "new" guard must be placed in each of the remaining five triangles $\triangle(bdf), \triangle(ced), \triangle(bcd), \triangle(cae)$, and $\triangle(abf)$ to observe the five small triangles $t_2, t_3, \ldots, t_6$, respectively. Hence, Lemma 1 holds for $n_3, n_4, \ldots, n_7$.

Now, we prove Lemma 1 for every $n_i \in \{9, 12, \ldots, 3i + 6, \ldots\}$. For every addition of the set of seven triangles $t_0, t_1, \ldots, t_6$, the numbers of vertices and guards increase by 21 and 6, respectively. Therefore, the number of guards grows proportional to $6n_j/21$ ($= 2n_j/7$) as the number of vertices $n_j = 3j + 6$ increases, where $j \in \{0, 7, 14, \ldots\}$. Thus, for every $n_i \in \{9, 12, \ldots, 3i + 6, \ldots\}$, there exists a constant $k \geq 0$ such that the lower bound can be represented as $\lfloor (2n_i - k)/7 \rfloor$. ($k$ will be fixed later.)

The lower bound of guards grows from 1 to 1, 2, 3, 4, 5, 6, 7 when the number of vertices increases from 6 to 9, 12, 15, 18, 21, 24, 27 (i.e., when $t_0, t_1, \ldots, t_6$ are added), respectively. In general, suppose that the number of vertices is $n_j = 21j + 6$, where $j \geq 0$ is an arbitrary integer. The lower bound of guards grows from $6j + 1$ to

$$6j + 1,\ 6j + 2,\ 6j + 3,\ 6j + 4,\ 6j + 5,\ 6j + 6,\ 6j + 7$$

when the number of vertices increases from $21j + 6$ to

$$21j + 9,\ 21j + 12,\ 21j + 15,\ 21j + 18,\ 21j + 21,$$
$$21j + 24,\ 21j + 27,$$

respectively. Let

$$g_{(j,l)} = 6j + (l + 1),$$
$$n_{(j,l)} = 21j + (3l + 9),$$

where $l \in \{0, 1, \ldots, 6\}$. If we fix $k = 5$, then

$$\begin{aligned}\lfloor (2n_{(j,l)} - 5)/7 \rfloor &= \lfloor (2 \cdot (21j + (3l + 9)) - 5)/7 \rfloor \\ &= \lfloor (42j + (6l + 13))/7 \rfloor \\ &= 6j + \lfloor (6l + 13)/7 \rfloor \\ &= 6j + (l + 1) \\ &= g_{(j,l)}.\end{aligned}$$

for every pair of $j \in \{0, 1, 2, \ldots\}$ and $l \in \{0, 1, \ldots, 6\}$. Therefore, Lemma 1 holds for every $n_i \in \{9, 12, \ldots, 3i + 6, \ldots\}$.

4. Conclusions

We studied the problem of determining the minimum number of face guards sufficient to cover the surface of a polyhedral terrain. We showed that $\lfloor (2n-5)/7 \rfloor$ guards are sometimes necessary for n-vertex triangulated polyhedral terrains. Reducing the gaps between the upper and lower bounds of the edge and face guarding problems remain open problems.

Reference

[1] Bose, P.: A note on the lower bound of edge guards of polyhedral terrains, *Int. J. Comput. Math.*, Vol.86, No.4, pp.577–583 (2009).

[2] Bose, P., Kirkpatrick, D. and Li, Z.: Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces, *Comput. Geom. Theory Appl.*, Vol.26, pp.209–219 (2003).

[3] Bose, P., Shermer, T., Toussaint, G. and Zhu, B.: Guarding polyhedral terrains, *Comput. Geom. Theory Appl.*, Vol.7, pp.173–185 (1997).

[4] Chvátal, V.: A combinatorial theorem in plane geometry, *J. Combin. Theory, Ser. B*, Vol.18, pp.39–41 (1975).

[5] Everett, H. and Rivera-Campo, E.: Edge guarding polyhedral terrains, *Comput. Geom. Theory Appl.*, Vol.7, pp.201–203 (1997).

[6] Kaučič, B., Žalik, B. and Novak, F.: On the lower bound of edge guards of polyhedral terrains, *Int. J. Comput. Math.*, Vol.80, No.7, pp.811–814 (2003).

Chuzo Iwamoto received his B.Eng. degree from Yamaguchi University in 1990, and M.Eng. and Dr. Eng. degrees from Kyushu University in 1992 and 1995, respectively. From 1995 to 1997, he was a lecturer at Kyushu Institute of Design. Since 1997, he has been an associate professor in the Graduate School of Engineering, Hiroshima University. His present interests include computational complexity and automata theory.

Junichi Kishi received his B.Eng. and M.Eng. degrees from Hiroshima University in 2009 and 2011, respectively. He is currently working in the City Office of Hiroshima.

Kenichi Morita received his B.Eng., M.Eng., and Dr. Eng. degrees from Osaka University in 1971, 1973, and 1978, respectively. From 1974 to 1987, he was a research associate of the Faculty of Engineering Science, Osaka University. From 1987 to 1990, he was an associate professor, and from 1990 to 1993 a professor of the Faculty of Engineering, Yamagata University. Since 1993, he has been a professor of the Graduate School of Engineering, Hiroshima University. He has been engaged in the research of automata theory (especially cellular automata), reversible computing, formal language theory, array grammars, and logic systems for knowledge and language processing.