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Online TSP in a Simple Polygon

Yuya Higashikawa[†1] and Naoki Katoh[†1]

We consider an online traveling salesman problem in a simple polygon where starting from a point in the interior of a simple polygon, the searcher is required to explore a simple polygon to visit its all vertices and finally return to the initial position as quickly as possible. The information of the polygon is given online. As the exploration proceeds, the searcher gains more information of the polygon. We give a 1.219-competitive algorithm for this problem. We also study the case of a rectilinear simple polygon, and give a 1.167-competitive algorithm.

1. Introduction

The Tohoku Earthquake attacked East Japan area on March 11, 2011. When such a big earthquake occurs in an urban area, it is predicted that many buildings and underground shopping areas will be heavily damaged, and it is seriously important to efficiently explore the inside of damaged areas in order to rescue human beings left there. With this motivation, we deal with *online traveling salesman problem* (**online TSP** for short) in a simple polygon. Given a simple polygon P, suppose the searcher is initially in the interior of P. Starting from the origin o, the aim of the searcher is to visit all vertices of P at least once and to return to the starting point as quickly as possible. The information of the polygon is given online. Namely, at the beginning, the searcher has only the information of a visible part of the polygon. As the exploration proceeds, the visible area changes. However, the information of the region which has once become visible is assumed to be accumulated. So, as the exploration proceeds, the searcher gains more information of the polygon, and determines which vertex to visit next based on the information obtained so far.

In general, the performance of an online algorithm is measured by a *competitive ratio* which is defined as follows. Let $\mathcal{S}$ denote a class of objects to be explored. When an online exploration algorithm ALG is used to explore an object $S \in \mathcal{S}$, let $|\mathsf{ALG}(S)|$ denote the tour length (cost) required to explore S by ALG. Let $|\mathsf{OPT}(S)|$ denote the tour length (cost) required to explore S by the offline optimal algorithm. Then the competitive ratio of ALG is defined as follows.

$$\sup_{S \in \mathcal{S}} \frac{|\mathsf{ALG}(S)|}{|\mathsf{OPT}(S)|}.$$

Previous work: Online TSP has been extensively studied for the case of graphs. Kalyanasundaram et al.[10] presented a 16-competitive algorithm for planar undirected graphs. Megow et al.[8] recently extended this result to undirected graphs with genus g and gave a $16(1+2g)$-competitive algorithm. For the case of a cycle, Miyazaki et al.[9] gave an optimal 1.37-competitive algorithm. All these results are concerned with a single searcher. For the case of $p(> 1)$ searchers, there are some results. Fraigniaud et al.[3] gave an $O(p/\log p)$-competitive algorithm for the case of a tree. Higashikawa et al.[6] gave $(p/\log p + o(1))$-competitive algorithm for this problem. Dynia et al.[2] showed a lower bound $\Omega(\log p/\log\log p)$ for any deterministic algorithm for this problem.

There are some papers that deal with online TSP in geometric regions (see survey paper[5]). Kalyanasundaram et al.[10] studied the case of a polygon with holes where all edges are required to traverse. They gave a 17-competitive algorithm for this case. Hoffmann et al.[7] studied the problem that asks to find a tour in a simple polygon such that every vertex is visible from some point on the tour, and gave a 26.5-competitive algorithm.

Our results: We will show 1.219-competitive algorithm for an online TSP in a simple polygon. We also study the case of a rectilinear simple polygon, and give a 1.167-competitive algorithm. We will give a lower bound result that the competitive ratio is at least 1.040 within a certain framework of exploration algorithms.

2. Strategy of AOE

In this report, we define a *simple polygon* as a closed polygonal chain with no self-intersction in the plane. In the followings, we use the term *polygon* to stand for a simple polygon. Also an *edge of a polygon* (or a *polygon edge*) is defined as

†1 Department of Architecture and Architectural Engineering, Kyoto University

a line segment forming a part of the polygonal chain, a *vertex of a polygon* (or a *polygon vertex*) as a point where two polygon edges meet and the *boundary of polygon* as a polygonal chain. Let P be a polygon and o be the origin. Sometimes we abuse the notation P to stand for the interior (including the boundary) of P. Let $V = \{v_1, v_2, ..., v_n\}$ be a polygon vertex set of P sorted in clockwise order along the boundary and $E = \{e_1, e_2, ..., e_n\}$ be a polygon edge set of P composed of $e_i = (v_i, v_{i+1}) = (v^1_{e_i}, v^2_{e_i})$ with $1 \le i \le n$ ($v_{n+1} = v_1$ is assumed). let $|e|$ denote the length of edge $e \in E$ and $L = \sum_{e\in E} |e|$ be the boundary length of P. For any two points $x, y \in P$, let $sp(x, y)$ denote the shortest path from x to y that lies in the inside of P, $|sp(x, y)|$ be its length and $|xy|$ be the Euclidean distance from x to y. Note that $sp(x, y) = sp(y, x)$ and $|xy| \le |sp(x, y)|$. Furthermore, for any two vertices $x, y \in V$, let $bp(x, y)$ denote the clockwise path along the boundary of P from x to y and $|bp(x, y)|$ be its length. The cost of a TSP tour is defined to be its length.

For a point $x \in P$ and an edge $e \in E$, let

$$cost(x, e) = |sp(x, v^1_e)| + |sp(x, v^2_e)| - |e|.$$

In the offline version of this problem, we will prove below that an optimal strategy is that starting from the origin o, the searcher first goes to one endpoint of some edge e, namely v^2_e, then follows the boundary path $bp(v^2_e, v^1_e)$ and finally comes back to o. The proof is given in the appendix.

Lemma 1. *For offline TSP in a polygon P, the cost of the offline optimal algorithm satisfies the following.*

$$|\mathsf{OPT}(P)| = L + \min_{e\in E} cost(o, e).$$

Let $e_{opt} \in E$ be an edge satisfying the following equation.

$$cost(o, e_{opt}) = \min_{e\in E} cost(o, e). \qquad (1)$$

For two points $x, y \in P$, we say that y is *visible* from x if the line segment xy lies in the inside of P. Then the *visibility polygon* $VP(P, x)$ is

$$VP(P, x) := \{y \in P \mid y\ is\ visible\ from\ x\}.$$

Note that an edge of the visibility polygon is not necessarily an edge of P. For a polygon vertex b and a point $x \in P$, we call b a *blocking vertex* with respect to x if b is visible from x and there is the unique edge incident to b such that any point on the edge except b is not visible from x. Let b^* be a point where the extension of the line segment xb towards b first intersects the boundary of P. Then we call b^* a *virtual vertex* and the line segment bb^* a *cut edge*. Without loss of generality we assume that b^* does not coincide with any vertex in V. Also let $\hat{e}$ be an edge of P containing b^* then we regard a visible part of $\hat{e}$ as a new edge, which we call a *virtual edge*. Note that a cut edge bb^* divides P in two areas, a polygon which contains $VP(P, x)$ and the other not. We call the latter area the *invisible polygon* $IP(P, x, b)$. Notice that $VP(P, x)$ and $IP(P, x, b)$ share a cut edge bb^*.

We assume that there is a blocking vertex b with respect to the origin o since otherwise an optimal solution can be found by Lemma 1. Then we have the following lemma.

Lemma 2. *For an invisible polygon $IP(P, o, b)$ defined by a blocking vertex b, let $e \in E$ be a polygon edge both endpoints of which are in $IP(P, o, b)$, and $w \in V$ be the polygon vertex adjacent to b which is not in $IP(P, o, b)$. Then*

$$cost(o, (b, w)) < cost(o, e).$$

Proof. First, we remark a simple fact. Let x, y, z be points in P such that both line segments xz and zy are lying in the inside of P. Then the following inequality obviously holds.

$$|sp(x, y)| \le |xz| + |zy|. \qquad (2)$$

Notice the equality holds only when either (i) $sp(x, y)$ is a line segment xy and z is on xy, or (ii) $sp(x, y)$ is composed of two line segments xz and zy, i.e., y is not visible from x (z is a blocking vertex with respect to x). From this observation and since b is visible from o (i.e., $|sp(o, b)| = |ob|$),

$$|sp(o, w)| < |ob| + |bw| = |sp(o, b)| + |bw|. \qquad (3)$$

Besides, from the triangle inequality with respect to b, v^1_e and v^2_e,

$$cost(b, e) = |sp(b, v^1_e)| + |sp(b, v^2_e)| - |e| \ge 0. \qquad (4)$$

Furthermore both $sp(o, v^1_e)$ and $sp(o, v^2_e)$ pass through b. Hence, we have

$$|sp(o, b)| + |sp(b, v^1_e)| = |sp(o, v^1_e)| \text{ and } |sp(o, b)| + |sp(b, v^2_e)| = |sp(o, v^2_e)|. \qquad (5)$$

Thus,

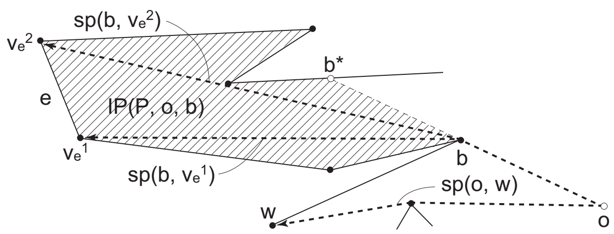

Fig. 1 Illustration of $sp(b, v_e^1)$, $sp(b, v_e^2)$ and $sp(o, w)$

$$\begin{aligned} cost(o,(b,w)) &= |sp(o,b)| + |sp(o,w)| - |bw| \\ &< |sp(o,b)| + |sp(o,b)| + |bw| - |bw| && \text{(from (3))} \\ &\le 2|sp(o,b)| + |sp(b,v_e^1)| + |sp(b,v_e^2)| - |e| && \text{(from (4))} \\ &= cost(o,e) && \text{(from (5))} \end{aligned}$$

holds. □

For e_{opt} defined by (1), the following corollary is immediate from Lemma 2.

Corollary 1. *For an invisible polygon $IP(P,o,b)$ defined by a blocking vertex b, let $e \in E$ be a polygon edge both endpoints of which are in $IP(P,o,b)$. Then e cannot be e_{opt} satisfying* (1).

Based on Corollary 1, candidates of e_{opt} are edges of $VP(P,o)$.

In what follows, we propose an online algorithm, AOE(Avoiding One Edge). By Lemma 1, the offline optimal algorithm chooses the edge e_{opt} which satisfies (1). But we cannot obtain the whole information about P. So, the seemingly best strategy based on the information of $VP(P,o)$ is to choose an edge in the same way as the offline optimal algorithm, assuming that there is no invisible polygon, namely $P = VP(P,o)$. Let E_1^* denote an edge set composed of all $e \in E$ such that both endpoints of e are visible from o, E_2^* denote a set of virtual edges on the boundary of $VP(P,o)$ and $E^* = E_1^* \cup E_2^*$. Also for a virtual edge $e \in E_2^*$, endpoints of e are labeled as v_e^1, v_e^2 in clockwise order around o and let $cost(o,e)$ denote the value of $|sp(o,v_e^1)|+|sp(o,v_e^2)|-|e|$. Let $e^* \in E^*$ be an edge satisfying the following equation.

$$cost(o,e^*) = \min_{e \in E^*} cost(o,e) \tag{6}$$

Then Algorithm AOE is described as follows.

Step 1: Choose $e^* \in E^*$ satisfying (6).

Step 2: If $e^* \in E_1^*$ then let $e_{\text{avoid}} = e^*$, or else let e_{avoid} be an edge of P containing e^*.

Step 3: Follow the tour $sp(o, v^2_{e_{\text{avoid}}}) \to bp(v^2_{e_{\text{avoid}}}, v^1_{e_{\text{avoid}}}) \to sp(v^1_{e_{\text{avoid}}}, o)$.

3. Competitive Analysis of AOE

First, we show the following lemma.

Lemma 3. *Let x be a point on the boundary of P and e^* be an edge satisfying* (6). *If x is visible from the origin o, then*

$$\frac{cost(o,e^*)}{2} \le |ox|.$$

Proof. Let $e' \in E^*$ be an edge of $VP(P,o)$ containing x. Then from (2), we have $|ox| \ge |sp(o,v_{e'}^1)| - |xv_{e'}^1|$ and $|ox| \ge |sp(o,v_{e'}^2)| - |xv_{e'}^2|$. Therefore, we obtain

$$\begin{aligned} 2|ox| &\ge |sp(o,v_{e'}^1)| + |sp(o,v_{e'}^2)| - |xv_{e'}^1| - |xv_{e'}^2| = |sp(o,v_{e'}^1)| + |sp(o,v_{e'}^2)| - |e'| \\ &\ge cost(o,e^*), \end{aligned}$$

namely $|ox| \ge cost(o,e^*)/2$. □

Furthermore, we show a lemma which plays a crucial role in our analysis.

Lemma 4. *Let L be the length of the boundary of P and e^* be an edge satisfying* (6). *Then the following inequality holds.*

$$L \ge \pi \cdot cost(o,e^*). \tag{7}$$

Proof. Let C be a circle centered at the origin o with the radius of $cost(o,e^*)/2$. From Lemma 3, any edge of P does not intersect C. Thus L is greater than the length of the circumference of C, namely

$$L \ge 2\pi \cdot \frac{cost(o,e^*)}{2} = \pi \cdot cost(o,e^*)$$

holds. □

Theorem 1. *The competitive ratio of Algorithm* AOE *is at most* 1.319.

Proof. The cost of Algorithm AOE obviously satisfies

$$|\mathsf{AOE}(P)| = L + cost(o, e^*).$$

On the other hand, the cost of the offline optimal algorithm satisfies $|\mathsf{OPT}(P)| = L + cost(o, e_{opt})$ holds from Lemma 1. By the triangle inequality, $cost(o, e_{opt}) \geq 0$, namely $|\mathsf{OPT}(P)| \geq L$ holds. Thus we have

$$\frac{|\mathsf{AOE}(P)|}{|\mathsf{OPT}(P)|} \leq \frac{L + cost(o, e^*)}{L} = 1 + \frac{cost(o, e^*)}{L}.$$

From this and (7),

$$\frac{|\mathsf{AOE}(P)|}{|\mathsf{OPT}(P)|} \leq 1 + \frac{cost(o, e^*)}{\pi \cdot cost(o, e^*)} = 1 + \frac{1}{\pi} \leq 1.319$$

is obtained. □

Theorem 1 gives an upper bound of the competitive ratio. In the followings, we will obtain a better bound by detailed analysis. First, we improve a lower bound of $|\mathsf{OPT}(P)|$. Note that for some points $x, y, z \in P$ such that both y and z are visible from x and the line segment yz is lying in P, we call $\angle yxz$ the *visual angle* at x formed by yz.

Lemma 5. *For an edge $e^* \in E^*$ satisfying (6), let $d = cost(o, e^*)$ and θ ($0 \leq \theta \leq \pi$) be a visual angle at o formed by a visible part of e_{opt}. Then*

$$|\mathsf{OPT}(P)| \geq L + d - d\sin\frac{\theta}{2}. \tag{8}$$

Proof. We first show the following claim.

Claim 1. *Let $b_1 \in V$ (resp. b_2) be the vertex visible from o such that the path $sp(o, v^1_{e_{opt}})$ (resp. $sp(o, v^2_{e_{opt}})$) passes through b_1 (resp. b_2) (see Fig. 2). Then*

$$cost(o, e_{opt}) \geq |ob_1| + |ob_2| - |b_1b_2|. \tag{9}$$

Proof. This follows from $|sp(o, v^1_{e_{opt}})| = |ob_1| + |sp(b_1, v^1_{e_{opt}})|$, $|sp(o, v^2_{e_{opt}})| = |ob_2| + |sp(b_2, v^2_{e_{opt}})|$ and $|e_{opt}| = |sp(v^1_{e_{opt}}, v^2_{e_{opt}})| \leq |sp(b_1, v^1_{e_{opt}})| + |b_1b_2| + |sp(b_2, v^2_{e_{opt}})|$. □

From (9), we have

$$\begin{aligned}|\mathsf{OPT}(P)| &= L + cost(o, e_{opt}) \\ &\geq L + |ob_1| + |ob_2| - |b_1b_2|. \end{aligned} \tag{10}$$

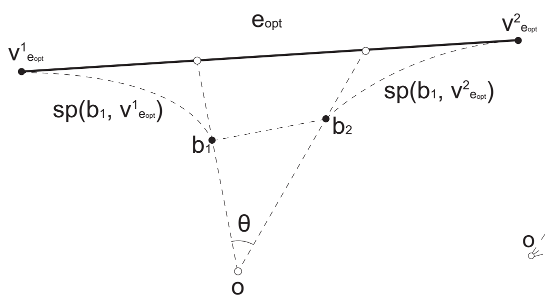

Fig. 2 A visible part of e_{opt} from o

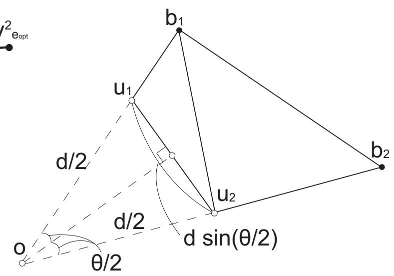

Fig. 3 u_1 and u_2

Furthermore b_1 and b_2 satisfy $|ob_1| \geq d/2$ and $|ob_2| \geq d/2$ from Lemma 3. Hence there exist points u_1, u_2 on line segments ob_1, ob_2 such that $|ou_1| = |ou_2| = d/2$ (see Fig. 3). Then, from the triangle inequality with respect to u_1, u_2 and b_1,

$$|u_1u_2| \geq |u_2b_1| - |b_1u_1| = |u_2b_1| - (|ob_1| - \frac{d}{2})$$

holds. Similarly we have

$$|u_2b_1| \geq |b_1b_2| - |u_2b_2| = |b_1b_2| - (|ob_2| - \frac{d}{2}).$$

Thus we have

$$\begin{aligned} d - |u_1u_2| &\leq d - \{|u_2b_1| - (|ob_1| - \frac{d}{2})\} = \frac{d}{2} + |ob_1| - |u_2b_1| \\ &\leq \frac{d}{2} + |ob_1| - \{|b_1b_2| - (|ob_2| - \frac{d}{2})\} = |ob_1| + |ob_2| - |b_1b_2|. \end{aligned} \tag{11}$$

In addition, the length of u_1u_2 satisfies the following equation.

$$|u_1u_2| = \frac{d}{2} \cdot 2\sin\frac{\theta}{2} = d\sin\frac{\theta}{2}. \tag{12}$$

By (10), (11) and (12),

$$|\mathsf{OPT}(P)| \geq L + d - |u_1u_2| = L + d - d\sin\frac{\theta}{2}.$$

is shown. □

Secondly, we show a better lower bound of L.

Lemma 6. *Let d and θ as defined in* Lemma 5. *Then*

$$L \geq d(\pi - \frac{\theta}{2} + \tan\frac{\theta}{2}). \tag{13}$$

Proof. Let C be a circle centered at o with radius $d/2$. From Lemma 3, any edge of P does not intersect C. Also let endpoints of a visible part of e_{opt} from o be w_1, w_2 in clockwise order around o. Then, we consider two cases; (Case 1) $\angle ow_1w_2 \leq \pi/2$ and $\angle ow_2w_1 \leq \pi/2$ and (Case 2) $\angle ow_1w_2 > \pi/2$ and $\angle ow_2w_1 \leq \pi/2$ (see Fig. 4, 5). Note that the case of $\angle ow_1w_2 \leq \pi/2, \angle ow_2w_1 > \pi/2$ can be treated in a manner similar to Case 2. Case 1: Let w_1^* (resp. w_2^*) be a point on

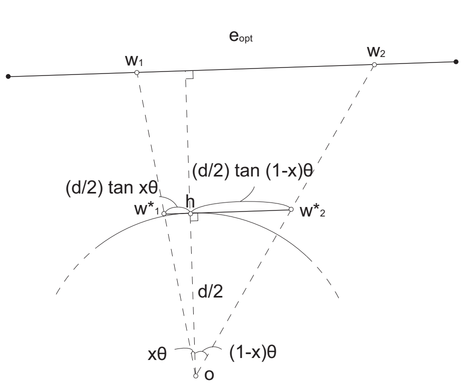

Fig. 4 Case 1

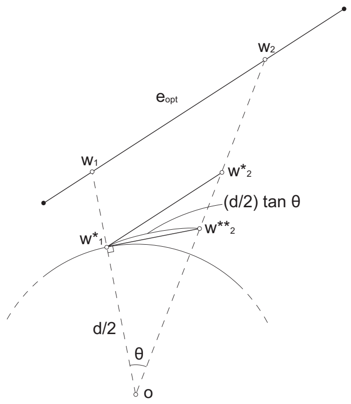

Fig. 5 Case 2

the line segment ow_1 (resp. ow_2) such that w_1w_2 is parallel to $w_1^*w_2^*$ and the line segment $w_1^*w_2^*$ touches the circle C and let h be a tangent point of $w_1^*w_2^*$ and C. Also let $\angle w_1oh = x\theta$ and $\angle w_2oh = (1-x)\theta$ with some x $(0 \leq x \leq 1)$. Then the length of $w_1^*w_2^*$ satisfies

$$|w_1^*w_2^*| = \frac{d}{2}\tan x\theta + \frac{d}{2}\tan(1-x)\theta.$$

The right-hand side of this equation attains the minimum value when $x = 1/2$. Thus

$$|w_1^*w_2^*| \geq \frac{d}{2}\tan\frac{\theta}{2} + \frac{d}{2}\tan\frac{\theta}{2} = d\tan\frac{\theta}{2}. \tag{14}$$

Furthermore the sum of the visual angle at o formed by a visible part of the boundary other than w_1w_2 is equal to $2\pi - \theta$. Hence we have

$$L \geq \frac{d}{2}(2\pi - \theta) + |w_1w_2|. \tag{15}$$

Since $|w_1w_2| \geq |w_1^*w_2^*|$ obviously holds, from (14) and (15), we obtain

$$L \geq \frac{d}{2}(2\pi - \theta) + d\tan\frac{\theta}{2} = d(\pi - \frac{\theta}{2} + \tan\frac{\theta}{2}).$$

Case 2: Let w_1^* (resp. w_2^*) be a point on the line segment ow_1 (resp. ow_2) such that w_1w_2 is parallel to $w_1^*w_2^*$ and $|ow_1^*| = d/2$ (the circumference of C passes through w_1^*). Also let w_2^{**} an intersection point of the line segment ow_2 and the lineperpendicular to the line segment ow_1 through w_1^*. Then

$$|w_1^*w_2^*| > |w_1^*w_2^{**}| = \frac{d}{2}\tan\theta \geq d\tan\frac{\theta}{2}.$$

In the same way as Case 1, we obtain $L \geq d(\pi - \theta/2 + \tan(\theta/2))$. □

By Lemma 5 and 6, we prove the following theorem.

Theorem 2. *The competitive ratio of Algorithm* AOE *is at most* 1.219.

Proof. Let d and θ as defined in Lemma 5. Since $|\mathsf{AOE}(P)| = L + d$ holds, from (8), (13), we have

$$\frac{|\mathsf{AOE}(P)|}{|\mathsf{OPT}(P)|} \leq \frac{L+d}{L+d-d\sin\frac{\theta}{2}} \leq \frac{d(\pi - \frac{\theta}{2} + \tan\frac{\theta}{2}) + d}{d(\pi - \frac{\theta}{2} + \tan\frac{\theta}{2}) + d - d\sin\frac{\theta}{2}}$$
$$= \frac{\pi - \frac{\theta}{2} + \tan\frac{\theta}{2} + 1}{\pi - \frac{\theta}{2} + \tan\frac{\theta}{2} + 1 - \sin\frac{\theta}{2}} \quad (0 \leq \theta \leq \pi). \tag{16}$$

In the followings, we compute the maximum value of (16),

$$\max_{0\le\theta\le\pi}\left\{z(\theta)=\frac{\pi-\frac{\theta}{2}+\tan\frac{\theta}{2}+1}{\pi-\frac{\theta}{2}+\tan\frac{\theta}{2}+1-\sin\frac{\theta}{2}}\right\}. \tag{17}$$

Generally the following fact about the fractional program is known[1),11)].

Fact 1. *Let $X\subseteq\mathbb{R}^n$, $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$. Let us consider the following fractional program formulated as*

$$\text{maximize}\left\{h(x)=\frac{f(x)}{g(x)}\;\middle|\; x\in X\right\}, \tag{18}$$

where $g(x)>0$ is assumed for any $x\in X$. Let $x^*\in\operatorname{argmax}_{x\in X}h(x)$ denote an optimal solution of (18) and $\lambda^*=h(x^*)$ denote the optimal value. Furthermore, with a real parameter λ, let $h_\lambda(x)=f(x)-\lambda g(x)$ and $M(\lambda)=\max_{x\in X}h_\lambda(x)$. Then $M(\lambda)$ is monotone decreasing for λ and the followings hold.
(i) $M(\lambda)<0\Leftrightarrow\lambda>\lambda^*$, (ii) $M(\lambda)=0\Leftrightarrow\lambda=\lambda^*$, (iii) $M(\lambda)>0\Leftrightarrow\lambda<\lambda^*$.

In the same way as Theorem 2, with a real parameter λ, we define $z_\lambda(\theta)$ and $M(\lambda)$ for $z(\theta)$ as follows.

$$z_\lambda(\theta)=\pi-\frac{\theta}{2}+\tan\frac{\theta}{2}+1-\lambda(\pi-\frac{\theta}{2}+\tan\frac{\theta}{2}+1-\sin\frac{\theta}{2})\quad(0\le\theta\le\pi),$$
$$M(\lambda)=\max_{0\le\theta\le\pi}z_\lambda(\theta).$$

From Fact 1 (ii), λ^* satisfying $M(\lambda^*)=0$ is equal to (17), i.e., the maximum value of $z(\theta)$. Hence we only need to compute λ^*.

Finally, let $\theta^*_\lambda\in\operatorname{argmax}_{0\le\theta\le\pi}z_\lambda(\theta)$, then we show θ^*_λ is unique. A derivative of $z_\lambda(\theta)$ is calculated as

$$\frac{dz_\lambda}{d\theta}=-\frac{\lambda-1}{2}\tan^2\frac{\theta}{2}+\frac{\lambda}{2}\cos\frac{\theta}{2}.$$

This derivative is monotone decreasing in the interval $0\le\theta\le\pi$, therefore $z_\lambda(\theta)$ is concave in this interval, then θ^*_λ is unique. Indeed when $\lambda=1.219$, $\theta^*_\lambda\simeq2.0706$ then $M(1.219)\simeq-0.0010<0$. Also when $\lambda=1.218$, $\theta^*_\lambda\simeq2.0718$ then $M(1.218)\simeq0.0029>0$. Thus we obtain $1.218<\lambda^*<1.219$. □

3.1 Lower Bound

Theorem 3. *The competitive ratio of Algorithm* AOE *is at least* 1.040.

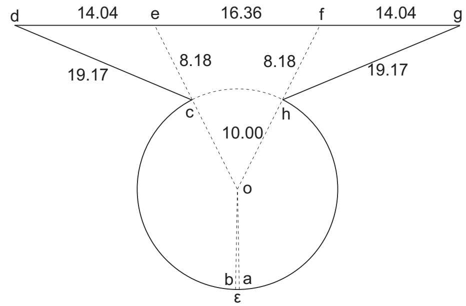

Fig. 6 Worst case polygon WP

Proof. We consider how Algorithm AOE works for a polygon WP illustrated in Fig. 6. We assume that the greater arc from h to c in clockwise ordering of a circle with radius 10.00 centered at o in the figure is in fact a chain composed of sufficiently many small edges of length ϵ. For each edge e along the arc hc, $cost(o,e)=20.00-\epsilon$ holds. Also the algorithm calculates the cost of a virtual edge (e,f) as $cost(o,(e,f))\simeq10.00+8.18+10.00+8.18-18.36=20.00$. Comparing these two values, the algorithm chooses an edge (a,b). Since $L\simeq136.26$ holds, the cost of Algorithm AOE for WP satisfies

$$|\mathsf{AOE}(WP)|\simeq136.26+20.00-\epsilon\ge156.26-\epsilon. \tag{19}$$

On the other hand, $(d,g)=e_{opt}$ because $cost(o,(d,g))\simeq13.89<20.00-\epsilon$ holds. Thus the cost of the offline optimal algorithm for WP satisfies

$$|\mathsf{OPT}(WP)|\simeq136.26+13.89\le150.16. \tag{20}$$

From (19) and (20), we obtain

$$\frac{|\mathsf{AOE}(WP)|}{|\mathsf{OPT}(WP)|}\ge\frac{156.26-\epsilon}{150.16}\ge1.0406-\frac{\epsilon}{150.16}.$$

By letting ϵ be sufficiently small, we prove the theorem. □

4. Competitive Analysis for Rectilinear Polygon

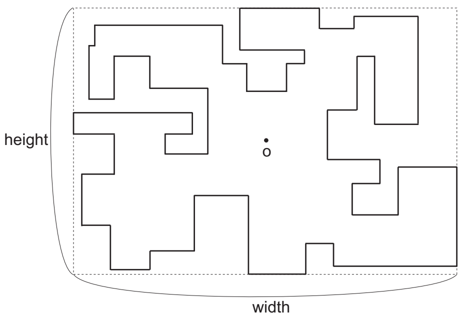

Fig. 7 A rectilinear polygon

In this section, we analyze the competitive ratio of AOE for a rectilinear polygon. Generally a rectilinear polygon is defined as a simple polygon all of whose interior angles are $\pi/2$ or $3\pi/2$. Edges of the rectilinear polygon are classified as horizontal or vertical edges. Let R be a rectilinear polygon and R' be the minimum enclosing rectangle of R. Then we define the height of R' as the height of R and also the width of R' as the width of R. Note that the searcher follows the Euclidean shortest path even if he/she is in the rectilinear polygon.

Lemma 7. *For an edge $e^* \in E^*$ satisfying (6), let $d = cost(o, e^*)$ and θ ($0 \leq \theta \leq \pi$) be a visual angle at o formed by a visible part of e_{opt}. Then*

$$L \geq \max\{4d, 2d + 2d \tan\frac{\theta}{2}\}. \tag{21}$$

Proof. First, we show $L \geq 4d$. Let C be a circle centered at o with the radius of $d/2$. From Lemma 3, any edge of R does not intersect C (see Fig. 8). Thus each of the height and width of R is greater than d (the diameter of C), namely $L \geq 4d$ holds. Secondly, we show $L \geq 2d + 2d\tan(\theta/2)$. Note that we should just consider the case of $4d \leq 2d + 2d\tan(\theta/2)$, namely $\pi/2 \leq \theta \leq \pi$ because $L \geq 4d$ has been proved. Without loss of generality we assume that e_{opt} is a horizontal edge. We label endpoints of a visible part of e_{opt} from o as w_1, w_2

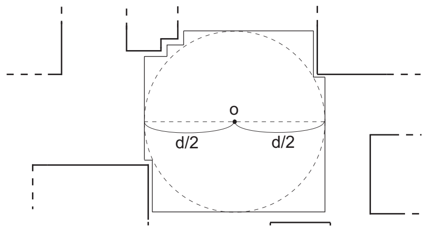

Fig. 8 $L \geq 4d$

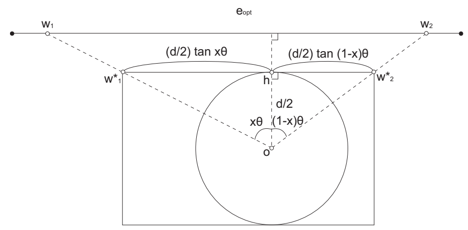

Fig. 9 $L \geq 2d + 2d\tan(\theta/2)$

in clockwise order around o. Let w_1^* (resp. w_2^*) be a point on the line segment ow_1 (resp. ow_2) such that w_1w_2 is parallel to $w_1^*w_2^*$ and the line segment $w_1^*w_2^*$ touches the circle C and h be a tangent point of $w_1^*w_2^*$ and C (see Fig. 9). Also let $\angle w_1oh = x\theta$ and $\angle w_2oh = (1-x)\theta$ with some x ($0 \leq x \leq 1$). Then the length of $w_1^*w_2^*$ satisfies

$$\begin{aligned}|w_1^*w_2^*| &= \frac{d}{2}\tan x\theta + \frac{d}{2}\tan(1-x)\theta \\ &\geq \frac{d}{2}\tan\frac{\theta}{2} + \frac{d}{2}\tan\frac{\theta}{2} = d\tan\frac{\theta}{2}.\end{aligned}$$

Thus the width of R is greater than $d\tan(\theta/2)$ and the height of R is greater than d, then $L \geq 2d + 2d\tan(\theta/2)$ holds. □

By Lemma 7, we prove the following theorem.

Theorem 4. *For a rectilinear polygon, the competitive ratio of Algorithm* AOE *is at most* 1.167.

Proof. Based on (21), we consider two cases; (Case 1) $0 \leq \theta < \pi/2$ and (Case 2) $\pi/2 \leq \theta \leq \pi$. Note that $4d > 2d + 2d\tan(\theta/2)$ holds in Case 1 and $4d \leq 2d + 2d\tan(\theta/2)$ holds in the other.

Case 1: From $L \geq 4d$, (8) and (13), we obtain

$$\begin{aligned}\frac{|\mathsf{AOE}(P)|}{|\mathsf{OPT}(P)|} &\leq \frac{L+d}{L+d-d\sin\frac{\theta}{2}} \leq \frac{4d+d}{4d+d-d\sin\frac{\theta}{2}} = \frac{5}{5-\sin\frac{\theta}{2}} \\ &< \frac{5}{5-\sin\frac{\pi}{4}} \leq 1.165.\end{aligned}$$

Case 2: From $L \geq 2d + 2d\tan(\theta/2)$, (8) and (13), we obtain

$$\frac{|\mathsf{AOE}(P)|}{|\mathsf{OPT}(P)|} \leq \frac{L+d}{L+d-d\sin\frac{\theta}{2}} \leq \frac{2d+2d\tan\frac{\theta}{2}+d}{2d+2d\tan\frac{\theta}{2}+d-d\sin\frac{\theta}{2}} = \frac{3+2\tan\frac{\theta}{2}}{3+2\tan\frac{\theta}{2}-\sin\frac{\theta}{2}}. \quad (22)$$

We will compute the maximum value of (22) as in the proof of Theorem 2 by defining $z_\lambda(\theta)$ and $M(\lambda)$ for a real parameter λ as follows.

$$z_\lambda(\theta) = 3+2\tan\frac{\theta}{2} - \lambda(3+2\tan\frac{\theta}{2}-\sin\frac{\theta}{2}) \quad (\frac{\pi}{2} \leq \theta \leq \pi)$$
$$M(\lambda) = \max_{\frac{\pi}{2}\leq\theta\leq\pi} z_\lambda(\theta)$$

Let $\theta^*_\lambda \in \mathrm{argmax}_{0\leq\theta\leq\pi}\, z_\lambda(\theta)$, then a derivative of $z_\lambda(\theta)$ is calculated as

$$\frac{dz_\lambda}{d\theta} = -(\lambda-1)\frac{1}{\cos^2\frac{\theta}{2}} + \frac{\lambda}{2}\cos\frac{\theta}{2}.$$

This derivative is monotone decreasing in the interval $\pi/2 \leq \theta \leq \pi$, therefore $z_\lambda(\theta)$ is concave in this interval, then θ^*_λ is unique. Indeed when $\lambda = 1.167$, $\theta^*_\lambda \simeq 1.7026$ then $M(1.167) \simeq -0.0044 < 0$. Also when $\lambda = 1.166$, $\theta^*_\lambda \simeq 1.7056$ then $M(1.166) \simeq 7.6 \times 10^{-5} > 0$. Thus we obtain $1.166 < \lambda^* < 1.167$. □

5. Discussion and Open Problems

We believe that the upper bound of the competitive ratio can be improved: the least upper bound could be close to the lower bound 1.04 given in Section 3.1.

As one of many variations of online TSP, we could consider online TSP with multiple searchers. In this problem, all searchers are initially at the same origin $o \in P$. The goal of the exploration is that each vertex is visited by at least one searcher and that all searchers return to the origin o. We regard the time when the last searcher comes back to the origin as the cost of the exploration. Note that our algorithm can be easily adapted to the case of online TSP with 2-searchers. For offline TSP with k-searchers, Frederickson et al.[4] proposed a $(e + 1 - 1/k)$-approximation algorithm, where e is the approximation ratio of some 1-searcher algorithm. Their idea is splitting a TSP tour given by some 1-searcher algorithm into k parts such that the cost of each part is equal, where the cost of a part is the length of the shortest tour from o which passes along the part. When $k = 2$, we can apply this idea to our algorithm as follows. First, choose similarly $e^* \in E^*$ satisfying (6). Then let one searcher go to $v^1_{e^*}$ and walk counterclockwise along the boundary of P, and let symmetrically the other go to $v^2_{e^*}$ and walk clockwise. When two searchers meet at a point on the boundary, two searchers come back together to o along the shortest path in the inside P. In this case, we obtain an upper bound 1.719. However, when $k \geq 3$, the above-mentioned idea cannot be directly applied. So, it remains open.

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