一般化行列固有値問題の精度保証付き評価
On verified estimation of eigenvalue of generalized matrix eigenvalue problem

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Matrix eigenvalue problem

Bounding eigenvalues of generalized eigenvalue problem:

$$Ax = \lambda Bx$$

where A, B are square symmetric and B positive definite.
Notations: (1) $\lambda_i(A,B)$: the i-th eigenvalue on increasing order of magnitude; $\lambda_i(A) := \lambda_i(A, I_n)$ (I_n: identity matrix); (2) $A \succ 0$: $\Leftrightarrow$ A is positive definite matrix.

Failure of approximate computation

We solve the Lapace eigenvalue problem over unit squre domain by variational method with polynomial of degree 10 as trial function. The approximate result shows that

$$\lambda_1^h = 19.7392088021787\underline{02}$$

However, this is in contradict to theoretial estimation:

$$\lambda_1^h \geq \lambda_1 = 2\pi^2 = 19.7392088021787\underline{16}...$$

The verified computation gives correct upper bound for λ_1:

$$\lambda_1^h = 19.7392088021\underline{82223}\,.$$

Verified computation

New perturbation theorem for eigenvalue problem:

Theorem 1. *Let B be symmetric positive matrix and A_1, A_2 symmetric ones. Suppose quantity ϵ satisfies*

$$\epsilon B - (A_2 - A_1) \succ 0, \quad \epsilon B - (A_1 - A_2) \succ 0 \qquad (1)$$

Then

$$|\lambda_i(A_1, B) - \lambda_i(A_2, B)| \leq \epsilon$$

Remark: The condition (1) is weaker than classical one:

$$\epsilon \leq \|A_1 - A_2\|_2 \,/\, \min \lambda_i(B)\,. \qquad (2)$$

Algorithm for bounding eigenvalue

Two steps for bounding eigenvalues

1) Verify index of eigenvalue through LDL fraction with error estimation (Theorem 1 needed);

2) Sharpen bound of eigenvalue through Lehmann-Behnke's method.

LDL fraction to verify eigenvalue roughly

1 Calculate $\lambda_k^* \approx \lambda_k(A,B)$.

2 Choose two small proper positive quantities σ_1 and σ_2 and perform approximate LDL fraction:

$$A-(\lambda_k^*-\sigma_1)B \approx L_1^T D_1 L_1, \quad A-(\lambda_k^*+\sigma_2)B \approx L_2^T D_2 L_2.$$

3 Compute η_1 and η_2 as small as possible such that

$$\eta_1 B - \left(A - (\lambda_n^* - \sigma_1)B - L_1^T D_1 L_1\right) \succ 0,$$

$$\eta_2 B - \left(A - (\lambda_n^* + \sigma_2)B - L_2^T D_2 L_2\right) \succ 0\,.$$

4 Denote by $p(\leq k-1)$ and $q(\geq k)$ the negative eigenvalues of D_1 and D_2 then

$$\{\lambda_{p+1}, \cdots, \lambda_q\} \subset [\rho - \sigma_1 - \eta_1, \rho + \sigma_2 + \eta_2)\,.$$

Lehmann-Behnke's method to sharpen the bounds

- Let $m \in \mathcal{N}$; $u_1, \cdots, u_m$ are linearly independent vectors of $\mathcal{R}^n$; let $v_i \in \mathcal{R}^n$ for $i = 1, \cdots, m$.
- Let $\sigma \in \mathcal{R}$. Define A_0, A_1, A_2, $\hat{A}$ and $\hat{B}$ by

$$A_0 := (u_i^T B u_k)_{i,k}, \quad A_1 := (u_i^T A u_k)_{i,k}$$

$$A_2 := (u_i^T A B^{-1} A u_k)_{i,k}\ (i,k = 1, \cdots, m)$$

$$\hat{A} := A_1 - \sigma A_0, \quad \hat{B} := A_2 - 2\sigma A_1 + \sigma^2 A_0$$

- In case $\hat{B} \succ 0$, denote by $\{\mu_i\}_{i=1,\cdots,m}$ the eigenvalues of $\hat{A}x = \mu\hat{B}x$ in increasing order, then there exit at least p eigenvlaues in

$$[\sigma + \frac{1}{\mu_p}, \sigma) \text{ for } p = 1, \cdots, m\,.$$

Implementation of the algorithm

The algorithm is implemented in Matlab language together with Interval toolbox. The code can deal with sparse matrix and the estimate clustered eigenvalues effectively.

Sample use:

```
%bounding largest three eigenvalues in magnitude
[bounds, index] = veigs(A,B,'lm',3)

%bounding smallest three eigenvalues in magnitude
[bounds, index] = veigs(A,B,'sm',3)

%bounding three eigenvalues nearest to 0.1
[bounds, index] = veigs(A,B,0.1,3)
```