IPSJ Transactions on Programming Vol. 4 No. 4 12–35 (Sep. 2011)

Regular Paper

Decidability of Reachability for Right-shallow Context-sensitive Term Rewriting Systems

Yoshiharu Kojima,[†1,†2] Masahiko Sakai,[†1]
Naoki Nishida,[†1] Keiichirou Kusakari[†1]
and Toshiki Sakabe[†1]

The reachability problem for an initial term, a goal term, and a rewrite system is to decide whether the initial term is reachable to goal one by the rewrite system or not. The innermost reachability problem is to decide whether the initial term is reachable to goal one by innermost reductions of the rewrite system or not. A context-sensitive term rewriting system (CS-TRS) is a pair of a term rewriting system and a mapping that specifies arguments of function symbols and determines rewritable positions of terms. In this paper, we show that both reachability for right-linear right-shallow CS-TRSs and innermost reachability for shallow CS-TRSs are decidable. We prove these claims by presenting algorithms to construct a tree automaton accepting the set of terms reachable from a given term by (innermost) reductions of a given CS-TRS.

1. Introduction

The reachability problem for two given terms s, t, and a reduction of a rewrite system R is to decide whether s is reachable to t by the reduction of R or not. Decision procedures of the problem for ordinary reductions of term rewriting systems (TRSs) are applicable to security protocol verification [7] and to solving other problems of TRSs. Since it is known that this problem is undecidable for general TRSs, efforts have been made to find subclasses of TRSs in which the reachability is decidable or undecidable [1),4),5),9),13)–18)], as shown in **Fig. 1**.

A context-sensitive TRS [12] (CS-TRS) is a pair of TRSs and a mapping from a function symbol to a set of natural numbers, where the mapping is used to specify that arguments are rewritable or not. CS-TRS is used in evaluating `if` `···` `then` `···` `else` `···` or `case` structures.

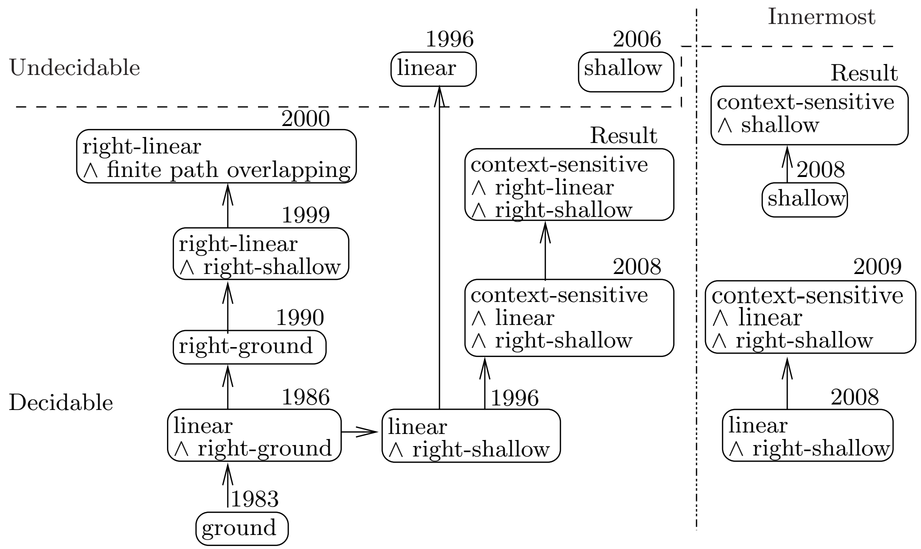

Fig. 1 Major subclasses of TRS in which reachability is decidable or undecidable.

We have already shown that reachability is decidable for linear right-shallow CS-TRSs [10]. However, linear right-shallow is not a large enough class to express practical programs (e.g., multiplication).

In this paper, we show that reachability is decidable for right-linear right-shallow CS-TRSs. Right-linear right-shallow, however, is still not enough to express practical programs precisely, but we can express programs closer to the precise one.

Innermost reduction is a strategy that rewrites innermost redexes, and is known as good at representing call-by-value computation widely used in most programming languages. Therefore, the languages that adopt call-by-value computation and `if` `···` `then` `···` `else` `···` structures (e.g., C languages) have computation models defined by the innermost reduction of CS-TRSs. For innermost reduction of TRSs and CS-TRSs, some decidable classes of reachability are known [6),8),10),11)]. However, these classes do not have a large enoughe similarly to the case of the ordinary reduction of CS-TRSs.

†1 Graduate School of Information Science, Nagoya University
†2 Research Fellow of the Japan Society for the Promotion of Science

In this paper, we also show that reachability for innermost reduction (innermost reachability) is decidable for shallow CS-TRSs.

We show the results of this paper by presenting two algorithms. The first algorithm constructs a tree automaton (TA) recognizing the set of terms reachable from a given term and a given right-linear right-shallow CS-TRS. This algorithm is based on the procedure for linear right-shallow CS-TRSs [10], and we introduce the idea in Refs. 14) and 17) to adopt the procedure to left-non-linear CS-TRSs as well. The second algorithm constructs a tree automaton with constraints between brothers (TACBB) that recognize the set of terms innermost reachable from a given term and a shallow CS-TRS. The second algorithm is achieved by introducing the TACBB $\mathcal{A}_{\mathrm{NF}}$ that accepts the set of normal forms to check whether each subterm is rewritable or unrewritable

2. Preliminary

We use the usual notations of rewrite systems [2] and tree automata [3]. Let F be a set of function symbols with fixed arity and X be an enumerable set of variables. The arity of function symbol f is denoted by $\mathrm{ar}(f)$. Function symbols with $\mathrm{ar}(f) = 0$ are *constants*. The set of *terms*, defined in the usual way, is denoted by $\mathcal{T}(F, X)$. A term is *linear* if no variable occurs more than once in the term. The set of variables occurring in t is denoted by $\mathrm{Var}(t)$. A term t is *ground* if $\mathrm{Var}(t) = \emptyset$. The set of ground terms is denoted by $\mathcal{T}(F)$.

A *position* in a term t is defined, as usual, as a sequence of positive integers, and the set of all positions in a term t is denoted by $\mathrm{Pos}(t)$, where the empty sequence ε is used to denote root position. The *depth* of a position p is defined as the length of p and denoted as $|p|$. The *height* $|t|$ of a term t is defined as $max(\{|p| \mid p \in \mathrm{Pos}(t)\})$. A term t is *shallow* if the depths of variable occurrences in t are all 0 or 1. The *subterm* of t at position p is denoted by $t|_p$, and $t[t']_p$ represents the term obtained from t by replacing the subterm $t|_p$ by t'. If a term s is a subterm of t and $s \neq t$, s is a *proper* subterm of t. We denote $s \trianglelefteq t$ ($s \triangleleft t$) such that a term s is a (proper) subterm of t. A *context* C is a term that contains the symbol $\Box$, and $C[t]_p$ represents the term obtained by replacing $\Box$ in the position p of C by t.

A *substitution* σ is a mapping from X to $\mathcal{T}(F, X)$ whose domain $\mathrm{Dom}(\sigma) = \{x \in X \mid x \neq \sigma(x)\}$ is finite. The term obtained by applying a substitution σ to a term t is written as $t\sigma$. The term $t\sigma$ is an *instance* of t.

A *rewrite rule* is an ordered pair of terms in $\mathcal{T}(F, X)$, written as $l \to r$, such that $l \notin X$ and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$. We say that variables in $\mathrm{Var}(l) \setminus \mathrm{Var}(r)$ are *erasing*. A *term rewriting system (over F)* (TRS) is a finite set of rewrite rules. *Rewrite relation* $\xrightarrow[R]{}$ induced by a TRS R is as follows: $s \xrightarrow[R]{} t$ if and only if $s = s[l\sigma]_p$, and $t = s[r\sigma]_p$ for some rule $l \to r \in R$, with substitution σ and position $p \in \mathrm{Pos}(s)$. We call $l\sigma$ a *redex*. We sometimes write $\xrightarrow[R]{}^{p}$ by presenting the position p explicitly.

A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*, *shallow*) if l is linear (resp. r is linear, l and r are linear, r is shallow, l and r are shallow). A TRS R is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*, *shallow*) if every rule in R is left-linear (resp. right-linear, linear, right-shallow, shallow).

Let $\to$ be a binary relation on a set $\mathcal{T}(F)$. We say $s \in \mathcal{T}(F)$ is a *normal form* (with respect to $\to$) if there exists no term $t \in \mathcal{T}(F)$ such that $s \to t$. If each proper subterm of redex is a normal form, we say the rewriting is *innermost*. We denote the innermost reduction of the relation $\to$ as $\to^{\mathrm{in}}$. We use $\circ$ to denote the composition of two relations. We write $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$. We also write $\xrightarrow{n}$ for the relation $\to \circ \cdots \circ \to$ composed of n $\to$'s. The set of *reachable terms* from a term in T with respect to the relation $\to$ is defined by $\to[T] = \{t \mid s \in T,\ s \xrightarrow{*} t\}$. The *reachability problem* (resp. *innermost reachability problem*) with respect to $\to$ is the problem that decides whether $s \xrightarrow{*} s'$ (resp. $s \xrightarrow{*}{}^{\mathrm{in}} s'$) or not, for given terms s and s'.

A mapping $\mu :\ F \to \mathcal{P}(\mathbb{N})$ is said to be a *replacement map* (or F-map) if $\mu(f) \subseteq \{1, \ldots, \mathrm{ar}(f)\}$ for all $f \in F$. A *context-sensitive term rewriting system* (CS-TRS) is the pair $\mathcal{R} = (R, \mu)$ of a TRS and a replacement map. The set of μ-*replacing positions* $\mathrm{Pos}^{\mu}(t)$ ($\subseteq \mathrm{Pos}(t)$) is recursively defined: $\mathrm{Pos}^{\mu}(t) = \{\varepsilon\}$ if t is a constant or a variable, otherwise $\mathrm{Pos}^{\mu}(f(t_1, \ldots, t_n)) = \{\varepsilon\} \cup \{ip \mid i \in \mu(f), p \in \mathrm{Pos}^{\mu}(t_i)\}$. The rewrite relation induced by a CS-TRS $\mathcal{R}$ is defined: $s \xhookrightarrow[\mathcal{R}]{} t$ if and only if $s \xrightarrow[R]{}^{p} t$ and $p \in \mathrm{Pos}^{\mu}(t)$. If a term t has no redex at $\mathrm{Pos}^{\mu}(t)$, we say t is a *context-sensitive normal form*. We denote the set of a context-sensitive normal form of $\mathcal{R}$ as $\mathrm{CS\text{-}NF}_{\mathcal{R}}$. If each proper subterm of redex is a context-sensitive normal form or not in a μ-replacing position, we say the rewriting with CS-TRS is *innermost*.

A *tree automaton* (TA) is a quadruple $\mathcal{A} = (F, Q, Q^f, \Delta)$ where F is a finite set of function symbols, Q is a finite set of states, $Q^f (\subseteq Q)$ is a set of final states, and Δ is a finite set of transition rules of the forms $f(q_1, \ldots, q_n) \to q$ or $q_1 \to q$ where $f \in F$ with $\mathrm{ar}(f) = n$, and $q_1, \ldots, q_n, q \in Q$. We sometimes omit F if it is not necessary to specify explicitly. We can regard Δ as a (ground) TRS over $F \cup Q$. The rewrite relation induced by Δ of $\mathcal{A}$ is called a *transition relation* denoted by $\xrightarrow[\Delta]{}$ or $\xrightarrow[\mathcal{A}]{}$. We denote $|\alpha|$ as the length of a transition sequence α (if α is $s \xrightarrow{n} t$, then $|\alpha| = n$). We say that a term s $(\in \mathcal{T}(F))$ is *accepted* by $q \in Q$ if $s \xrightarrow[\Delta]{*} q$, and if $q \in Q^f$, we also say that s is accepted by $\mathcal{A}$. The set of all terms accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$. We say $\mathcal{A}$ *recognizes* $\mathcal{L}(\mathcal{A})$. A set of terms T is *regular* if there exists a TA that recognizes T. We use a notation $\mathcal{L}(\mathcal{A}, q)$ or $\mathcal{L}(\Delta, q)$ to represent the set $\{s \mid s \xrightarrow[\Delta]{*} q, s \in \mathcal{T}(F)\}$. A TA $\mathcal{A}$ is *deterministic* if $s \xrightarrow[\Delta]{*} q$ and $s \xrightarrow[\Delta]{*} q'$ implies $q = q'$ for any $s \in \mathcal{T}(F)$. A TA $\mathcal{A}$ is *complete* if there exists $q \in Q$ such that $s \xrightarrow[\Delta]{*} q$ for any $s \in \mathcal{T}(F)$. A state $q \in Q$ of $\mathcal{A}$ is *accessible* if $\mathcal{L}(\mathcal{A}, q) \neq \emptyset$, and if every state in Q is accessible, $\mathcal{A}$ is *reduced*.

A tree *automaton with constraints between brothers* (TACBB) is a extended TA in which transition rules have *constraints between brothers*. Constraints between brothers are recursively defined: $\top$, $\bot$, equality $i = j$, and disequality $i \neq j$ are constraints between brothers where $i, j \in \mathbb{N}$, and if c_1 and c_2 are constraints between brothers, then a conjunction $c_1 \wedge c_2$ and a disjunction $c_1 \vee c_2$ are also constraints between brothers. A term $f(t_1, \ldots, t_n)$ *satisfies* the constraints between brothers c if c is true by assigning true to $\top$, equality $i = j$ if $t_i = t_j$, and disequality $i \neq j$ if $t_i \neq t_j$, and false to $\bot$, equality $i = j$ if $t_i \neq t_j$, and disequality $i \neq j$ if $t_i = t_j$. Each transition rule is of the form $f(q_1, \ldots, q_n) \xrightarrow{c} q$ or $q_1 \xrightarrow{\top} q$ where c is a constraint between brothers. A term $f(t_1, \ldots, t_n)$ can reach to a state q by the transition rule $f(q_1, \ldots, q_n) \xrightarrow{c} q$ Δ of a TACBB if $t_i \xrightarrow[\Delta]{*} q_i$ for $1 \leq i \leq n$ and $f(t_1, \ldots, t_n)$ satisfies c.

The following properties on TA and TACBB are known [3)].

Theorem 1 All of the following holds for TAs and TACBBs:
(1) For a given TA (TACBB) $\mathcal{A}$, there exists a deterministic complete reduced TA (TACBB) $\mathcal{A}'$ that recognizes $\mathcal{L}(\mathcal{A})$.
(2) The class of recognizable tree languages is closed under union, intersection, and complementation.
(3) The membership problem and the emptiness problem are decidable.

3. Decidability of Reachability for Right-linear Right-shallow CS-TRSs

In this section, we prove that reachability for right-linear right-shallow CS-TRSs is decidable. To this end, we show the algorithm $\mathrm{P_{cs}}$ that constructs a tree automaton recognizing the set of terms reachable by a right-linear right-shallow CS-TRS from an input term.

The algorithm $\mathrm{P_{cs}}$ is based on the algorithm in Ref. 9). In Ref. 9), if a term t matches both a rewrite rule and a transition rule, then we produce transition rules to accept the term obtained by the rewriting. For example, if we have the rewrite rule $a \to b$ and the transition rule $a \to q$, then we produce the transition rule $b \to q$. However this algorithm can only deal with linear right-shallow TRSs. Therefore, we introduce the ideas in Refs. 14) and 17) to deal with the left-non-linear system, and the idea in Ref. 10) to deal with context-sensitive TRSs.

In Refs. 14) and 17), to deal with left-non-linear TRSs, we use subsets of the set of states of input automata as the set of states of output automata. In Ref. 10), to deal with context-sensitive TRSs, each state q of input automata is divided to $\langle q, \mathtt{a}\rangle$ and $\langle q, \mathtt{i}\rangle$[⋆1]. We merge and modify these ideas to deal with right-linear right-shallow CS-TRSs. We show an example of automata construction in the following where it can be seen that the automaton obtained by $\mathrm{P_{cs}}$ recognizes the set of terms reachable from an input term correctly.

Example 2 Let $R = \{a \to b, b \to d, c \to d, f(x, x) \to g(x, c), g(x, x) \to h(x)\}$, $\mu(f) = \{1\}$, $\mu(g) = \{1, 2\}$, $\mu(h) = \emptyset$, and $\mathcal{A} = \langle Q, Q^f, \Delta\rangle$ where $Q = \{q^a, q^b, q^c, q^{f(a,b)}\}$, $Q^f = \{q^{f(a,b)}\}$, $\Delta = \{a \to q^a, b \to q^b, c \to q^c, f(q^a, q^b) \to q^{f(a,b)}\}$, and hence $\mathcal{L}(\mathcal{A}) = \{f(a, b)\}$. $\mathrm{P_{cs}}$ output the automaton $\mathcal{A}_* = \langle Q_*, Q^f_*, \Delta_*\rangle$ that recognizes $\xrightarrow[\mathcal{R}]{}[\{f(a, b)\}]$.

⋆1 In Ref. 10), divided states are denoted as $\tilde{q}$ and q.

The set of states Q_* is the set $\{\langle P, \mathtt{a}\rangle, \langle \{p\}, \mathtt{i}\rangle\}$ where $P \subseteq Q$, $P \neq \emptyset$, and $p \in Q$, Q^f_* is $\{\langle P^f, \mathtt{a}\rangle \mid P^f \subseteq Q, P^f \cap Q^f \neq \emptyset\}$ and the set of transition rules Δ_* is

$$\Delta_* = \left\{ \begin{array}{rcl} a & \to & \langle \{q^a\}, \mathtt{x}\rangle, \\ b & \to & \langle \{q^b\}, \mathtt{x}\rangle, \\ b & \to & \langle \{q^a\}, \mathtt{a}\rangle, \\ c & \to & \langle \{q^c\}, \mathtt{x}\rangle, \\ d & \to & \langle P_1, \mathtt{a}\rangle, \\ f(\langle \{q^a\}, \mathtt{x}\rangle, \langle \{q^b\}, \mathtt{i}\rangle) & \to & \langle \{q^{f(a,b)}\}, \mathtt{x}\rangle, \\ g(\langle \{q^b\}, \mathtt{a}\rangle, \langle \{q^c\}, \mathtt{a}\rangle) & \to & \langle \{q^{f(a,b)}\}, \mathtt{a}\rangle, \\ h(\langle \{q^b, q^c\}, \mathtt{a}\rangle) & \to & \langle \{q^{f(a,b)}\}, \mathtt{a}\rangle \end{array} \right\}$$

where $P_1 \subseteq \{q^a, q^b, q^c\}$ and $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$. We obtain $\mathcal{L}(\mathcal{A}_*) = \{f(a,b), f(b,b), f(d,b), g(b,c), g(d,c), g(b,d), g(d,d), h(d)\} = \hookrightarrow_{\mathcal{R}}[\mathcal{L}(\mathcal{A})]$ □

Here we describe Example 2. First, we obtain Q_* by augmenting parameter a or i to each state and taking subset of Q for the first components of the states. From the set of transition rules Δ_*, it can be seen that $\langle \{q^a\}, \mathtt{i}\rangle$, $\langle \{q^b\}, \mathtt{i}\rangle$, $\langle \{q^c\}, \mathtt{i}\rangle$, and $\langle \{q^{f(a,b)}\}, \mathtt{i}\rangle$ only accept the terms accepted by q^a, q^b, q^c and $q^{f(a,b)}$, and $\langle \{q^a\}, \mathtt{a}\rangle$, $\langle \{q^b\}, \mathtt{a}\rangle$, $\langle \{q^c\}, \mathtt{a}\rangle$, and $\langle \{q^{f(a,b)}\}, \mathtt{a}\rangle$ accept the terms reachable by $\mathcal{R}$ from the terms accepted by q^a, q^b, q^c, and $q^{f(a,b)}$, that is a, b, c, and $f(a,b)$, respectively. From $\mu(f) = \{1\}$, the state in the second argument of f in the transition rule must have i as its second component. In this way, $\mathcal{A}_*$ does not accepts the terms obtained by rewriting the second argument of f. Moreover, we have $\mathcal{L}(\mathcal{A}_*, \langle \{q^b, q^c\}, \mathtt{a}\rangle) = (\mathcal{L}(\mathcal{A}_*, \langle \{q^b\}, \mathtt{a}\rangle) \cap \mathcal{L}(\mathcal{A}_*, \langle \{q^c\}, \mathtt{a}\rangle))$. Indeed, the state $\langle \{q^b, q^c\}, \mathtt{a}\rangle$ accepts only d, and the term reachable from b and c is only d, too. Since the term that is reachable from $f(a,b)$ and matches $f(x,x)$ is only $f(b,b)$, we produce the transition rule $g(\langle \{q^b\}, \mathtt{a}\rangle, \langle \{q^c\}, \mathtt{a}\rangle) \to \langle \{q^{f(a,b)}\}, \mathtt{a}\rangle$ from the rewrite rule $f(x,x) \to g(x,c)$. Since the term that is reachable from $f(a,b)$ and matches $g(x,x)$ is only $g(d,d)$, we produce the transition rule $h(\langle \{q^b, q^c\}, \mathtt{a}\rangle) \to \langle \{q^{f(a,b)}\}, \mathtt{a}\rangle$ from the rewrite rule $g(x,x) \to h(x)$.

Concrete definition of the algorithm P_{cs} is the following.

Algorithm $\mathbf{P}_{\mathrm{cs}}$:

Input A term t and a right-shallow CS-TRS $\mathcal{R} = (R, \mu)$.

Output The TA $\mathcal{A}_* = \langle Q_*, Q^f_*, \Delta_*\rangle$ such that $\mathcal{L}(\mathcal{A}_*) = \hookrightarrow_{\mathcal{R}}[\{t\}]$, if $\mathcal{R}$ is right-linear.

Step 1 (initialize) (1) Prepare a TA $\mathcal{A} = \langle Q, Q^f, \Delta\rangle$ where each state q^s accepts $s \in \{t\} \cup \mathrm{RS}(R)$, and $\mathrm{RS}(R)$ is the set of a proper ground subterm of the right-hand sides of R. Here we assume $Q = \{q^s \mid s \trianglelefteq s', s' \in \{t\} \cup \mathrm{RS}(R)\}$, $Q^f = \{q^t\}$, and $\mathcal{L}(\mathcal{A}_{\mathrm{RS}}, q^s) = \{s\}$ for all q^s.

(2) Let
- $k := 0$,
- $Q_* := \{P(\subseteq Q) \mid P \neq \emptyset\} \times \{\mathtt{a}\} \cup \{\{q\} \mid q \in Q\} \times \{\mathtt{i}\}$,
- $Q^f_* = \{P^f(\subseteq Q) \mid P^f \cap Q^f \neq \emptyset\} \times \{\mathtt{a}\}$, and
- $\Delta_0 := \{f(\langle \{q_1\}, \mathtt{i}\rangle, \ldots, \langle \{q_n\}, \mathtt{i}\rangle) \to \langle \{q\}, \mathtt{i}\rangle \mid f(q_1, \ldots, q_n) \to q \in \Delta\}$
$$\cup \left\{ f(\langle \{q_1\}, \mathtt{x}_i\rangle, \ldots, \langle \{q_n\}, \mathtt{x}_n\rangle) \to \langle \{q\}, \mathtt{a}\rangle \;\middle|\; \begin{array}{l} f(q_1, \ldots, q_n) \to q \in \Delta, \\ \mathtt{x}_i = \left\{ \begin{array}{l} \mathtt{a} \cdots \text{if } i \in \mu(f), \\ \mathtt{i} \cdots \text{otherwise} \end{array} \right. \end{array} \right\}$$

Step 2 Let Δ_{k+1} be the set of transition rules produced by augmenting transition rules of Δ_k by the following inference rules. Let C be the context that has no variable:

(1) If there exists $\sigma : X \to T(F)$ such that $x_i\sigma \xrightarrow[\Delta_k]{*} \langle P_i, \mathtt{x}_i\rangle$ for all $1 \leq i \leq n$, we apply the following inference rule:

$$\frac{C[x_1, \ldots, x_n] \to g(r_1, \ldots, r_m) \in R, C[\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle] \xrightarrow[\Delta_k]{*} \langle \{q\}, \mathtt{a}\rangle}{g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \to \langle \{q\}, \mathtt{a}\rangle \in \Delta_{k+1}}$$

Let $I_j = \{i \mid x_i = r_j\}$. Each P'_j and $\mathtt{x}'_j$ is determined as follows:

- $$P'_j = \left\{ \begin{array}{ll} \{q^{r_j}\} & \cdots \text{if } r_j \notin X, \\ P_i & \cdots \text{if } r_j \in X \wedge \exists i \in I_j.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i \in I_j} P_i & \cdots \text{if } r_j \in X \wedge \forall i \in I_j.\mathtt{x}_i = \mathtt{a}. \end{array} \right.$$
- $$\mathtt{x}'_j = \left\{ \begin{array}{l} \mathtt{i} \cdots \text{if } j \notin \mu(g) \wedge (r_j \in X \Rightarrow \exists i \in I_j.\mathtt{x}_i = \mathtt{i}), \text{ and} \\ \mathtt{a} \cdots \text{otherwise.} \end{array} \right.$$

(2) If there exists $\sigma : X \to T(F)$ such that $x_i\sigma \xrightarrow[\Delta_k]{*} \langle P_i, \mathtt{x}_i\rangle$ for all $1 \leq i \leq n$, we apply the following inference rule:

$$\frac{C[x_1,\ldots,x_n] \to x_i \in R, C[\langle P_1,\mathtt{x}_1\rangle,\ldots,\langle P_n,\mathtt{x}_n\rangle] \xrightarrow[\Delta_k]{*} \langle\{q\},\mathtt{a}\rangle}{\langle P',\mathtt{a}\rangle \to \langle\{q\},\mathtt{a}\rangle \in \Delta_{k+1}}$$

Let $I = \{j \mid x_j = x_i\}$. P' is determined as follows:

- $P' = \begin{cases} P_i & \cdots \text{ if } \exists i \in I.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i\in I} P_i & \cdots \text{ if } \forall i \in I.\mathtt{x}_i = \mathtt{a}. \end{cases}$

Step 3 For all states $\langle P^1 \cup P^2, \mathtt{a}\rangle \in Q_*$ where $P^1 \neq P^2$, we add new transition rule to Δ_{k+1} as follows:

(1) $f(\langle P_1,\mathtt{x}_1\rangle,\ldots,\langle P_n,\mathtt{x}_n\rangle) \to \langle P^1 \cup P^2,\mathtt{a}\rangle \in \Delta_{k+1}$ where

- $P_i = \begin{cases} P_i^j & \cdots \text{ if } \mathtt{x}_i^j = \mathtt{i} \text{ for some } j \in \{1,2\} \text{ and } \mathcal{L}(\Delta_k,\langle P_i^1,\mathtt{x}_i^1\rangle) \cap \mathcal{L}(\Delta_k,\langle P_i^2,\mathtt{x}_i^2\rangle) \neq \emptyset, \text{ and} \\ P_i^1 \cup P_i^2 & \cdots \text{ if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \end{cases}$
- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{ if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a}, \text{ and} \\ \mathtt{i} \cdots \text{ if otherwise.} \end{cases}$

if $f(\langle P_1^j,\mathtt{x}_1^j\rangle,\ldots,\langle P_n^j,\mathtt{x}_n^j\rangle) \to \langle P^j,\mathtt{a}\rangle \in \Delta_k$ for $j \in \{1,2\}$.

Note that if $\mathcal{L}(\Delta_k,\langle P_i^1,\mathtt{x}_i^1\rangle) \cap \mathcal{L}(\Delta_k,\langle P_i^2,\mathtt{x}_i^2\rangle) = \emptyset$ and $\mathtt{x}_i^j = \mathtt{i}$ for some $j \in \{1,2\}$, then the transition rule is not produced.

(2) $\langle P_1' \cup P_2',\mathtt{a}\rangle \to \langle P_1 \cup P_2,\mathtt{a}\rangle \in \Delta_{k+1}$ if $\langle P_1',\mathtt{a}\rangle \to \langle P_1,\mathtt{a}\rangle \in \Delta_k$, and, $\langle P_2',\mathtt{a}\rangle \to \langle P_2,\mathtt{a}\rangle \in \Delta_k$ or $P_2' = P_2$.

Step 4 If $\Delta_{k+1} = \Delta_k$ then stop and set $\Delta_* := \Delta_k$; Otherwise $k := k+1$ and go to Step 2.

Example 3 Let us follow how the algorithm $\mathrm{P_{cs}}$ works. We input the right-linear right-shallow CS-TRS $\mathcal{R}$ of Example 2 and the term $f(a,b)$.

In the initializing step, at (1) of Step 1, we construct the automaton $\mathcal{A}$ of Example 2, and at (2) of Step 1, we have $Q_* = \{\langle P,\mathtt{a}\rangle, \langle\{p\},\mathtt{i}\rangle\}$ where $P \subseteq Q$, $P \neq \emptyset$, and $p \in Q$, $Q_*^f = \{\langle P_f,\mathtt{a}\rangle\}$ where $P_f \subseteq Q$ and $P_f \cap Q^f \neq \emptyset$, and $\Delta_0 = \{a \to \langle\{q^a\},\mathtt{x}\rangle, b \to \langle\{q^b\},\mathtt{x}\rangle, f(\langle\{q^a\},\mathtt{x}\rangle,\langle\{q^b\},\mathtt{i}\rangle) \to \langle\{q^{f(a,b)}\},\mathtt{x}\rangle\}$ where $\mathtt{x} \in \{\mathtt{a},\mathtt{i}\}$.

In the saturation step at $k = 0$, we produce the transition rules $\{b \to \langle\{q^a\},\mathtt{a}\rangle, d \to \langle\{q^b\},\mathtt{a}\rangle, d \to \langle\{q^c\},\mathtt{a}\rangle, g(\langle\{q^b\},\mathtt{a}\rangle,\langle\{q^a\},\mathtt{a}\rangle) \to \langle\{q^{f(a,b)}\},\mathtt{a}\rangle\}$ at Step 2.

At $k = 1$, we produce the transition rules $\{d \to \langle\{q^a\},\mathtt{a}\rangle, h(\langle\{q^b,q^c\},\mathtt{a}\rangle) \to \langle\{q^{f(a,b)}\},\mathtt{a}\rangle\}$ at Step 2 and $\{b \to \langle\{q^a,q^b\},\mathtt{a}\rangle, d \to \langle\{q^b,q^c\},\mathtt{a}\rangle\}$ at Step 3.

At $k = 2$, we produce the transition rules $\{d \to \langle\{q^a,q^b\},\mathtt{a}\rangle, d \to \langle\{q^a,q^c\},\mathtt{a}\rangle, d \to \langle\{q^a,q^b,q^c\},\mathtt{a}\rangle\}$ at Step 3.

The saturation steps stop at $k = 3$, and we have $\Delta_* = \Delta_3$. □

The algorithm $\mathrm{P_{cs}}$ eventually terminates at some k, because rewrite rules in R and states in Q_* are finite and hence possible transitions rules are finite. Apparently $\Delta_0 \subset \cdots \subset \Delta_k = \Delta_{k+1} = \cdots$.

Here we have two remarks.

Our first remark is that the state in which the second parameter is $\mathtt{a}$ does not always occur at rewritable positions. In Example 3, we have both $\mu(h) = \emptyset$ and the transition rule $h(\langle\{q^b,q^c\},\mathtt{a}\rangle) \to \langle q^{f(a,b)},\mathtt{a}\rangle$. However, this causes no problem. Indeed, the rewriting $h(b) \underset{\mathcal{R}}{\hookrightarrow} h(c)$ is forbidden but $h(c)$ is reachable from $g(b)$ as $g(b) \underset{\mathcal{R}}{\hookrightarrow} g(c) \underset{\mathcal{R}}{\hookrightarrow} h(c)$.

Our second remark is that the former part of the input for $\mathrm{P_{cs}}$ is a term while it is an arbitrary tree automaton in Refs. 14), 17), and 10). Otherwise, $\mathrm{P_{cs}}$ may output an incorrect automaton as shown in the following example:

Example 4 Let R be $\{a \to b, a \to d, c \to d, f(x,x) \to g(x)\}$, $\mu(f) = \mu(g) = \{1\}$, and $\mathcal{A} = \langle Q, Q^f, \Delta\rangle$ where $Q = \{q_1,q_2,q^f\}$, $Q^f = \{q^f\}$, $\Delta = \{a \to q_1, b \to q_2, c \to q_2, f(q_1,q_2) \to q^f\}$, and hence $\mathcal{L}(\mathcal{A}) = \{f(a,b), f(a,c)\}$. Thus, $\overset{*}{\underset{\mathcal{R}}{\hookrightarrow}}[\mathcal{L}(\mathcal{A})]$ is the set $\{f(a,b), f(b,b), f(d,b), f(a,c), f(b,c), f(d,c), g(b)\}$.

Then, $\mathrm{P_{cs}}$ output the automaton $\mathcal{A}_*$ of which transition rules in Δ_* are $\{a \to \langle\{q_1\},\mathtt{x}\rangle, b \to \langle\{q_2\},\mathtt{i}\rangle, b \to \langle P,\mathtt{a}\rangle, c \to \langle\{q_2\},\mathtt{x}\rangle, d \to \langle P,\mathtt{a}\rangle, f(\langle\{q_1\},\mathtt{x}\rangle,\langle\{q_2\},\mathtt{i}\rangle) \to \langle\{q^f\},\mathtt{x}\rangle, g(\langle\{q_2\},\mathtt{a}\rangle) \to \langle\{q^f\},\mathtt{a}\rangle\}$ where $P \in \{\{q_1\},\{q_2\},\{q_1,q_2\}\}$ and $\mathtt{x} \in \{\mathtt{a},\mathtt{i}\}$. Hence, $\mathcal{A}_*$ accepts the terms $g(d)$ that is not in $\overset{*}{\underset{\mathcal{R}}{\hookrightarrow}}[\mathcal{L}(\mathcal{A})]$. □

As for Example 4, preparing another state that accepts only b to construct a correct automaton is enough. However, guaranteeing the termination of a procedure if a new state is added in the procedure is difficult.

In the following, we show the correctness of $\mathrm{P_{cs}}$.

First, we show several propositions that are trivially derived from the definition of $\mathrm{P_{cs}}$.

Proposition 5 Let $t \in \mathcal{T}(F)$. For $q^t \in Q$, $t \xrightarrow[\Delta_0]{*} \langle\{q^t\}, \mathtt{i}\rangle$ iff $t \xrightarrow[\Delta]{*} q^t$.

Proof: Direct consequence of the construction of Δ, and Δ_0. □

Proposition 6 Let $t \in \mathcal{T}(F)$. For any k, if $t \xrightarrow[\Delta_k]{*} \langle P, \mathtt{i}\rangle \in Q_*$, then $t \xrightarrow[\Delta_0]{*} \langle P, \mathtt{i}\rangle$. Moreover, P is of the form $\{q\}$.

Proof: The first claim follows from the fact that the transition rules in which right-hand-sides is the state having $\mathtt{i}$ are not added at Step 2 or Step 3. The second claim follows from the construction of states. □

Proposition 7 Let $t \in \mathcal{T}(F)$. Then, $t \xrightarrow[\Delta_0]{*} \langle P, \mathtt{i}\rangle \in Q_*$ iff $t \xrightarrow[\Delta_0]{*} \langle P, \mathtt{a}\rangle \in Q_*$.

Proof: Direct consequence of the construction of Δ_0. □

Proposition 8 Let $t \in \mathcal{T}(F)$. For any k, If $t \xrightarrow[\Delta_k]{*} \langle P, \mathtt{i}\rangle$, then $t \xrightarrow[\Delta_k]{*} \langle P, \mathtt{a}\rangle$.

Proof: Let $t \xrightarrow[\Delta_k]{*} \langle P, \mathtt{i}\rangle$, then $t \xrightarrow[\Delta_0]{*} \langle P, \mathtt{i}\rangle$ from Proposition 6. This proposition follows from Proposition 7 and $\Delta_0 \subseteq \Delta_k$. □

Next we show several technical lemmas. Lemmas 9, 10, 12, and 14 below are necessary to prove Lemmas 15 and 19 that are key lemmas to prove completeness and soundness. Lemmas 11 and 13 are auxiliary lemmas for Lemmas 12 and 14, respectively.

Lemma 9 Let $s, t \in \mathcal{T}(F)$, $s \xrightarrow[\Delta_0]{*} \langle P, \mathtt{x}\rangle$, and $t \xrightarrow[\Delta_0]{*} \langle P', \mathtt{x}'\rangle$. Then, $P = P'$ iff $s = t$.

Proof: First we have s $\xrightarrow[\Delta_0]{*} \langle P, \mathtt{i}\rangle$, $t \xrightarrow[\Delta_0]{*} \langle P', \mathtt{i}\rangle$, $P = \{q\}$, and $P' = \{q'\}$ for some $q, q' \in Q$ from Proposition 6 and Proposition 7. Then, we have $s \xrightarrow[\Delta]{*} q^s = q$ and $t \xrightarrow[\Delta]{*} q^t = q'$ from Proposition 5 and the construction of $\mathcal{A}$. Thus, we have $P = P'$ iff $s = t$. □

Lemma 10 If $\alpha : t[t']_p \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ and $p \in \mathrm{Pos}^\mu(t)$, then there exists $\langle P', \mathtt{a}\rangle$ such that $t' \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ and $t[\langle P', \mathtt{a}\rangle]_p \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$.

Proof: We show this lemma by induction on $|\alpha|$. Let $p \in \mathrm{Pos}^\mu(t)$.

(1) If $p = \varepsilon$, then $t = t'$, and hence $t' \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ follows.

(2) Consider the case $p = ip'$ for some $i \in \mathbb{N}$. Then α can be represented as $t[t']_p \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}\rangle$ or $t[t']_p = f(\ldots, t_{i-1}, t_i[t']_{p'}, t_{i+1}, \ldots) \xrightarrow[\Delta_*]{*} f(\ldots, \langle P_{i-1}, \mathtt{x}_{i-1}\rangle, \langle P_i, \mathtt{x}_i\rangle, \langle P_{i+1}, \mathtt{x}_{i+1}\rangle, \ldots) \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}\rangle$.
In the former case, this lemma holds from the induction hypothesis.
In the latter case, since $ip' = p \in \mathrm{Pos}^\mu(t)$, we have $i \in \mu(f)$. Hence $\mathtt{x}_i = \mathtt{a}$ follows from the construction of Δ_*.
From the induction hypothesis, there exists $\langle P', \mathtt{a}\rangle \in Q_*$ such that $t' \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ and $t_i[\langle P', \mathtt{a}\rangle]_{p'} \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{a}\rangle$. Thus we have $t[\langle P', \mathtt{a}\rangle]_p = f(\ldots, t_{i-1}, t_i[\langle P', \mathtt{a}\rangle]_{p'}, t_{i+1}, \ldots) \xrightarrow[\Delta_*]{*} f(\ldots, \langle P_{i-1}, \mathtt{x}_{i-1}\rangle, \langle P_i, \mathtt{a}\rangle, \langle P_{i+1}, \mathtt{x}_{i+1}\rangle, \ldots) \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}\rangle$. □

Lemma 11 If $\langle P_1', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_1, \mathtt{a}\rangle$ and $\langle P_2', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_2, \mathtt{a}\rangle$, then we have $\langle P_1' \cup P_2', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_1 \cup P_2, \mathtt{a}\rangle$.

Proof: We can assume $\langle P_1', \mathtt{a}\rangle \xrightarrow[\Delta_*]{n} \langle P_1, \mathtt{a}\rangle$ and $\langle P_2', \mathtt{a}\rangle \xrightarrow[\Delta_*]{n} \langle P_2'', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_2, \mathtt{a}\rangle$ without loss of generality.

First, we prove the claim that $\langle P_1' \cup P_2', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_1 \cup P_2'', \mathtt{a}\rangle$. If $n = 0$, the claim trivially holds. If $n = 1$, the claim holds from (2) of Step 3 of $\mathrm{P_{cs}}$. If $n > 1$, the claim holds by repeating the process for $n = 1$.

Moreover, we can show the claim that $\langle P_1 \cup P_2'', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P_1 \cup P_2, \mathtt{a}\rangle$ similarly to the previous claim. □

Lemma 12 If $t \xrightarrow[\Delta_*]{*} \langle P^j, \mathtt{a}\rangle$ for $1 \leq j \leq m$, then we have $t \xrightarrow[\Delta_*]{*} \langle \bigcup_{1\leq j\leq m} P^j, \mathtt{a}\rangle$.

Proof: The proof for $m = 1$ is trivial. We show the proof for $m = 2$ by induction on $|t|$. By applying the proof for $m = 2$ repeatedly, we can show this lemma.

Let $t = f(t_1, \ldots, t_n)$. Then, each transition sequence is represented as $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1^j, \mathtt{x}_1^j\rangle, \ldots, \langle P_n^j, \mathtt{x}_n^j\rangle) \xrightarrow[\Delta_*]{} \langle P_r^j, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P^j, \mathtt{a}\rangle$ for $j \in \{1, 2\}$. From Lemma 11, we have $\langle P_r^1 \cup P_r^2, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P^1 \cup P^2, \mathtt{a}\rangle$. Therefore, we show that $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} \langle P_r^1 \cup P_r^2, \mathtt{a}\rangle$.

From (1) of Step 3 of $\mathrm{P_{cs}}$, we have the transition rule $f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \to \langle P_r^1 \cup P_r^2, \mathtt{a}\rangle \in \Delta_*$ where

- $P_i = \begin{cases} P_i^j & \cdots \text{ if } \mathtt{x}_i^j = \mathtt{i} \text{ for some } j \in \{1, 2\}, \text{ and} \\ P_i^1 \cup P_i^2 & \cdots \text{ if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a}. \end{cases}$
- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a}, \text{ and} \\ \mathtt{i} \cdots \text{otherwise.} \end{cases}$

Here we show that $t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ for $1 \le i \le n$.

- For i such that $\mathtt{x}_i = \mathtt{i}$, P_i is P_i^1 or P_i^2 and hence we have $t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$.
- For i such that $\mathtt{x}_i = \mathtt{a}$, P_i is $P_i^1 \cup P_i^2$ and hence we have $t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ from the induction hypothesis.

Thus, we have the transition $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \xrightarrow[\Delta_*]{} \langle P_r, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$. □

Lemma 13 If $\langle P_1, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$, then there exists $P_1' \subseteq P_1$ such that $\langle P_1', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ for all $P' \subseteq P$ where $P' \neq \emptyset$.

Proof: By the induction on $|P| + |P_1|$, we show the proof for the case of $\langle P_1, \mathtt{a}\rangle \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}\rangle$. If $\langle P_1, \mathtt{a}\rangle = \langle P, \mathtt{a}\rangle$, then this lemma holds trivially. If $|\langle P_1, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle| > 1$, then we can prove this lemma by applying the proof for $\langle P_1, \mathtt{a}\rangle \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}\rangle$.

Let $P' = P \backslash P''$. We show that if P_1' exists such that $\langle P_1', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ where $P_1' \subseteq P_1$. If $|P| = 1$, then the claim holds trivially. If $|P| > 1$, we can assume that the transition rule $\langle P_1, \mathtt{a}\rangle \to \langle P, \mathtt{a}\rangle \in \Delta_*$ is produced by the rules $\langle P_1^j, \mathtt{a}\rangle \to \langle P^j, \mathtt{a}\rangle \in \Delta_*$ where $j \in \{1, 2\}$, $P^1 \cup P^2 = P$, and $P_1^1 \cup P_1^2 = P_1$ by (2) of Step 3 of $\mathrm{P_{cs}}$. Note that we have $|P^j| + |P_1^j| < |P| + |P_1|$ for $j \in \{1, 2\}$ because if $|P^j| + |P_1^j| = |P| + |P_1|$ then we have $P^1 = P^2$ and $P_1^1 = P_1^2$, and hence the rule $\langle P_1, \mathtt{a}\rangle \to \langle P, \mathtt{a}\rangle \in \Delta_*$ is the same as $\langle P_1^j, \mathtt{a}\rangle \to \langle P^j, \mathtt{a}\rangle \in \Delta_*$ for $j \in \{1, 2\}$.

For each j, we also have the transition rule $\langle P_1'^j, \mathtt{a}\rangle \to \langle P^j \backslash P'', \mathtt{a}\rangle \in \Delta_*$ for some $P_1'^j \subseteq P_1^j$ from the induction hypothesis.

Thus, we obtain $\langle P_1'^1 \cup P_1'^2, \mathtt{a}\rangle \xrightarrow[\Delta_*]{} \langle (P^1 \cup P^2) \backslash P'', \mathtt{a}\rangle = \langle P', \mathtt{a}\rangle$ where $P_1'^1 \cup P_1'^2 \subseteq P_1^1 \cup P_1^2 = P_1$ by (2) of Step 3 of $\mathrm{P_{cs}}$. □

Lemma 14 If $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$, then $t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ for any $P' \subseteq P$.

Proof: We can assume that the transition $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ is represented as $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \xrightarrow[\Delta_*]{} \langle P_\mathtt{r}, \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$. From Lemma 13, there exists $P_\mathtt{r}' \subseteq P_\mathtt{r}$ such that $\langle P_\mathtt{r}', \mathtt{a}\rangle \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}\rangle$ for all $P' \subseteq P$. Therefore, we show that we have $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \xrightarrow[\Delta_*]{} \langle P_\mathtt{r}', \mathtt{a}\rangle$. Let $P_\mathtt{r}' = P_\mathtt{r} \backslash P_\mathtt{r}''$. We show the claim by induction on $\Sigma_{i=1}^n |P_i| + |P_\mathtt{r}|$ and $|t|$. If $|P_\mathtt{r}| = 1$, then the claim holds trivially. If $|P_\mathtt{r}| > 1$, the transition rule $f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \to \langle P_\mathtt{r}, \mathtt{a}\rangle$ is produced from the transition rules $f(\langle P_1^j, \mathtt{x}_1\rangle, \ldots, \langle P_n^j, \mathtt{x}_n\rangle) \to \langle P_\mathtt{r}^j, \mathtt{a}\rangle$ where $j \in \{1, 2\}$ by (1) of Step 3 of $\mathrm{P_{cs}}$ and $P_\mathtt{r}$, P_i's, and $\mathtt{x}_i$'s are represented as follows:

- $P_\mathtt{r} = P_\mathtt{r}^1 \cup P_\mathtt{r}^2$,
- $P_i = \begin{cases} P_i^j & \cdots \text{ if } \mathtt{x}_i^j = \mathtt{i} \text{ for some } j \in \{1, 2\}, \text{ and} \\ P_i^1 \cup P_i^2 & \cdots \text{ if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a}. \end{cases}$
- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a}, \text{ and} \\ \mathtt{i} \cdots \text{otherwise.} \end{cases}$

Here, we show that $t \xrightarrow[\Delta_*]{*} \langle P_i^j, \mathtt{x}_i^j\rangle$ for $j \in \{1, 2\}$ and $1 \le i \le n$.

- For i such that $\mathtt{x}_i = \mathtt{a}$, we have $\mathtt{x}_i^j = \mathtt{a}$ and $P_i^j \subseteq P_i$. Thus, we have $t \xrightarrow[\Delta_*]{*} \langle P_i^j, \mathtt{x}_i^j\rangle$ from the induction hypothesis.
- For i such that $\mathtt{x}_i = \mathtt{i}$, we have $\mathcal{L}(\Delta_*, \langle P_i^1, \mathtt{x}_i^1\rangle) \cap \mathcal{L}(\Delta_*, \langle P_i^2, \mathtt{x}_i^2\rangle) \neq \emptyset$, and, $\mathtt{x}_i^1 = \mathtt{i}$ or $\mathtt{x}_i^2 = \mathtt{i}$. From Lemma 9, t_i is the only term accepted by $\langle P_i^j, \mathtt{i}\rangle$ where j is 1 or 2, and from $\mathcal{L}(\Delta_*, \langle P_i^1, \mathtt{x}_i^1\rangle) \cap \mathcal{L}(\Delta_*, \langle P_i^2, \mathtt{x}_i^2\rangle) \neq \emptyset$, we have $t_i \xrightarrow[\Delta_*]{*} \langle P_i^j, \mathtt{x}_i\rangle$ for both $j = 1$ and $j = 2$.

Thus, we have $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1^j, \mathtt{x}_1^j\rangle, \ldots, \langle P_n^j, \mathtt{x}_n^j\rangle) \xrightarrow[\Delta_*]{} \langle P_\mathtt{r}^j, \mathtt{a}\rangle$ for both $j = 1$ and $j = 2$.

Moreover, we have $\Sigma_{i=1}^n |P_i^j| + |P_\mathtt{r}^j| < \Sigma_{i=1}^n |P_i| + |P_\mathtt{r}|$ for both $j = 1$ and $j = 2$ because if it does not hold, then the rule for $j = 1$ or $j = 2$ become the same one as the rule $f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \to \langle P, \mathtt{a}\rangle$. Hence, we have $t \xrightarrow[\Delta_*]{*} \langle P_\mathtt{r}^j \backslash P_\mathtt{r}''\rangle$ for both $j = 1$ and $j = 2$ from the induction hypothesis.

Thus, we have $t \xrightarrow[\Delta_*]{*} \langle P_\mathtt{r}^1 \cup P_\mathtt{r}^2 \backslash P_\mathtt{r}'', \mathtt{a}\rangle$ from Lemma 12. □

The following lemma is a key lemma for completeness of $\mathrm{P_{cs}}$.

Lemma 15 Let $\mathcal{R}$ be right-shallow CS-TRS. Then $s \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ and $s \xhookrightarrow[\mathcal{R}]{*} t$ implies $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$.

Proof: We present the proof in the case of $s \underset{\mathcal{R}}{\hookrightarrow} t$ because the proof in the case of $s = t$ is trivial and in the case of $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t' \underset{\mathcal{R}}{\hookrightarrow} t$, we can prove it by applying the proof for $s \underset{\mathcal{R}}{\hookrightarrow} t$ repeatedly. Let $s \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ and $s = s[l\sigma]_p \underset{\mathcal{R}}{\hookrightarrow} s[r\sigma]_p = t$ for some rewrite rule $l \to r \in R$, where $p \in \mathrm{Pos}^{\mu}(s)$. We have a transition sequence $s[l\sigma]_p \xrightarrow[\Delta_*]{*} s[\langle P', \mathtt{a}\rangle]_p \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ for some $\langle P', \mathtt{a}\rangle \in Q_*$ by Lemma 10.

From Lemma 14, we have $l\sigma \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle$ for all $q \in P$. Therefore, we prove that $r\sigma \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle$ for all $q \in P$, because if we can prove this, we have $s[r\sigma]_p \xrightarrow[\Delta_*]{*} s[\langle P', \mathtt{a}\rangle]_p \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ from Lemma 12.

(1) Consider the case where the rewrite rule is of the form $C[x_1, \ldots, x_n] \to g(r_1, \ldots, r_m)$ where C has no variable. The diagram of this case is shown in **Fig. 2**. Here, $C[x_1, \ldots, x_n]\sigma \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle$ is represented in $C[x_1, \ldots, x_n]\sigma \xrightarrow[\Delta_*]{*} C[\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle] \xrightarrow[\Delta_*]{} \langle \{q\}, \mathtt{a}\rangle$ for some $\langle P_i, \mathtt{x}_i\rangle \in Q_*$ for $1 \le i \le n$. Since we have $C[x_1, \ldots, x_n] \to g(r_1, \ldots, r_m) \in R$, $C[\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle] \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle$, and $\sigma : X \to \mathcal{T}(F)$ such that $x_i\sigma \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ for all $1 \le i \le n$, Δ_* has the transition rule $g(\langle P', \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \to \langle \{q\}, \mathtt{a}\rangle \in \Delta_*$ such that

- $$P'_j = \begin{cases} \{q^{r_j}\} & \cdots \text{ if } r_j \notin X, \\ P_i & \cdots \text{ if } r_j \in X \wedge \exists i \in I_j.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i \in I_j} P_i & \cdots \text{ if } r_j \in X \wedge \forall i \in I_j.\mathtt{x}_i = \mathtt{a}. \end{cases}$$
- $$\mathtt{x}'_j = \begin{cases} \mathtt{i} \cdots \text{ if } j \notin \mu(g) \wedge (r_j \in X \Rightarrow \exists i \in I_j.\mathtt{x}_i = \mathtt{i})\text{), and} \\ \mathtt{a} \cdots \text{ otherwise.} \end{cases}$$

where $I_j = \{i \mid x_i = r_j\}$.

Here, we show that $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle$ for $1 \le j \le m$.

(a) For j such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$. From the shallowness of $\mathcal{R}$, we have $r_j\sigma = r_j$. Moreover, we have $r_j \xrightarrow[\Delta]{*} q^{r_j}$ from the construction of Δ and hence we have $r_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, \mathtt{x}\rangle$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$ from Proposition 5 and Proposition 7.

(b) For j such that $r_j \in X$ and there exists $i \in I_j$ such that $\mathtt{x}_i = \mathtt{i}$, we have $r_j\sigma = x_i\sigma$ and hence $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{i}\rangle$. If $j \notin \mu(g)$, we have $\mathtt{x}'_i = \mathtt{i}$ and hence $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_i\rangle = \langle P_i, \mathtt{i}\rangle$. If $j \in \mu(g)$, we have $\mathtt{x}'_j = \mathtt{a}$ and hence $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle = \langle P_i, \mathtt{a}\rangle$ from Proposition 8.

(c) For j such that $r_j \in X$ and there exists no $i \in I_j$ such that $\mathtt{x}_i = \mathtt{i}$,

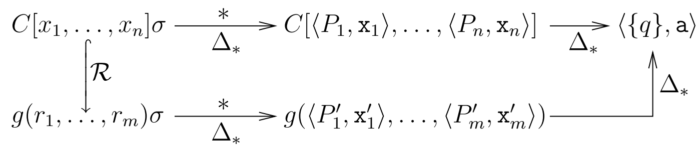

Fig. 2 The diagram of the proof of lemma 15.

since we have $r_j\sigma = x_{i'}\sigma \xrightarrow[\Delta_k]{*} \langle P_{i'}, \mathtt{x}_{i'}\rangle = \langle P_{i'}, \mathtt{a}\rangle$ for all $i' \in I_j$, $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle = \langle \bigcup_{i' \in I_j} P_{i'}, \mathtt{a}\rangle$ follows from Lemma 12.

Therefore we have $g(r_1, \ldots, r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \xrightarrow[\Delta_*]{} \langle \{q\}, \mathtt{a}\rangle$.

By applying the above statement for all $q \in P$, this lemma holds.

(2) In the case where the rewrite rule is of the form $C[x_1, \ldots, x_n] \to x_i$, we can prove this lemma similarly to the previous case.

□

The following lemma shows completeness of $\mathrm{P_{cs}}$.

Lemma 16 If $\mathcal{R}$ is right-shallow CS-TRS, then $\mathcal{L}(\mathcal{A}_*) \supseteq \underset{\mathcal{R}}{\hookrightarrow}[\mathcal{L}(\mathcal{A})]$.

Proof: Let $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t$ and $s \xrightarrow[\Delta]{*} q \in Q^f$. Since we have $s \xrightarrow[\Delta_0]{*} \langle \{q\}, \mathtt{i}\rangle$ from Proposition 5, we also have $s \xrightarrow[\Delta_0]{*} \langle \{q\}, \mathtt{a}\rangle$ from Proposition 7. Hence $t \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle \in Q^f_*$ follows by Lemma 15. □

To prove the soundness of $\mathrm{P_{csin}}$, we define the following measures of transition and order. These are necessary to prove soundness of $\mathrm{P_{cs}}$.

Definition 17 Let $||t \xrightarrow[\Delta_*]{*} P||$ be the sequence of integers defined as follows[⋆1]:

⋆1 Sometimes we have $k - 1 < 0$ in this definition. Δ_k for $k < 0$ is undefined but we assume it as an empty set.

$$||t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}\rangle|| = \begin{cases} k.||t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{x}'\rangle|| & \cdots \text{ if } t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{x}'\rangle, \text{ and } \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle P, \mathtt{x}\rangle, \\ & \quad \text{if } t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle) \\ k.||t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle|| \cdots & \quad \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle P, \mathtt{x}\rangle \text{ and} \\ & \quad \forall j \neq i.||t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle|| \geq_{\text{lex}} ||t_j \xrightarrow[\Delta_*]{*} \langle P_j, \mathtt{x}_j\rangle|| \end{cases}$$

Definition 18 Let $\sqsupseteq$ and $\sqsupset$ be the order for transition sequences as follows:

$$\alpha \sqsupseteq \beta \text{ iff } \begin{cases} (1)\ \beta \text{ occurs in } \alpha, \text{or} \\ (2)\ \alpha \text{ does not occur in } \beta \text{ and } ||\alpha|| \geq_{\text{lex}} ||\beta||. \end{cases}$$

$$\alpha \sqsupset \beta \text{ iff } \begin{cases} (1)\ \beta \text{ occurs in } \alpha \text{ and } \alpha \text{ does not occur in } \beta, \text{or} \\ (2)\ \alpha \text{ does not occur in } \beta \text{ and } ||\alpha|| >_{\text{lex}} ||\beta||. \end{cases}$$

Note that $\sqsupset$ is well-founded, $\alpha \sqsupset \beta$ implies $\alpha \sqsupseteq \beta$, and if $\alpha \sqsupseteq \beta$ then $\beta \not\sqsupset \alpha$. The minimal components in the order $\sqsupset$ are the transitions of the form $a \xrightarrow[\Delta_0]{} \langle P, \mathtt{x}\rangle$ where a is a constant.

The following lemma is a key lemma for soundness of $\mathrm{P_{cs}}$.

Lemma 19 Let $\mathcal{R}$ be a right-linear right-shallow CS-TRS. Then, $\alpha : t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}\rangle$ implies that there exists s and $q \in P$ such that $s \xrightarrow[\mathcal{R}]{*} t$ and $s \xrightarrow[\Delta_0]{*} \langle \{q\}, \mathtt{a}\rangle$.

Proof: We show this lemma by induction on α with respect to $\sqsupset$. Since we have $t \xrightarrow[\Delta_*]{*} \langle \{q\}, \mathtt{a}\rangle$ for all $q \in P$ from Lemma 14, we show the proof in the case where P is of the form $\{q\}$.

(1) Consider the case where α is represented as $t = g(t_1, \ldots, t_m) \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle \{q\}, \mathtt{a}\rangle$.

(a) If $k = 0$, the transition rule $g(\langle P'_1, \mathtt{a}\rangle, \ldots, \langle P'_m, \mathtt{a}\rangle) \to \langle \{q\}, \mathtt{a}\rangle$ is produced at Step 1, and hence each P'_j is of the form $\{q_j\}$ and we have $j \in \mu(g)$ iff $\mathtt{x}'_j = \mathtt{a}$. For $j \in \mu(g)$, we have $\mathtt{x}'_j = \mathtt{a}$ and hence there exists s_j such that $s_j \xrightarrow[\mathcal{R}]{*} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{a}\rangle = \langle \{q_j\}, \mathtt{a}\rangle$ from the induction hypothesis. For $j \notin \mu(g)$, we have $\mathtt{x}'_j = \mathtt{i}$ and $t_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{i}\rangle$ from Proposition 6. We take $s_j = t_j$ for $j \notin \mu(g)$. Finally, we obtain $g(s_1, \ldots, s_m) \xrightarrow[\mathcal{R}]{*} g(t_1, \ldots, t_m) = t$ and $g(s_1, \ldots, s_m) \xrightarrow[\Delta_0]{*} g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \xrightarrow[\Delta_0]{} \langle \{q\}, \mathtt{a}\rangle$. Thus, this lemma holds in the case $k = 0$.

(b) If $k > 0$, the transition rule $g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \to \langle \{q\}, \mathtt{a}\rangle \in \Delta_k \backslash \Delta_{k-1}$ is produced at (1) of Step 2. The diagram of this case is shown in **Fig. 3**. From the production of the transition rule, we have $C[x_1, \ldots, x_n] \to g(r_1, \ldots, r_m) \in R$ where C has no variable and $C[\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle] \xrightarrow[\Delta_{k-1}]{*} \langle \{q\}, \mathtt{a}\rangle$, and $\sigma' : X \to \mathcal{T}(F)$ such that $x_i\sigma' \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i\rangle$ for all $1 \leq i \leq n$, and each $\langle P'_j, \mathtt{x}'_j\rangle$ is represented as follows:

- $$P'_j = \begin{cases} \{q^{r_j}\} & \cdots \text{ if } r_j \notin X, \\ P_i & \cdots \text{ if } r_j \in X \wedge \exists i \in I_j.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i \in I_j} P_i & \cdots \text{ if } r_j \in X \wedge \forall i \in I_j.\mathtt{x}_i = \mathtt{a}. \end{cases}$$
- $$\mathtt{x}'_j = \begin{cases} \mathtt{i} \cdots \text{ if } j \notin \mu(g) \wedge (r_j \in X \Rightarrow \exists i \in I_j.\mathtt{x}_i = \mathtt{i}), \text{ and} \\ \mathtt{a} \cdots \text{ otherwise.} \end{cases}$$

where $I_j = \{i \mid x_i = r_j\}$. In the following, we show that there exists the substitution σ such that $g(r_1, \ldots, r_m)\sigma \xrightarrow[\mathcal{R}]{*} g(t_1, \ldots, t_m)$ and $\alpha' : g(r_1, \ldots, r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m\rangle) \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle \{q\}, \mathtt{a}\rangle$ where $\alpha' \sqsubseteq \alpha$.

(i) For $j \notin \mu(g)$ such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$ and $\mathtt{x}'_j = \mathtt{i}$. Since $t_j \xrightarrow[\Delta_0]{*} \langle \{q^{r_j}\}, \mathtt{i}\rangle$ is from Proposition 7, we have $t_j = r_j$ from Proposition 5 and the construction of $\mathcal{A}$.

(ii) For $j \in \mu(g)$ such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$ and $\mathtt{x}'_j = \mathtt{a}$. Hence, we have $s_j \xrightarrow[\mathcal{R}]{*} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{x}'_j\rangle = \langle \{q^{r_j}\}, \mathtt{a}\rangle$ from the induction hypothesis. Since we have $s_j \xrightarrow[\Delta_0]{*} \langle \{q^{r_j}\}, \mathtt{i}\rangle$ from Proposition 7, we have $s_j = r_j$ from Proposition 5 and the construction of $\mathcal{A}$.

(iii) For j such that $r_j \in X$, $j \notin \mu(g)$, and there exists $i \in I_j$ such that $\mathtt{x}_i = \mathtt{i}$, we have $t_j \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle = \langle P_i, \mathtt{i}\rangle$. Hence, we have $t_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{x}'_j\rangle$ from Proposition 7, and let $r_j\sigma = t_j$.

(iv) For j such that $r_j \in X$, $j \in \mu(g)$, and there exists $i \in I_j$ such that $\mathtt{x}_i = \mathtt{i}$, we have $t_j \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle = \langle P_i, \mathtt{a}\rangle$. Since we have $\langle P_i, \mathtt{i}\rangle = \langle P_i, \mathtt{x}_i\rangle$ where P_i is of the form $\{q_i\}$ from

Proposition 6, there exists a s_j such that $s_j \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{x}'_j\rangle$ from the induction hypothesis. Let s_j be $r_j\sigma$.

(v) For j such that $r_j \in X$ and there exists no $i \in I_j$ such that $\mathtt{x}_i = \mathtt{i}$, we take $r_j\sigma = t_j$.

Note that σ is well defined from the right-linearity of $\mathcal{R}$ and we have $\alpha' \sqsubseteq \alpha$ because α does not occur in α', and we have $(r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle) \sqsubseteq (t_j \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j\rangle)$ for all $1 \le j \le m$.

Next, we define a substitution $\sigma'' : \mathrm{Var}(f(l_1, \ldots, l_n)) \to \mathcal{T}(F)$ as follows:

$$x\sigma'' = \begin{cases} x\sigma \cdots & \text{if there exists } r_j \text{ such that } r_j = x \\ x\sigma' \cdots & \text{otherwise.} \end{cases}$$

Here, we show that we have $x_i\sigma'' \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ for all $1 \le i \le n$.

(i) For i such that there exists j such that $x_i = r_j$ and $i' \in I_j$ such that $\mathtt{x}_{i'} = \mathtt{i}$, we have $x_{i'}\sigma' \xrightarrow[\Delta_0]{*} \langle P_{i'}, \mathtt{i}\rangle = \langle P'_j, \mathtt{i}\rangle$ from Proposition 5. Thus we have $x_i\sigma' = x_i\sigma''$ from Lemma 9 and hence $x_i\sigma'' \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i\rangle$.

(ii) For i such that there exists j such that $x_i = r_j$ and no $i' \in I_j$ such that $\mathtt{x}_{i'} = \mathtt{i}$, we have $P_i \subseteq P'_j$ and hence $x_i\sigma'' \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ from Lemma 12.

(iii) For i such that there exists no j such that $x_i = r_j$, we have $x_i\sigma'' = x_i\sigma' \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i\rangle$ from the construction of the rule.

Thus, we have $\beta : C[x_1, \ldots, x_n]\sigma'' \xrightarrow[\Delta_*]{*} C[\langle P_1, \mathtt{x}_1\rangle, \ldots, \langle P_n, \mathtt{x}_n\rangle] \xrightarrow[\Delta_{k-1}]{*} \langle\{q\}, \mathtt{a}\rangle$.

Note that $\beta \sqsubset \alpha'$ because transition rules in $\Delta_k \backslash \Delta_{k-1}$ are not applied at β except for the transitions $x_i\sigma'' \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i\rangle$ where there exists j such that $x_i = r_j$. However, since $x_i\sigma''$ is a proper subterm of $g(r_1, \ldots, r_m)\sigma''$, α' does not occur in β.

Finally, we have the term s such that $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} C[x_1, \ldots, x_n]\sigma'' \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t$ and $s \xrightarrow[\Delta_0]{*} \langle\{q\}, \mathtt{a}\rangle$ from the induction hypothesis.

(2) If the last transition rule is of the form $\langle P', \mathtt{a}\rangle \to \langle P, \mathtt{a}\rangle$, this lemma holds similarly to the previous case. □

The following lemma shows soundness of $\mathrm{P_{cs}}$.

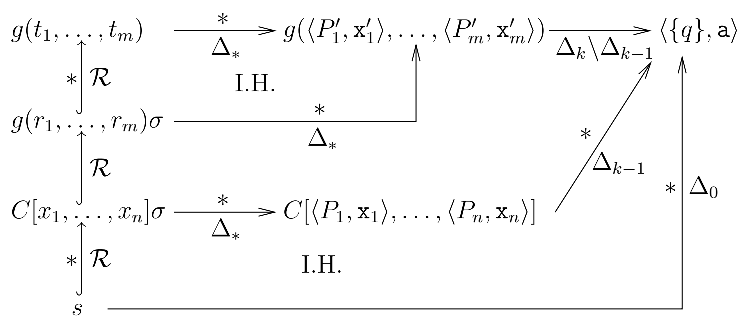

Fig. 3 The diagram of the proof of Lemma 19.

Lemma 20 If $\mathcal{R}$ be right-linear and right-shallow, then $\mathcal{L}(\mathcal{A}_*) \subseteq \overset{*}{\underset{\mathcal{R}}{\hookleftarrow}}[\mathcal{L}(\mathcal{A})]$.

Proof: Let t be $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}\rangle \in Q^f_*$ where P contains the state $q^f \in Q^f$. Then, we have $t \xrightarrow[\Delta_*]{*} \langle\{q^f\}, \mathtt{x}\rangle$ from Lemma 14. If $\mathtt{x} = \mathtt{i}$, we have $t \xrightarrow[\Delta_0]{*} \langle\{q^f\}, \mathtt{i}\rangle \in Q^f_*$ from Proposition 6. If $\mathtt{x} = \mathtt{a}$, there exists the term s such that $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}} t$ and $s \xrightarrow[\Delta_0]{*} \langle\{q^f\}, \mathtt{a}\rangle$ from Lemma 19 and we have $s \xrightarrow[\Delta_0]{*} \langle\{q^f\}, \mathtt{i}\rangle$ from Proposition 7. Thus, we have $s \xrightarrow[\Delta]{*} q^f$ from Proposition 5. □

The following theorem is proved by Lemma 16 and 20.

Theorem 21 For any right-linear right-shallow CS-TRS $\mathcal{R}$, we can construct a TA recognizing the set of terms that is reachable from a term. Thus, context-sensitive reachability is decidable for right-linear right-shallow TRSs.

4. Decidability of Innermost Reachability for Shallow CS-TRSs

In this section, we show that innermost reachability for shallow CS-TRSs is decidable. Similarly to the previous section, we show the algorithm $\mathrm{P_{csin}}$ that constructs the tree automaton accepting the set of terms reachable by innermost reduction of a shallow CS-TRS from an input term. The algorithm $\mathrm{P_{csin}}$ is a modification of the algorithm $\mathrm{P_{cs}}$ by the idea in Ref. 11). States of output automata obtained by $\mathrm{P_{csin}}$ have three components, while the one by $\mathrm{P_{cs}}$ has two components. Since we must check whether each proper subterm of redex is a context-sensitive normal form or not in the innermost case, we augment the parameter that shows whether the state accepts the context-sensitive normal form or not. Therefore, first we show the construction of a deterministic complete reduced tree automata accepting the set of context-sensitive normal forms, and then we show P_{csin}.

4.1 Tree Automata Accepting Context-sensitive Normal Forms

In this subsection, we give an algorithm to construct a deterministic complete reduced tree automata recognizing the set of context-sensitive normal forms for shallow CS-TRS $\mathcal{R}$. However, in general, ordinary tree automata cannot recognize the set of context-normal forms for shallow CS-TRS. Therefore we use tree automata with constraints between brothers (TACBB) [3].

The procedure is similar to the ones for TRSs [3]. The steps of the algorithm to construct TACBB $\mathcal{A}_{NF}$ are as follows:

(1) Construct the TACBB $\mathcal{A}_l$ that recognizes the set of terms having a redex $l\sigma$ at a μ-replacing position for each $l \to r \in R$ and determinize it.

(2) Construct the union of all $\mathcal{A}_l$'s and convert the TA into complete and reduced TA $\mathcal{A}$.

(3) We output a TA recognizing the complement of $\mathcal{L}(\mathcal{A})$ as $\mathcal{A}_{NF}$.

The steps (2) and (3) are obviously possible from Theorem 1. Now we show the details of step (1).

Each component of $\mathcal{A}_l$ is as follows.

- $Q_l = \{u^\circ, u_\perp\} \cup \{u_t \mid t \lhd l, t \notin X\}$.
- $Q_l^f = \{u^\circ\}$
- Δ_l consists of the following transition rules:
  - **(i)** $f(u_\perp, \dots, u_\perp) \xrightarrow{\top} u_\perp$ for each $f \in F$,
  - **(ii)** $f(u_{t_1}, \dots, u_{t_n}) \xrightarrow{\top} u_{f(t_1,\dots,t_n)}$ for each $f \in F$ and state $u_{f(t_1,\dots,t_n)}$.
  - **(iii)** $f(u_{s_1}, \dots, u_{s_n}) \xrightarrow{c} u^\circ$ where $f(s_1, \dots, s_n)$ is the term obtained by replacing all variables in $l = f(l_1, \dots, l_n)$ by $\perp$, and c is the conjunction of all equalities $i = j$ where $l_i = l_j \in X$.
  - **(iv)** $f(u_1, \dots, u_n) \to u^\circ$ for each $f \in F$ if exactly one u_j such that $j \in \mu(f)$ is u° and the other u_i's are $u_\perp$.

  Each state u_t is associated with a proper non-variable subterm t of l. From the shallowness of R, t in u_t has no variable.

We obtain the following lemmas for $\mathcal{A}_l$.

Lemma 22 $\mathcal{L}(\mathcal{A}_l, u_t)$ is equal to the singleton set that consists of $t \lhd l$, (that is, $\mathcal{L}(\mathcal{A}_l, u_t) = \{t \in \mathcal{T}(F)\}$.)

Proof:

($\supseteq$) By induction on the height $|t|$ of t, we prove the claim that $t \xrightarrow[\Delta_l]{*} u_t$ for the proper non-variable subterm t of l. We can represent t as $f(t_1, \dots, t_n)$ where $n \geq 0$. From the construction of (ii) of Δ_l, we have the transition rule $f(u_{t_1}, \dots, u_{t_n}) \xrightarrow{\top} u_{f(t_1,\dots,t_n)}$

From the induction hypothesis, we have $t_i \xrightarrow[\Delta_l]{*} u_{t_i}$ for all $1 \leq i \leq n$. Thus, we have $t\sigma = f(t_1, \dots, t_n)\sigma \xrightarrow[\Delta_l]{*} f(u_{t_1}, \dots, u_{t_n}) \xrightarrow[\Delta_l]{} u_t$.

($\subseteq$) We show that if $\alpha : t \xrightarrow[\Delta_l]{*} u_{f(t_1,\dots,t_n)}$ then we have $t = f(t_1, \dots, t_n)$ by induction on $|\alpha|$. From the construction of Δ_l, the last transition rule applied in α is represented as $f(u_{t_1}, \dots, u_{t_n}) \xrightarrow{c} u_{f(t_1,\dots,t_n)}$.

From the induction hypothesis, we have $t_i \xrightarrow[\Delta_l]{*} u_{t_i}$, for all $1 \leq i \leq n$.

Thus, we have $f(t_1, \dots, t_n) = f(l_1, \dots, l_n)\sigma$. □

Lemma 23 $\mathcal{L}(\mathcal{A}_l) = \{t[s]_p \mid t \in \mathcal{T}(F),\ s$ is a ground instance of l, $p \in \mathrm{Pos}^\mu(t)\}$.

Proof:

($\supseteq$) Let $l = f(l_1, \dots, l_n)$. First, we show that $s = f(l_1, \dots, l_n)\sigma \xrightarrow[\Delta_l]{*} u^\circ$. For $l_i \notin X$, l_i is ground from shallowness of l and $l_i \xrightarrow[\Delta_l]{*} u_{l_i}$. For $l_i \in X$, we have $l_i\sigma \xrightarrow[\Delta_l]{*} u_\perp$. From (iii) of construction of Δ_l, we have the transition rule $f(u_{s_1}, \dots, u_{s_n}) \xrightarrow{c} u^\circ$ where $s_i = l_i$ for $i \notin X$ and $s_i = \perp$ for $i \in X$, and c is the conjunction of all equalities $i = j$ where $l_i = l_j \in X$. Since we have $l_i\sigma = l_j\sigma$ for $l_i = l_j \in X$, s satisfies c. Thus, we have $s \xrightarrow[\Delta_l]{*} u^\circ$ and from (i) and (iv) of construction of Δ_l, we have $t[s]_p \xrightarrow[\Delta_l]{*} t[u^\circ]_p \xrightarrow[\Delta_l]{} u^\circ$.

($\subseteq$) Let $l = f(l_1, \dots, l_n)$ and $t \xrightarrow[\Delta_l]{*} u^\circ$, then we have $t \xrightarrow[\Delta_l]{*} t[f(u_{s_1}, \dots, u_{s_n})]_p \xrightarrow[\Delta_l]{} t[u^\circ]_p \xrightarrow[\Delta_l]{*} u^\circ$ where $s_i = l_i$ for $l_i \notin X$, $s_i = \perp$ for $l_i \in X$, and $p \in \mathrm{Pos}^\mu(t)$ from (iii) and (iv) of the construction of Δ_l. From Lemma 22, we have $t|_{pi} = l_i \notin X$. Moreover, since the transition $t[f(u_{s_1}, \dots, u_{s_n})]_p \xrightarrow[\Delta_l]{} t[u^\circ]_p$ has the constraint c that is the conjunction of all equalities $i = j$ where $l_i = l_j \in X$, we have $t|_{pi} = t|_{pj}$ for $l_i = l_j \in X$. Hence $t|_p$ is a ground substitution of l. Thus we have $t = t[s]_p$ for some ground instance s of l. □

The method of determinization is the so called "*subset construction*." The claim "$t \xrightarrow[\mathcal{A}^d]{*} S$ iff $S = \{q \mid t \xrightarrow[\mathcal{A}]{*} q\}$" holds where $\mathcal{A}^d$ is determinized from $\mathcal{A}$ by subset construction.

Therefore, the following lemma holds.

Lemma 24 Let s be a proper subterm of l, u_s and S'' be a state and a subset of the set of states of TACBB $\mathcal{A}_l$ respectively, and $\mathcal{A}_l^d$ be a determinized TACBB from $\mathcal{A}_l$ by subset construction. Then, $t \xrightarrow[\mathcal{A}_l^d]{*} \{u_s\} \cup S'$ iff $t = s$.

Proof: From Lemma 22 and shallowness of l, $t \xrightarrow[\mathcal{A}_l]{*} q_s$ iff $t = s$.

Thus, this lemma holds from the above claim. □

As shown in Lemma 23, the TACBB $\mathcal{A}_l$ recognizes the set of terms having a redex $l\sigma$ at a μ-replacing position. Now we obtain the following lemma.

Lemma 25 For a CS-TRS $\mathcal{R}$, we can construct a deterministic complete reduced TACBB $\mathcal{A}_{\mathrm{NF}}$ that recognizes CS-NF$_{\mathcal{R}}$.

Proof: By step (1) of the algorithm, we obtain a TACBB $\mathcal{A}_l$ for each $l \to r \in R$, and we can determinize them. Let the determinized TACBB from A_l be $A_l^d = \langle Q_l^d, Q_l^{df}, \Delta_l^d \rangle$.

We can obtain a TACBB $\mathcal{A}' = \langle F, Q', Q'^f, \Delta' \rangle$ that recognizes the following set:

$$\bigcup_{l \to r \in R} \mathcal{L}(\mathcal{A}_l').$$

Let $R = \{l_i \to r_i \mid 1 \leq i \leq m\}$. The concrete construction of the TACBB $\mathcal{A}' = \langle Q', Q'^f, \Delta' \rangle$ is as follows:

- $Q' = \{\langle u_1, \ldots, u_n \rangle \mid u_i \in Q_{l_i}^d\}$,
- $Q'^f = \{\langle u_1, \ldots, u_n \rangle (\in Q') \mid \exists i. u_i \in Q_{l_i}^{df}\}$
- $f(\langle u_{11}, \ldots, u_{1m} \rangle, \ldots, \langle u_{n1}, \ldots, u_{nm} \rangle) \xrightarrow{c} \langle u_1, \ldots, u_m \rangle \in \Delta'$ where $f(u_{1i}, \ldots, u_{mi}) \xrightarrow{c_i} u_i \in \Delta_{l_i}^d$ and $c = c_1 \wedge \cdots \wedge c_m$.

This is the construction of the union of all $\mathcal{A}_l'$'s. Since this construction preserves determinacy of TACBB, the constructed TACBB $\mathcal{A}'$ is deterministic.

Converting $\mathcal{A}'$ to complete one is not so difficult. By adding the new state q_{e} and new transition rules such that $f(q_1, \ldots, q_n) \xrightarrow{c} q_{\mathsf{e}}$ where $q_1, \ldots, q_n \in Q'$ and $c = \top$ if $f(q_1, \ldots, q_n)$, which does not occur in any transition rule of Δ', otherwise c is equivalent to $\neg(c_1 \vee \cdots \vee c_k)$ where $f(q_1, \ldots, q_n) \xrightarrow{c_j} q \in \Delta'$ for some $q \in Q'$.

Since the emptiness problem of TACBB is decidable from Theorem 1, we can check whether each state is accessible or not and hence we can construct a reduced TACBB A'' by erasing the inaccessible state of $\mathcal{A}'$.

Finally, since $\mathcal{A}''$ is deterministic and complete, we can easily obtain the TA $\mathcal{A}'''$ that accepts complementation of $\mathcal{A}''$ by replacing the final state.

From Lemma 23, $\mathcal{L}(\mathcal{A}'')$ is the set of terms having redex at a μ-replacing position. Thus we can obtain the deterministic, complete, and reduced TACBB $\mathcal{A}_{\mathrm{NF}}$ recognizing CS-NF$_{\mathcal{R}}$ by the algorithm. □

We show an example of $\mathcal{A}_{\mathrm{NF}}$ in Appendix A.1.1.

For the constructed TA $\mathcal{A}_{\mathrm{NF}}$, the following proposition holds from Lemmas 24 and 25.

Proposition 26 Let $t \in \mathcal{T}(F)$, $u \in Q_{\mathrm{NF}}$, and $t \xrightarrow[\Delta_{\mathrm{NF}}]{*} u$. If t is a proper subterm of some l of $l \to r \in R$ where $\mathcal{R} = (R, \mu)$ is a shallow CS-TRS, then u accepts no term other than t.

Proof: Let $\mathcal{A}_l^d$ be the deterministic TACBB obtained in step (1) of the algorithm. From Lemma 22, Lemma 24, and shallowness of R, we have $t \xrightarrow[\Delta_l]{*} S \in Q_l^d$ where S contains $u_t \in Q_l$ and there exists no term other than t accepted by u_t. Thus, from the construction of Δ_{NF}, there exists no term accepted by u other than t. □

Proposition 27 If $f(u_1, \ldots, u_n) \xrightarrow{c} u \in \Delta_{\mathrm{NF}}$ and $u \in Q_{\mathrm{NF}}^f$, then $i \in \mu(f)$ implies $u_i \in Q_{\mathrm{NF}}^f$.

Proof: Let $f(u_1, \ldots, u_n) \xrightarrow{c} u \in \Delta_{\mathrm{NF}}$, $u \in Q_{\mathrm{NF}}^f$, and assume $u_i \notin Q_{\mathrm{NF}}^f$ for some $i \in \mu(f)$. Since $\mathcal{A}_{\mathrm{NF}}$ is a reduced TACBB from Lemma 25, there exists ground terms $t_1, \ldots, t_n$ such that $t_j \xrightarrow[\Delta_{\mathrm{NF}}]{*} u_j$ for each j $(1 \leq j \leq n)$. Hence we have $f(t_1, \ldots, t_n) \xrightarrow[\Delta_{\mathrm{NF}}]{*} f(u_1, \ldots, u_n) \xrightarrow[\Delta_{\mathrm{NF}}]{} u$.

Here $f(t_1, \ldots, t_n) \in$ CS-NF$_{\mathcal{R}}$ and $t_i \notin$ CS-NF$_{\mathcal{R}}$ is from Lemma 25. Since t_i is not a context-sensitive normal form and $i \in \mu(f)$, the term $f(t_1, \ldots, t_n)$ is not a context-sensitive normal form, which contradicts $f(t_1, \ldots, t_n) \in \text{CS-NF}_{\mathcal{R}}$. □

4.2 An Algorithm to Construct the Set of Reachable Terms by Context-Sensitive Innermost Reduction

In this section, we show the concrete definition of $\mathrm{P}_{\mathrm{csin}}$ to construct a TACBB that recognizes the set of reachable terms by innermost reduction of a shallow CS-TRS. $\mathrm{P}_{\mathrm{csin}}$ is a modification of P_{cs}. The main difference between $\mathrm{P}_{\mathrm{csin}}$ and P_{cs} is the number of components of each state of output automata. States of output TA by $\mathrm{P}_{\mathrm{csin}}$ have an extra component that is a state of $\mathcal{A}_{\mathrm{NF}}$. Since $\mathcal{A}_{\mathrm{NF}}$ is TACBB, output automata by $\mathrm{P}_{\mathrm{csin}}$ are also TACBB.

Algorithm $\mathrm{P}_{\mathrm{csin}}$:

Input A term t and a shallow CS-TRS $\mathcal{R} = (R, \mu)$ that has no erasing variable.

Output A TA $\mathcal{A}_* = \langle Q_*, Q_*^f, \Delta_* \rangle$ such that $\mathcal{L}(\mathcal{A}_*) = \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.

Step 1 (initialize) (1) Prepare a TACBB $\mathcal{A}_{\mathrm{NF}}$ obtained by the algorithm in previous section and a TA $\mathcal{A} = \langle Q, Q^f, \Delta \rangle$ where each state q^s accepts $s \in \{t\} \cup \mathrm{RS}(R)$, $\mathrm{RS}(R)$ is the set of the proper ground subterm of the right-hand sides of R. Here we assume $Q = \{q^s \mid s \trianglelefteq s', s' \in \{t\} \cup \mathrm{RS}(R)\}$, $Q^f = \{q^t\}$, and $\mathcal{L}(\mathcal{A}_{\mathrm{RS}}, q^s) = \{s\}$ for all q^s.

(2) Let
- $k := 0$,
- $Q_* = \{P(\subseteq Q)\} \times \{\mathtt{a}\} \times Q_{\mathrm{NF}} \cup \{\{q\} \mid q \in Q\} \times \{\mathtt{a}\} \times Q_{\mathrm{NF}}$,
- $Q_*^f = \{P(\subseteq Q) \mid P \cap Q^f \neq \emptyset\} \times \{\mathtt{a}\} \times Q_{\mathrm{NF}}$, and
- Δ_0 as follows:
  (a) $f(\langle\{q_1\}, \mathtt{i}, u_1\rangle, \ldots, \langle\{q_n\}, \mathtt{i}, u_n\rangle) \xrightarrow{c} \langle\{q\}, \mathtt{i}, u\rangle \in \Delta_0$ where $f(q_1, \ldots, q_n) \to q \in \Delta$ and $f(u_1, \ldots, u_n) \xrightarrow{c} u \in \Delta_{\mathrm{NF}}$, and
  (b) $f(\langle\{q_1\}, \mathtt{x}_1, u_1\rangle, \ldots, \langle\{q_n\}, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle\{q\}, \mathtt{a}, u\rangle \in \Delta_0$ where $f(q_1, \ldots, q_n) \to q \in \Delta$, $f(u_1, \ldots, u_n) \xrightarrow{c} u \in \Delta_{\mathrm{NF}}$, and if $i \in \mu(f)$ then $\mathtt{x}_i = \mathtt{a}$, otherwise $\mathtt{x}_i = \mathtt{i}$.

Step 2 Let Δ_{k+1} be the set of transition rules produced by augmenting transition rules of Δ_k by the following inference rules.

(1) If there exists $\sigma : X \to \mathcal{T}(F)$ such that $f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_k]{*} f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_k]{} \langle\{q\}, \mathtt{a}, u\rangle$ and we have $u_i \in Q_{\mathrm{NF}}^f$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$, then we apply the following inference rules:

$$\frac{f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R,\ f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle\{q\}, \mathtt{a}, u\rangle \in \Delta_k}{g(\langle P_1', \mathtt{x}_1', u_1'\rangle, \ldots, \langle P_m', \mathtt{x}_m', u_m'\rangle) \xrightarrow{c'} \langle\{q\}, \mathtt{a}, u'\rangle \in \Delta_{k+1}}$$

Let $I_j = \{i \mid l_i = r_j\}$. Each P_j', $\mathtt{x}_j'$, u_j', c', and u' is determined as follows:

- – $P_j' = \begin{cases} \{q^{r_j}\} & \cdots \text{ if } r_j \notin X, \\ P_i & \cdots \text{ if } r_j \in X \wedge \exists i \in I_j.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i \in I_j} P_i & \cdots \text{ if } r_j \in X \wedge \forall i \in I_j.\mathtt{x}_i = \mathtt{a}. \end{cases}$
  – $\mathtt{x}_j' = \begin{cases} \mathtt{i} \cdots \text{ if } j \notin \mu(g) \wedge (r_j \in X \Rightarrow \exists i \in I_j.\mathtt{x}_i = \mathtt{i}), \text{ and} \\ \mathtt{a} \cdots \text{ if otherwise.} \end{cases}$
  – $u_j' = \begin{cases} u_i & \cdots \text{ if } r_j \in X \wedge (j \in \mu(g) \Rightarrow \forall i \in I_j.\mathtt{x}_i = \mathtt{a}) \\ v \in Q_{\mathrm{NF}} & \cdots \text{ otherwise} \end{cases}$
- $c' = c_1 \wedge c_2 \wedge c_3$ that is a satisfiable constraint, where
  – $c_1 = \bigwedge_{r_i = r_j \in X, \neg\exists k \in I_j.\mathtt{x}_k = \mathtt{i}} i = j$
  – c_2 is obtained from c by replacing equality and disequality between i and j in c as follows. Let i' and j' be as $l_i = r_{i'}$ and $l_j = l_{j'}$.
    * If $u_i \neq u_j$, we replace $i = j$ in c by $\perp$ and $i \neq j$ by $\top$.
    * If $u_i = u_j \in Q_{\mathrm{NF}} \backslash Q_{\mathrm{NF}}^f$, we consider the following two cases:
      · In the subcase of $P_i = P_j$, we replace $i = j$ in c by $\top$ and $i \neq j$ by $\perp$.
      · In the subcase of $P_i \neq P_j$, we replace $i = j$ in c by $\perp$ and $i \neq j$ by $\top$.
    * If $u_i = u_j \in Q_{\mathrm{NF}}^f$, we consider the following two cases:
      · In the subcase of $l_i \neq l_j$ and $l_i, l_j \in X$, we replace $i = j$ in c by $i' = j'$ and $i \neq j$ by $i' \neq j'$.
      · Otherwise, we replace $i = j$ in c by $\top$ and $i \neq j$ by $\perp$.
    * If $i > n$ or $j > n$, then replace $i = j$ and $i \neq j$ by $\perp$.
  – $g(u_1', \ldots, u_m') \xrightarrow{c_3} u' \in \Delta_{\mathrm{NF}}$.

Note that c' is not unique because we may choose more than one constraint for c_3, and also that the role of c_2 is to preserve the constraints for variables in the rewrite rule applied at the inference rule.

(2) If there exists $\sigma : X \to \mathcal{T}(F)$ such that $f(l_1,\ldots,l_n)\sigma \xrightarrow[\Delta_k]{*} f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow[\Delta_k]{*} \langle\{q\},\mathtt{a},u\rangle$ and $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$, we apply the following inference rule:

$$\frac{f(l_1,...,l_n) \to x \in R,\ f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c} \langle\{q\},\mathtt{a},u\rangle \in \Delta_k}{\langle P',\mathtt{a},u\rangle \xrightarrow{\top} \langle\{q\},\mathtt{a},u\rangle \in \Delta_{k+1}}$$

Let $I = \{i \mid l_i = x\}$. P' is determined as the follows:

- $P' = \begin{cases} P_i & \cdots \text{ if } \exists i \in I.\mathtt{x}_i = \mathtt{i}, \text{ and} \\ \bigcup_{i\in I} P_i & \cdots \text{ if } \forall i \in I.\mathtt{x}_i = \mathtt{a}. \end{cases}$

Step 3 For all states $\langle P^1 \cup P^2,\mathtt{a},u\rangle \in Q_*$ where $P^1 \ne P^2$, we add the new transition rules to Δ_{k+1} as follows:[⋆1]

(1) $f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c'} \langle P^1 \cup P^2,\mathtt{a}\rangle \in \Delta_{k+1}$ where

- $P_i = \begin{cases} P_i^j & \cdots \begin{array}{l}\text{if } \mathtt{x}_i^j = \mathtt{i} \text{ for some } j \in \{1,2\} \text{ and} \\ \mathcal{L}(\Delta_k,\langle P_i^1,\mathtt{x}_i^1,u_i\rangle) \cap \mathcal{L}(\Delta_k,\langle P_i^2,\mathtt{x}_i^2,u_i\rangle) \ne \emptyset\end{array} \\ P_i^1 \cup P_i^2 & \cdots \text{ if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \end{cases}$
- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \\ \mathtt{i} \cdots \text{otherwise} \end{cases}$
- $c' = c^1 \wedge c^2$.

if $f(\langle P_1^j,\mathtt{x}_1^j,u_1\rangle,\ldots,\langle P_n^j,\mathtt{x}_n^j,u_1\rangle) \xrightarrow{c^j} \langle P^j,\mathtt{a},u\rangle \in \Delta_k$ for $j \in \{1,2\}$.
Note that if $\mathcal{L}(\Delta_k,\langle P_i^1,\mathtt{x}_i^1,u_i\rangle) \cap \mathcal{L}(\Delta_k,\langle P_i^2,\mathtt{x}_i^2,u_i\rangle) = \emptyset$ and $\mathtt{x}_i^j = \mathtt{i}$ for some $j \in \{1,2\}$, then the transition rule is not produced.

(2) $\langle P_1' \cup P_2',\mathtt{a},u\rangle \xrightarrow{\top} \langle P_1 \cup P_2,\mathtt{a},u\rangle \in \Delta_{k+1}$ if $\langle P_1',\mathtt{a},u\rangle \xrightarrow{\top} \langle P_1,\mathtt{a},u\rangle \in \Delta_k$, and, $\langle P_2',\mathtt{a},u\rangle \xrightarrow{\top} \langle P_2,\mathtt{a},u\rangle \in \Delta_k$ or $P_2' = P_2$.

Step 4 If $\Delta_{k+1} = \Delta_k$ then stop and set $\Delta_* = \Delta_k$. Otherwise, $k := k+1$, and go to Step 2. □

We show an example that shows how $\mathrm{P}_{\mathrm{csin}}$ works in Appendix A.1.2. This procedure $\mathrm{P}_{\mathrm{csin}}$ eventually terminates at some k and apparently $\Delta_0 \subset \cdots \subset \Delta_k = \Delta_{k+1} = \cdots$ similarly to P_{cs}.

In the following, we show the correctness of $\mathrm{P}_{\mathrm{csin}}$.

⋆1 This step is almost the same as the step of P_{cs} because we do not need to be concerned about third components of states in each transition rule $f(\langle P_1^j,\mathtt{x}_1^j,u_1\rangle,\ldots,\langle P_n^j,\mathtt{x}_n^j,u_n\rangle) \xrightarrow{c^j} \langle P^j,\mathtt{a},u\rangle$.

First, we show several propositions. Since Propositions 28–31 below are similar to the case of P_{cs}, we abbreviate their proofs.

Proposition 28 Let $s \in \mathcal{T}(F)$, Then $s \xrightarrow[\Delta]{*} q^s \in Q$ iff $s \xrightarrow[\Delta_0]{*} \langle\{q^s\},\mathtt{i},u\rangle$ for some $u \in Q_{\mathrm{NF}}$.

Proposition 29 Let $t \in \mathcal{T}(F)$. For any k, if $t \xrightarrow[\Delta_k]{*} \langle P,\mathtt{i},u\rangle$, then $t \xrightarrow[\Delta_0]{*} \langle P,\mathtt{i},u\rangle$. Moreover, P is of the form $\{q\}$.

Proposition 30 Let $t \in \mathcal{T}(F)$. Then, for any k, $t \xrightarrow[\Delta_0]{*} \langle P,\mathtt{a},u\rangle$ iff $t \xrightarrow[\Delta_0]{*} \langle P,\mathtt{i},u\rangle$.

Proposition 31 Let $t \in \mathcal{T}(F)$. Then, $t \xrightarrow[\Delta_k]{*} \langle P,\mathtt{i},u\rangle$ implies $t \xrightarrow[\Delta_k]{*} \langle P,\mathtt{a},u\rangle$.

Proposition 32 If the rule $f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c} \langle P,\mathtt{i},u\rangle$ is in Δ_*, then it is also in Δ_0. Moreover, $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$.

Proof: Such rules are introduced at Step 1 and hence the claim follows from the construction of Δ_0. □

Proposition 33 If the rule $f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c} \langle P,\mathtt{a},u\rangle$ is in Δ_*, then $i \in \mu(f)$ implies $\mathtt{x}_i = \mathtt{a}$.

Proof: From the construction of the transition rule. □

Next we show several technical lemmas. These are necessary to prove completeness and soundness of $\mathrm{P}_{\mathrm{csin}}$. We abbreviate the proofs of Lemma 35 and 39–40 since their proofs are similar to the case of P_{cs}.

Lemma 34 For any k, if $\alpha : t \xrightarrow[\Delta_k]{*} \langle P,\mathtt{x},u\rangle$, then $t \xrightarrow[\Delta_{\mathrm{NF}}]{*} u$.

Proof: We show the proof by induction on $|\alpha|$. If the last transition rule applied in α is of the form $\langle P',\mathtt{x},u\rangle \xrightarrow{\top} \langle P,\mathtt{x},u\rangle$, then we have $t \xrightarrow[\Delta_{\mathrm{NF}}]{*} u$ from the induction hypothesis. Otherwise, let the last transition rule applied in $|\alpha|$ is $f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c} \langle P,\mathtt{x},u\rangle$.

If $f(\langle P_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle P_n,\mathtt{x}_n,u_n\rangle) \xrightarrow{c} \langle P,\mathtt{x},u\rangle \in \Delta_k$, then we have

$f(u_1,\ldots,u_n) \xrightarrow{c'} u \in \Delta_{\mathrm{NF}}$ where c is of the form $c = c'' \wedge c'$ for some c'. Thus, if t satisfies c then c' is also satisfied. Since we have $t|_i \xrightarrow[\Delta_{\mathrm{NF}}]{*} u_i$ from the induction hypothesis, we have $t \xrightarrow[\Delta_{\mathrm{NF}}]{*} f(u_1,\ldots,u_n) \xrightarrow[\Delta_{\mathrm{NF}}]{} u$. □

Lemma 35 Let $s,t \in \mathcal{T}(F)$, $s \xrightarrow[\Delta_0]{*} \langle P, \mathtt{x}, u\rangle$, and $t \xrightarrow[\Delta_0]{*} \langle P', \mathtt{x}', u'\rangle$. Then, $P = P'$ iff $s = t$.

Proof: Similar to the proof of Lemma 9. □

Lemma 36 Let $\alpha:\ t \xrightarrow[\Delta_*]{*} t[\langle P, \mathtt{a}, u\rangle]_p \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, u'\rangle$ and $p \in \mathrm{Pos}^{\mu}(s)$. Then $u' \in Q^f_{\mathrm{NF}}$ implies $u \in Q^f_{\mathrm{NF}}$.

Proof: We show this lemma by induction on $|\alpha|(>0)$.

(1) Consider the case where the last transition rule applied in α is (of the form) $f(\langle P_1, \mathtt{x}_1, u_1\rangle,\ldots,\langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle P, \mathtt{a}, u'\rangle \in \Delta_*$. Then α can be represented as $t \xrightarrow[\Delta_*]{*} t[\langle P, \mathtt{a}, u\rangle]_p \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1\rangle,\ldots,\langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_*]{} \langle P, \mathtt{a}, u'\rangle$.
In this case, the position p can be represented as ip' for $1 \le i \le m$. From the construction of the transition rule $f(\langle P_1, \mathtt{x}_1, u_1\rangle,\ldots,\langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle P, \mathtt{a}, u'\rangle$, we have the transition rule $f(u_1,\ldots,u_n) \xrightarrow{c'} u' \in \Delta_{\mathrm{NF}}$ for c'. Therefore, from $i \in \mu(g)$ and Proposition 27, we have $u_i \in Q^f_{\mathrm{NF}}$ and hence we also have $u \in Q^f_{\mathrm{NF}}$ from the induction hypothesis.

(2) In the case where the last transition rule applied in α is (in the form of) $\langle P', \mathtt{a}, u'\rangle \xrightarrow{c} \langle P, \mathtt{a}, u\rangle \in \Delta_k$, we have $u' = u$ from the construction of Δ_0 or the second inference rule of Step 2. Hence this lemma holds by the induction hypothesis. □

Lemma 37 If $j \notin \mu(g)$ and $g(\ldots,\langle P'_j, \mathtt{x}'_j, u'_j\rangle,\ldots) \xrightarrow{c} \langle P, \mathtt{x}', u'\rangle \in \Delta_k$, then $u'_j \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}'_j = \mathtt{i}$.

Proof:

(1) If $k = 0$, then $\mathtt{x}'_j = \mathtt{i}$ from the construction of Δ_0

(2) Consider the case of $k > 0$. We can assume $g(\ldots,\langle P'_j, \mathtt{x}'_j, u'_j\rangle,\ldots) \xrightarrow{c} \langle P, \mathtt{x}', u'\rangle \in \Delta_k \backslash \Delta_{k-1}$ without loss of generality. This rule is introduced by (1) of Step 2 or (1) of Step 3. In the latter case, if $\mathtt{x}'_j = \mathtt{a}$, then we have $g(\ldots,\langle P''_j, \mathtt{a}, u'_j\rangle,\ldots) \xrightarrow{c'} \langle \{q\}, \mathtt{x}', u'\rangle \in \Delta_k$ for any $q \in P$ where this rule is in Δ_0 or produced by (1) of Step 2. Therefore, if we prove the former case, we can also prove the latter case. In the former case, $\mathtt{x}'_j = \mathtt{a}$ implies $\mathtt{x}' = \mathtt{a}$ from Proposition 32, and there exist $f(l_1,\ldots,l_n) \to g(r_1,\ldots,r_m) \in R$ and $f(\langle P_1, \mathtt{x}_1, u_1\rangle,\ldots,\langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle P, \mathtt{a}, u\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$. If $j \notin \mu(g)$ and $\mathtt{x}'_j = \mathtt{a}$, then there exists some i such that $u_i = u'_j$ and $\mathtt{x}_{i'} = \mathtt{a}$ for all i' such that $l_{i'} = r_j$. Hence we have $u_i = u'_j \in Q^f_{\mathrm{NF}}$. □

Lemma 38 Let $\alpha:\ t[t']_p \xrightarrow[\Delta_*]{*} t[\langle P, \mathtt{a}, u\rangle]_p \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, u'\rangle$. If $u \in Q_{\mathrm{NF}} \setminus Q^f_{\mathrm{NF}}$ and $p \in \mathrm{Pos}^{\mu}(t)$, then there exists $v' \in Q_{\mathrm{NF}}$ such that $t[t']_p \xrightarrow[\Delta_*]{*} t[\langle P, \mathtt{a}, v\rangle]_p \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, v'\rangle$ for any $v \in Q_{\mathrm{NF}}$.

Proof: We prove this lemma by induction on $|\alpha|(>0)$.

(1) Consider the case where the last transition rule applied in α is (of the form) $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c} \langle P', \mathtt{x}', u'\rangle \in \Delta_*$. Then α can be represented as $t[t']_p \xrightarrow[\Delta_*]{*} t[\langle P, \mathtt{x}, u\rangle]_p \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow[\Delta_*]{} \langle P', \mathtt{x}', u'\rangle$. Let $p = jp'$ where $1 \le j \le n$.
If the rule $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c} \langle P, \mathtt{x}', u\rangle$ is in Δ_0, the rule is produced at (2) of Step 1 of $\mathrm{P}_{\mathrm{csin}}$. Therefore, for any $u''_j \in Q_{\mathrm{NF}}$, there exists the constraints c'' and $u'' \in Q_{\mathrm{NF}}$ such that $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_j, \mathtt{x}'_j, u''_j\rangle,\ldots,\langle P'_n, \mathtt{x}'_n, u'_n\rangle) \xrightarrow{c''} \langle P, \mathtt{x}, u''\rangle \in \Delta_0$ where t satisfies c'' from the completeness of $\mathcal{A}_{\mathrm{NF}}$. Hence this lemma holds from the induction hypothesis.
Consider the case where the rule $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c} \langle P, \mathtt{x}', u\rangle$ is in $\Delta_k \backslash \Delta_{k-1}$ for $k > 0$. In this case, $j \in \mu(g)$ from $p \in \mathrm{Pos}^{\mu}(s)$, and we have $\mathtt{x}'_j = \mathtt{a}$ from $\mathtt{x} = \mathtt{a}$ and Proposition 33. For $\alpha_j:\ (t|_j)[t']_{p'} \xrightarrow[\Delta_*]{*} (t|_j)[\langle P, \mathtt{a}, u\rangle]_{p'} \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{x}'_j, u'_j\rangle$, we have $(t|_j)[t']_{p'} \xrightarrow[\Delta_*]{*} (t|_j)[\langle P, \mathtt{a}, v\rangle]_{p'} \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{a}, v'_j\rangle$ for some $v'_j \in Q_{\mathrm{NF}}$ from the induction hypothesis. Note that we have $u'_j \notin Q^f_{\mathrm{NF}}$ from $u \notin Q^f_{\mathrm{NF}}$ and Lemma 36. Thus, we prove there exists the transition rule $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_j, \mathtt{a}, v'_j\rangle,\ldots,\langle P'_n, \mathtt{x}'_n, u'_n\rangle) \xrightarrow{c'} \langle P', \mathtt{x}, v'\rangle \in \Delta_*$.
Here, there are two cases in which the rule $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle,\ldots,\langle P'_j, \mathtt{a}, u'_j\rangle,\ldots,\langle P'_n,$

$\mathtt{x}'_n, u'_n\rangle) \xrightarrow{c'} \langle P', \mathtt{x}, u\rangle \in \Delta_*$ is produced in (1) of Step 2 or (1) of Step 3 of $\mathrm{P_{csin}}$. In the former case, there exist $f(l_1,\ldots,l_n) \to g(r_1,\ldots,r_m) \in R$ and $f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c'} \langle P', \mathtt{a}, u''\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$, and the constraint c is of the form $c_1 \wedge c_2$ where $g(\ldots, u'_j, \ldots) \xrightarrow{c_2} u \in \Delta_{\mathrm{NF}}$.
In the subcase $r_j \notin X$, we have $g(\ldots, \langle P'_j, \mathtt{a}, v'_j\rangle, \ldots) \xrightarrow{c'} \langle P', \mathtt{a}, v'\rangle \in \Delta_k \setminus \Delta_{k-1}$ for any v'_j and c' is of the form $c_1 \wedge c'_2$ where $g(\ldots, v'_j, \ldots,) \xrightarrow{c'_2} v' \in \Delta_{\mathrm{NF}}$. Moreover, from the completeness of Δ_{NF}, we have c'_2 that is satisfied by $t[t']_p$.
In the remaining subcase $r_j \in X$, we have $l_i = r_j$ such that $\mathtt{x}_i = \mathtt{i}$ for some i; otherwise we have $u'_j = u_i$ from $j \in \mu(g)$ and $\mathtt{x}_i = \mathtt{a}$ for any i such that $l_i = r_j$. Hence we have $u_i \in Q^f_{\mathrm{NF}}$. This contradicts $u'_j = u_i$ and $u'_j \notin Q^f_{\mathrm{NF}}$. Thus, we have the transition rule $g(\ldots, \langle P'_j, \mathtt{a}, v'_j\rangle, \ldots) \xrightarrow{c'} \langle P', \mathtt{a}, v'\rangle$ for any v'_j and $s[s']_p$ satisfies c' by the same reason in the case of $r_j \notin X$.
If the transition rule is produced at (1) of Step 3, we have $g(\ldots, \langle P''_j, \mathtt{a}, u_j\rangle, \ldots) \xrightarrow{c''} \langle \{q\}, a, u'\rangle$ where c is of the form $c'' \wedge c'''$ for any $q \in P$ and some c'''. From the former case, we have $g(\ldots, \langle P''_j, \mathtt{a}, v_j\rangle, \ldots) \xrightarrow{c'} \langle \{q\}, a, v'\rangle$ for each q and c' that is satisfied by $t[t']_p$. Thus, we have $g(\ldots, \langle P'_j, \mathtt{a}, v_j\rangle, \ldots) \xrightarrow{c'} \langle P, a, v'\rangle$.

(2) In the case where the last transition rule applied in α is (of the form) $\langle P', \mathtt{x}', u'\rangle \to \langle P, \mathtt{x}, u\rangle \in \Delta_k$, we have $u' = u$ from the construction of any of Δ_0, (2) of Step 2, or (2) of Step 3. Hence this lemma holds from the induction hypothesis. □

Lemma 39 If $\alpha : t[t']_p \xrightarrow[\Delta_k]{*} \langle P, \mathtt{a}, u\rangle$ and $p \in \mathrm{Pos}^\mu(t)$, then there exists $\langle P', \mathtt{a}, u'\rangle$ such that $t' \xrightarrow[\Delta_k]{*} \langle P', \mathtt{a}, u'\rangle$ and $t[\langle P', \mathtt{a}, u'\rangle]_p \xrightarrow[\Delta_k]{*} \langle P, \mathtt{a}, u\rangle$.

Proof: Similar to the proof of Lemma 10. □

Proofs of the following Lemmas 40– 43 are similar to the ones of Lemma 11–14, because we do not need to consider the third components of the states.

Lemma 40 If $\langle P'_1, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P_1, \mathtt{a}, u\rangle$ and $\langle P'_2, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P_2, \mathtt{a}, u\rangle$, then we have $\langle P'_1 \cup P'_2, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P_1 \cup P_2, \mathtt{a}, u\rangle$.

Proof: Similar to Lemma 11. □

Lemma 41 If $\langle P_1, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u\rangle$, then there exists $P'_1 \subseteq P_1$ such that $\langle P'_1, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, u\rangle$ for all $P' \subseteq P$.

Proof: Similar to of Lemma 13. □

Lemma 42 If $t \xrightarrow[\Delta_*]{*} \langle P^j, \mathtt{a}, u\rangle$ for $1 \le j \le m$, then we have $t \xrightarrow[\Delta_*]{*} \langle \bigcup_{1\le j\le m} P^j, \mathtt{a}, u\rangle$.

Proof: Since the proof of this lemma is similar to the proof of Lemma 12, we describe a concrete proof in Appendix A.2.1. The difference between proofs of this lemma and Lemma 12 is in the constraints. However, since the constraint of the transition rule produced at (1) of Step 3 of $\mathrm{P_{csin}}$ is simple, the difference does not cause difficulty. □

Lemma 43 If $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u\rangle$, then $t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, u\rangle$ for any $P' \subseteq P$.

Proof: Since the proof of this lemma is similar to the proof of Lemma 14, we describe a concrete proof in Appendix A.2.2. The difference between proofs of this lemma and Lemma 14 is in the constraints. However, since the constraint of the transition rule produced at (1) of Step 3 of $\mathrm{P_{csin}}$ is simple, the difference does not cause difficulty. □

The following lemma is a key lemma to prove completeness of $\mathrm{P_{csin}}$.

Lemma 44 Let $\mathcal{R}$ be shallow CS-TRS. Then $s \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u\rangle$ and $s \xleftrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t$ imply $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u'\rangle$ for some $u' \in Q_{\mathrm{NF}}$.

Proof: Since the proof of this lemma is similar to the proof of Lemma 15 and the proof of this lemma is long, here we show the outline of this proof and the detail about constraints of transition rules which is the most important point. We the other points at Appendix A.3.

Similarly to Lemma 15, we show the proof in the case where $s \xleftrightarrow[\mathcal{R}]{}_{\mathrm{in}} t$, and from Lemmas 39, 42, and 43. This proof is sufficient to prove the transition $f(l_1,\ldots,l_n)\sigma \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_*]{} \langle \{q\}, \mathtt{a}, u\rangle$ and the rewrite rule $f(l_1,\ldots,l_n) \to g(r_1,\ldots,r_m) \in R$, we have the transition $g(r_1,\ldots,r_m)\sigma \xrightarrow[\Delta_*]{*}$

$g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow[\Delta_*]{} \langle\{q\}, \mathtt{a}, u'\rangle$.

Similarly to the proof of Lemma 15, we have the transition rules $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c'} \langle\{q\}, \mathtt{a}, u'\rangle$ (see (1) of Appendix A.3) and each component of states is determined as the definition of $\mathrm{P_{csin}}$. Moreover, we have $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_1, \mathtt{x}'_1, u'_1\rangle$ for each j similarly to the proof of Lemma 15.

However, we must show that the term $g(r_1, \ldots, r_m)\sigma$ satisfies the constraint c', which is the point that the proof of Lemma 15 does not have. Here, we show that if $f(l_1, \ldots, l_n)\sigma$ satisfies the constraint c of the transition rule $f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_*]{} \langle\{q\}, \mathtt{a}, u\rangle$, then there exists c' satisfied by $g(r_1, \ldots, r_m)\sigma$. In the following, we assume the constraints c_1, c_2, and c_3 are the same as the definition of $\mathrm{P_{csin}}$.

(1) $g(r_1, \ldots, r_m)\sigma$ trivially satisfies c_1 because we have $r_i\sigma = r_j\sigma$ for $r_i = r_j \in X$ obviously.

(2) Here, we show the claim that if $f(l_1, \ldots, l_n)\sigma$ satisfies c then $g(r_1, \ldots, r_m)\sigma$ satisfies c_2. We describe the constraints replaced by equality, disequality, or $\bot$.

- For i and j such that $u_i \neq u_j$, we have $l_i\sigma \neq l_j\sigma$ from Lemma 34 and the determinacy of $\mathcal{A}_{\mathrm{NF}}$. Thus, $i = j$ is not satisfied by $f(l_1, \ldots, l_n)\sigma$, and hence, there is no problem replacing $i = j$ in c by $\bot$ in c_2.
- For i and j such that $u_i = u_j \in Q_{\mathrm{NF}} \backslash Q^f_{\mathrm{NF}}$, we have $\mathtt{x}_i = \mathtt{x}_j = \mathtt{i}$ and hence $l_i\sigma \xrightarrow[\Delta_0]{*} \langle P_i, \mathtt{x}_i, u_i\rangle$ and $l_j\sigma \xrightarrow[\Delta_0]{*} \langle P_j, \mathtt{x}_j, u_j\rangle$ from Proposition 29. Therefore, we have $P_i = P_j$ iff $l_i\sigma = l_j\sigma$ from Lemma 35. Thus, $i \neq j$ is not satisfied by $f(l_1, \ldots, l_n)\sigma$ if $P_i = P_j$ and $i = j$ is not satisfied by $f(l_1, \ldots, l_n)\sigma$ if $P_i \neq P_j$, and therefore there is no problem replacing $i \neq j$ in c by $\bot$ in c_2 if $P_i = P_j$ and $i = j$ in c by $\bot$ in c_2 if $P_i \neq P_j$.
- For i and j such that $u_i = u_j \in Q^f_{\mathrm{NF}}$, we consider the following three cases. Let i' and j' be as $l_i = r_{i'}$ and $l_j = r_{j'}$.
  - If $l_i \neq l_j$ and $l_i, l_j \in X$, then we have $l_i\sigma = l_j\sigma$ iff $r_{i'}\sigma = r_{j'}\sigma$. Thus, we have $f(l_1, \ldots, l_n)\sigma$ satisfies $i = j$ in c iff $g(r_1, \ldots, r_m)\sigma$ satisfies $i' = j'$ in c_2, and $f(l_1, \ldots, l_n)\sigma$ satisfies $i \neq j$ in c iff $g(r_1, \ldots, r_m)\sigma$ satisfies $i' \neq j'$ in c_2.
  - If $l_i = l_j \in X$, then we have $l_i\sigma = l_j\sigma$ and $r_{i'}\sigma = r_{j'}\sigma$. Thus, there is no problem to replace $i \neq j$ in c by $\bot$ in c_2.
  - If $l_i \notin X$, then we have $l_i = l_j\sigma$ from Lemma 34 and Proposition 26. Thus, there is no problem replacing $i \neq j$ in c by $\bot$ in c_2.
- For i and j such that $i > 0$ or $j > 0$, the constraints $i = j$ or $i \neq j$ is not satisfied by $f(l_1, \ldots, l_n)\sigma$ and therefore, there is no problem replacing these constraints by $\bot$.

Moreover, we have a constraint c_3 that is satisfied by $g(r_1, \ldots, r_m)\sigma$ from the completeness of Δ_{NF}. Thus, we have a constraint c' that is satisfied by $g(r_1, \ldots, r_m)\sigma$ and hence we have the transition $g(r_1, \ldots, r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow[\Delta_*]{} \langle\{q\}, \mathtt{a}, u'\rangle$. □

The following lemma shows the completeness of $\mathrm{P_{csin}}$.

Lemma 45 Let $\mathcal{R}$ be shallow. Then $\mathcal{L}(\mathcal{A}_*) \supseteq \hookrightarrow_{\mathcal{R}\,\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.

Proof: Let $s \overset{*}{\hookrightarrow}_{\mathcal{R}\,\mathrm{in}} t$ and $s \xrightarrow[\Delta]{*} q \in Q^f$. Since $s \xrightarrow[\Delta_0]{*} \langle\{q\}, \mathtt{i}, u\rangle \in Q^f_*$ from Proposition 28, we have $s \xrightarrow[\Delta_0]{*} \langle\{q\}, \mathtt{a}, u\rangle \in Q^f_*$ by Proposition 30. Hence $t \xrightarrow[\Delta_*]{*} \langle\{q\}, \mathtt{a}, u'\rangle \in Q^f_*$ for some $u' \in Q_{\mathrm{NF}}$ by Lemma 44. □

Next we define the measure and order of transition in order to prove the soundness. These definitions are similar to the case of $\mathrm{P_{cs}}$.

Definition 46 Let $||t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}, u\rangle||$ be the sequence of integer defined as follows:

$$||t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}, u\rangle|| = \begin{cases} k.||t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{x}', u'\rangle|| & \cdots \text{ if } t \xrightarrow[\Delta_*]{*} \langle P', \mathtt{x}', u'\rangle \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle P, \mathtt{x}, u\rangle \\ & \quad \text{if } t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\ldots, \langle P_i, \mathtt{x}_i, u_i\rangle, \ldots) \\ k.||t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i, u_i\rangle|| & \cdots \quad \xrightarrow[\Delta_k \backslash \Delta_{k-1}]{} \langle P, \mathtt{x}, u\rangle, \text{ and} \\ & \quad \forall i \neq j.||t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i, u_i\rangle|| \\ & \quad \geq_{\mathrm{lex}} ||t_j \xrightarrow[\Delta_*]{*} \langle P_j, \mathtt{x}_j, u_j\rangle|| \end{cases}$$

The order $\sqsupseteq$ and $\sqsupset$ for transition sequences is defined similarly to Definition 18.

The following lemma is the key lemma to prove soundness of $\mathrm{P_{csin}}$.

Lemma 47 Let Δ_* be generated from a shallow CS-TRS $\mathcal{R}$. Then $\alpha :\ t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}, u'\rangle$ implies that both $s \overset{*}{\hookrightarrow}_{\mathcal{R}\,\mathrm{in}} t$ and $\beta :\ s \xrightarrow[\Delta_0]{*} \langle\{q\}, \mathtt{x}, u\rangle$ for some term s, $q \in P$, and $u \in Q_{\mathrm{NF}}$.

Proof: Similarly to Lemma 44, we describe some points of the proof in Appendix A.3.2. Here we show the proof in the case of the last transition rule in α is in $\Delta_k \backslash \Delta_{k-1}$ for $k > 0$ and $|P| = 1$. We abbreviate the proof in the case for $k = 0$ or $|P| > 0$ because it is similar to the proof of Lemma 19.

Assume that $t = g(t_1, \ldots, t_m)$ and the last transition rule in α is $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c'} \langle\{q\}, \mathtt{x}, u'\rangle \in \Delta_k \backslash \Delta_{k+1}$. Since this rule is introduced at (1) of Step 2, there exist $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$, $f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c} \langle\{q\}, \mathtt{a}, u\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$, $\sigma' : X \to \mathcal{T}(F)$ such that $l_i\sigma' \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i, u_i\rangle$, and $\langle P'_j, \mathtt{x}'_j, u'_j\rangle$, c', and u' are given as the definition of $\mathrm{P}_{\mathrm{csin}}$. Then, we have the substitution σ such that $g(r_1, \ldots, r_m)\sigma \xleftrightarrow[\mathcal{R}]{*} g(t_1, \ldots, t_m)$ and $\alpha' : g(r_1, \ldots, r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, v'_m\rangle) \xrightarrow[\Delta_k]{} \langle\{q\}, \mathtt{a}, v'\rangle$ for some v', where $\alpha' \sqsubseteq \alpha$ similarly to the proof of Lemma 19 (see Appendix A.3.2).

Note that the substitution σ is well-defined because for all $r_j \in X$ such that there exists i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, we have the term $s_j(= t_j$ for $j \notin \mu(g))$ such that $s_j \xleftrightarrow[\mathcal{R}]{*} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j, \mathtt{x}'_j, u'_j\rangle$. Since there is no term other than s_j that transits to $\langle P'_j, \mathtt{x}'_j, u'_j\rangle$, all s_j's are the same for such j. For all $r_j \in X$ such that there is no i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, the constraint c' (c_1 in the procedure) has the equality that implies all t_j's are the same for such j.

Next we show that $g(r_1, \ldots, r_m)\sigma$ satisfies c_1 and c_2 of c' defined as the definition of $\mathrm{P}_{\mathrm{csin}}$.

Obviously, $g(r_1, \ldots, r_m)\sigma$ satisfies c_1 because $r_i\sigma = r_j\sigma$ for all $r_i = r_j \in X$. Moreover, it is not so difficult to show that $g(r_1, \ldots, r_m)\sigma$ satisfies c_2. This is because for all $r_j\sigma \ne t_j$, there exists i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$ and we have $u'_j = u_i$ from Lemma 34 and the determinacy of $\mathcal{A}_{\mathrm{NF}}$. In this case, we have $u_i = u'_j \notin Q^f_{\mathrm{NF}}$ and hence there is no equality or disequality that contain such j. From the completeness of Δ_{NF}, we have c'_3 that is satisfied by $g(r_1, \ldots, r_m)\sigma$. Thus we have the transition rule $g(\langle P'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, v'_m\rangle) \xrightarrow{c''} \langle\{q\}, \mathtt{a}, v'\rangle$ where $g(r_1, \ldots, r_m)\sigma$ satisfies c''.

On the other hand, we have $f(l_1, \ldots, l_n)\sigma \hookrightarrow_{\mathcal{R}} g(r_1, \ldots, r_m)\sigma$. Here, we show that we can construct $\beta : f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_{k-1}]{} \langle\{q\}, \mathtt{a}, v\rangle$ and hence $\beta \sqsubset \alpha$. For $l_i \notin X$, $l_i\sigma = l_i \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i, u_i\rangle$ from Step 2 of $\mathrm{P}_{\mathrm{csin}}$. For $l_i \in X$ and there exists h such that $l_h = l_i$ and $\mathtt{x}_h = \mathtt{i}$, since there is no term other than $l_h\sigma$ that transits to $\langle P_h, \mathtt{x}_h, u_h\rangle$ from Lemma 35, we have $l_i\sigma = l_i\sigma'$ and hence $l_i\sigma \xrightarrow[\Delta_{k-1}]{*} \langle P_i, \mathtt{x}_i, u_i\rangle$. For $l_i \in X$ such that there is no k such that $l_h = l_i$ and $\mathtt{x}_h = \mathtt{i}$, we have $l_i\sigma \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i, u_i\rangle$ from Lemma 42. In the following, we show that if $g(r_1, \ldots, r_m)\sigma$ satisfies c'' then $f(l_1, \ldots, l_n)\sigma$ satisfies c.

For an $\top$ in c_2, the constraint c has an equality or disequality. We consider the following three cases:

- Consider the case where $\top$ in c_2 is obtained by replacing $i \ne j$ in c where $u_i \ne u_j$. In this case, we have $l_i \ne l_i\sigma$ and hence $f(l_1, \ldots, l_n)\sigma$ satisfies $i \ne j$.
- Consider the case where $\top$ in c_2 is obtained by replacing $i = j$ or $i \ne j$ in c where $u_i = u_j \in Q_{\mathrm{NF}} \backslash Q^f_{\mathrm{NF}}$. Then, we have $\mathtt{x}_i = \mathtt{x}_j = \mathtt{i}$. In this case, if $P_i = P_j$ then we have $i = j$ but $f(l_1, \ldots, l_n)\sigma$ satisfies it from Proposition 29 and Lemma 35, and if $P_i \ne P_j$ then we have $i \ne j$ but $f(l_1, \ldots, l_n)\sigma$ satisfies it.
- Consider the case where $\top$ in c_2 is obtained by replacing $i = j$ in c where $u_i = u_j \in Q^f_{\mathrm{NF}}$.
  - If $l_i = l_j \in X$, we have $i = j$ in c but $f(l_1, \ldots, l_n)\sigma$ satisfies it trivially.
  - If $l_i \ne l_j$ and $l_i, l_j \in X$, then c does not have equality or disequality replaced by $\top$ in c_2.
  - If $l_i \notin X$, we have $i = j$ in c but we have $l_i = l_j\sigma$ from Lemma 34 and Proposition 26.

Moreover, we have $i = j$ or $i \ne j$ in c for $i' = j'$ or $i' \ne j'$ in c_2. These kinds of constraints are satisfied by $f(l_1, \ldots, l_n)\sigma$ similarly to the statement in Lemma 44.

Since $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all i, $l_i\sigma$ is a normal form or $i \notin \mu(f)$ for each i from Lemma 34 and the procedure. Hence we have $f(l_1, \ldots, l_n)\sigma \hookrightarrow_{\mathcal{R}}{}_{\mathrm{in}}\ g(r_1, \ldots, r_m)\sigma$. Here $\alpha \sqsupseteq \alpha' \sqsupset \beta$ follows. Thus, we have $s \xleftrightarrow[\mathcal{R}]{*}{}_{\mathrm{in}}\ f(l_1, \ldots, l_n)\sigma \hookrightarrow_{\mathcal{R}}{}_{\mathrm{in}}\ g(r_1, \ldots, r_m)\sigma \xleftrightarrow[\mathcal{R}]{*}{}_{\mathrm{in}}\ g(t_1, \ldots, t_m) = t$ and $s \xrightarrow[\Delta_0]{*} \langle\{q\}, \mathtt{a}, u\rangle$ for some u by the induction hypothesis. □

If a CS-TRS has an erasing variable, we cannot prove Lemma 47 as the above proof. Assume that the transition rule $g(\langle P'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow{c'} \langle\{q\}, \mathtt{a}, u'\rangle \in \Delta_{k+1}$ is produced from $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$ and $f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_m, \mathtt{x}_m, u_m\rangle) \xrightarrow{c} \langle\{q\}, \mathtt{a}, u\rangle \in \Delta_{k-1}$. If we have the equality or the disequality between i and j such that $l_i, l_j \in X$, l_i is the erasing variable, and there exists j' such that $l_j = r_{j'}$, then the equality or the disequality is not preserved to the produced rule.

The following lemma shows soundness of $\mathrm{P_{csin}}$.

Lemma 48 If $\mathcal{R}$ be shallow, then $\mathcal{L}(\mathcal{A}_*) \subseteq \hookrightarrow_{\mathcal{R}}{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.

Proof: Let $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{x}, u' \rangle \in Q^f_*$ then we have $s \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t$ and $s \xrightarrow[\Delta_0]{*} \langle \{q\}, \mathtt{x}, u \rangle \in Q^f_*$ for some $q \in P$ from Lemma 47. Since $s \xrightarrow[\Delta_0]{*} \langle \{q\}, \mathtt{i}, u \rangle$ from Proposition 30, we have $s \xrightarrow[\Delta]{*} q \in Q^f$ from Proposition 28. □

Finally we obtain the following theorems from Lemma 45 and 48.

Theorem 49 For any shallow CS-TRS $\mathcal{R}$, we can construct a TACBB recognizing the set of terms that is innermost reachable from a term. Thus, innermost reachability is decidable for shallow CS-TRSs.

However, in general, we cannot always construct a TACBB recognizing the innermost reachable set from a regular set of terms for a CS-TRS, while we can construct a TACBB in the case of the ordinary TRS [6].

Theorem 50 There exists a regular set L and a shallow CS-TRS $\mathcal{R}$ such that $\xhookrightarrow[\mathcal{R}]{*}[L]$ cannot be recognized by any TACBB.

Proof: Let $\mathrm{ar}(a) = 0$, $\mathrm{ar}(f) = \mathrm{ar}(h) = \mathrm{ar}(i) = 1$, $\mathrm{ar}(g) = 2$, $L = \{f(t) \mid t \in \mathcal{T}(\{h, a\})\}$, $\mathcal{R} = (R, \mu)$ where $R = \{f(x) \to g(x, x), h(x) \to i(x)\}$, and $\mu(f) = \mu(h) = \mu(i) = \emptyset$, $\mu(g) = \{1, 2\}$. Then, we have $\xhookrightarrow[\mathcal{R}]{*}[L] \cap \mathcal{T}(g, h, i, a) = \{g(t_1, t_2) \mid t_1, t_2 \in \mathcal{T}(\{h, i, a\}), |t_1| = |t_2|\}$. Since $\xhookrightarrow[\mathcal{R}]{*}[L] \cap \mathcal{T}(g, h, i, a)$ cannot be recognized by any TACBB, $\mathcal{T}(g, h, i, a)$ is regular, and TACBB is closed under intersection, there exists no TACBB that recognizes $\xhookrightarrow[\mathcal{R}]{*}[L]$. □

5. Conclusion

In this paper, we proved that both reachability for right-linear right-shallow CS-TRSs and innermost reachability for shallow CS-TRSs are decidable.

One of our future works is to construct a TA that recognizes the set of reachable terms from a regular set for a right-linear right-shallow CS-TRS. We described this problem in Section 3, but that does not mean that it is impossible to construct a correct TA. Since we have not found a TA and right-linear right-shallow CS-TRSs of which reachable sets cannot be recognized by any TA, this problem is still open.

Another future work is to find other subclasses that reachability, innermost reachability, or reachability of other strategies is decidable for TRSs or CS-TRSs. One of the candidates is reachability for right-linear finite pass overlapping CS-TRSs where it is known that reachability is decidable for ordinary TRSs in Ref. 17). However, the class is complex and hence we think this is not easy. Innermost reachability for right-linear right-shallow TRSs is also a candidate. In the case of this class, to recognize the set of normal forms, we need TA with equality or disequality constraints. This automata has more complex constraints than that of TACBB and sometimes more complex constraints than constraints nests. Therefore, we think that this problem is much more complex than the result of this paper. Moreover, outermost reachability is a candidate. Outermost reduction is a strategy that rewrites outermost redexes. Today, no class is known such that outermost reachability is decidable.

Acknowledgments We would like to thank the anonymous referees for their helpful comments and remarks. This work is partly supported by MEXT. KAKENHI #10J08166, #20300010, #20500008, and #21700011.

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Appendix

A.1 Examples for Section 4

A.1.1 An Example of TACBB Accepting the Set of Context-sensitive Normal Forms

Let CS-TRS $\mathcal{R} = (R, \mu)$ be as $R = \{a \to b, a \to c, f(x, b) \to g(x, a), g(x, x) \to h(x, x)\}$ and $\mu(f) = \emptyset$, $\mu(g) = \{1, 2\}$, $\mu(h) = \{1, 2\}$. We construct the TACBB $\mathcal{A}_{\mathrm{NF}}$ such that $\mathcal{L}(\mathcal{A}_{\mathrm{NF}}) = \text{CS-NF}_{\mathcal{R}}$ by the algorithm shown in Section 4.1.

First, we construct a deterministic TACBB $\mathcal{A}_a$, $\mathcal{A}_{f(x,a)}$, and $\mathcal{A}_{g(x,x)}$ at the first step of the algorithm.

The set of final states of $\mathcal{A}_a$ is $Q_a^f = \{U^\circ\}$ and the set of transition rules is $\Delta_a = \{a \xrightarrow{\top} U^\circ, b \xrightarrow{\top} U_\bot, c \xrightarrow{\top} U_\bot, f(U, U) \xrightarrow{\top} U_\bot, g(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, g(U_1, U_2) \xrightarrow{\top} U^\circ, h(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, h(U_1, U_2) \xrightarrow{\top} U^\circ\}$ where $U, U_1, U_2 \in \{U_\bot, U^\circ\}$ and one of U_1 and U_2 is U°.

The set of final states of $\mathcal{A}_{f(x,a)}$ is $Q_{f(x,b)}^f = \{U^\circ\}$ and the set of transition rules is $\Delta_{f(x,b)} = \{a \xrightarrow{\top} U_\bot, b \xrightarrow{\top} U_b, c \xrightarrow{\top} U_\bot, f(U, U_b) \xrightarrow{\top} U^\circ, f(U, U') \xrightarrow{\top} U_\bot, g(U'_1, U'_2) \xrightarrow{\top} U_\bot, g(U_1, U_2) \xrightarrow{\top} U^\circ, g(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, h(U'_1, U'_2) \xrightarrow{\top} U_\bot, h(U_1, U_2) \xrightarrow{\top} U^\circ\}$ where $U, U_1, U_2 \in \{U_\bot, U^\circ, U_b\}$, $U', U'_1, U'_2 \in \{U_\bot, U_b\}$, and one U_1 or U_2 is U°.

The set of final states of $\mathcal{A}_{g(x,x)}$ is $Q_{g(x,x)}^f = \{U^\circ\}$ and the set of transition rules is $\Delta_{g(x,x)} = \{a \xrightarrow{\top} U_\bot, b \xrightarrow{\top} U_b, c \xrightarrow{\top} U_\bot f(U) \xrightarrow{\top} U_\bot, g(U_\bot, U_\bot) \xrightarrow{1=2} U^\circ, g(U_1, U_2) \xrightarrow{\top} U^\circ, g(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, h(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, h(U_1, U_2) \xrightarrow{\top} U^\circ\}$ where $U, U_1, U_2 \in \{U_\bot, U^\circ\}$ and one U_1 or U_2 is U°.

At the second step, we construct the TACBB $\mathcal{A}'$ accepting all unions of $\mathcal{A}_a$, $\mathcal{A}_{f(x,a)}$, and $\mathcal{A}_{g(x,x)}$.

The set of final states of $\mathcal{A}'$ is $Q'^f = \{\langle U_1, U_2, U_3 \rangle\}$ where $U_1, U_2, U_3 \in \{U_\bot, U^\circ\}$ and one U_1, U_2, or U_3 is U°.

The set of transition rules of $\mathcal{A}'$ is $\Delta' = \{a \xrightarrow{\top} \langle U^\circ, U_\bot, U_\bot\rangle, b \xrightarrow{\top} \langle U_\bot, U_b, U_\bot\rangle, c \xrightarrow{\top} \langle U_\bot, U_\bot, U_\bot\rangle, f(U, U_b) \xrightarrow{\top} \langle U_\bot, U^\circ, U_\bot\rangle, f(U, U') \xrightarrow{\top} \langle U_\bot, U_\bot, U_\bot\rangle, g(\langle U_\bot, U_1', U_\bot\rangle, \langle U_\bot, U_2', U_\bot\rangle) \xrightarrow{1=2} \langle U_\bot, U_\bot, U_\bot\rangle, g(\langle U_\bot, U_1', U_\bot\rangle, \langle U_\bot, U_2', U_\bot\rangle) \xrightarrow{1\neq 2} \langle U_\bot, U_\bot, U^\circ\rangle, g(U_1, U_2) \to \langle U_\bot, U_\bot, U^\circ\rangle, h(U_\bot, U_\bot) \xrightarrow{\top} U_\bot, h(U_1, U_2) \xrightarrow{\top} U^\circ\}$, where $U', U_1', U_2' \in \{U_\bot, U^\circ\}$ and one of U_1 and U_2 is U°. We abbreviate the conversion to complete and reduced TACBB because the number of transition rules becomes huge. Let $\mathcal{A}''$ be the TACBB obtained by converting $\mathcal{A}'$ to a complete and reduced TACBB.

Finally, at the third step of the algorithm, we obtain $\mathcal{A}_{\mathrm{NF}}$ from $\mathcal{A}''$ by replacing the final state. We show the set of final states and the set of transition rules of $\mathcal{A}_{\mathrm{NF}}$ in the following. However, since $\mathcal{A}_{\mathrm{NF}}$ originally obtained from the algorithm is huge, we show a minified one. If we minify TACBB obtained by the algorithm, Proposition 26 may not hold. Therefore, we should not minify the TACBB obtained by the algorithm. In the case of following TACBB, Proposition 26 holds.

The set of states is $Q_{\mathrm{NF}} = \{u_b, u_\bot, u^\circ\}$, the set of final states is $Q^f_{\mathrm{NF}} = \{u_b, u_\bot\}$, and the set of transition rules is $\Delta_{\mathrm{NF}} = \{a \xrightarrow{\top} u^\circ, b \xrightarrow{\top} u_b, c \xrightarrow{\top} u_\bot, f(u^\circ, u^b) \xrightarrow{\top} u^\circ, f(u_1, u_2) \xrightarrow{\top} u_\bot, g(u_3, u_3) \xrightarrow{1=2} u^\circ, g(u_3, u_3) \xrightarrow{1\neq 2} u_\bot, g(u^\circ, u^\circ) \xrightarrow{\top} u^\circ, g(u_1, u_4) \xrightarrow{\top} u^\circ, h(u_1, u_1) \xrightarrow{\top} u_\bot, h(u_1, u_4) \xrightarrow{\top} u_\bot\}$ where $u_1, u_4 \in Q_{\mathrm{NF}}$, $u_2 \in \{u^\circ, u_\bot\}$, $u_3 \in Q^f_{\mathrm{NF}}$, and $u_1 \neq u_4$.

□

A.1.2 An Example of TACBB Obtained by $\mathbf{P_{csin}}$

Let CS-TRS $\mathcal{R}$ be the CS-TRS of A.1.1. We input the term $f(a, b)$ and the shallow CS-TRS $\mathcal{R}$ to $\mathrm{P_{csin}}$. Here, we have $\xhookrightarrow[\mathcal{R}]{}[\{f(a,b)\}] = \{f(a,b), g(a,a), g(b,a), g(a,b), g(b,b), g(c,a), g(c,b), g(a,c), g(b,c), g(c,c), h(b,b), h(c,c)\}$.

In the initializing step, at (1) of Step 1 of $\mathrm{P_{csin}}$, we have the TA $\mathcal{A} = \langle Q, Q^f, \Delta\rangle$ where $Q = \{q^a, q^b, q^{f(a,b)}\}$, $Q^f = \{q^{f(a,b)}\}$, and $\Delta = \{a \to q^a, b \to q^b, f(q^a, q^b) \to q^{f(a,b)}\}$, and TACBB $\mathcal{A}_{\mathrm{NF}}$ as a previous subsection. At (2) of Step 1, we have $Q_* = \{\langle P, \mathtt{a}, u\rangle, \langle \{p\}, \mathtt{i}, u\rangle\}$ where $P \subseteq Q$, $P \neq \emptyset$, $p \in Q$, and $u \in Q_{\mathrm{NF}}$ $Q^f_* = \{\langle P^f, \mathtt{a}, u\rangle\}$ where $P^f \cap Q^f \neq \emptyset$ and $u \in Q_{\mathrm{NF}}$, and Δ_0 is as follows:

$$\Delta_0 = \left\{\begin{array}{rcl} a & \xrightarrow{\top} & \langle \{q^a\}, \mathtt{x}, u^\circ\rangle, \\ b & \xrightarrow{\top} & \langle \{q^b\}, \mathtt{x}, u_b\rangle, \\ f(\langle \{q^a\}, \mathtt{i}, u^\circ\rangle, \langle \{q^b\}, \mathtt{i}, u_b\rangle) & \xrightarrow{\top} & \langle \{q^{f(a,b)}, \mathtt{x}, u^\circ\}\rangle \\ f(\langle \{q^a\}, \mathtt{i}, u_1\rangle, \langle \{q^b\}, \mathtt{i}, u_2\rangle) & \xrightarrow{\top} & \langle \{q^{f(a,b)}, \mathtt{x}, u_\bot\}\rangle \end{array}\right\}$$

where $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$, $u_1 \in Q_{\mathrm{NF}}$, and $u_2 \in \{u^\circ, u_\bot\}$.

In the saturation step, at $k = 0$, we produce the transition rules

$$\left\{\begin{array}{rcl} b & \xrightarrow{\top} & \langle \{q^a\}, \mathtt{x}, u_b\rangle, \\ c & \xrightarrow{\top} & \langle \{q^a\}, \mathtt{x}, u_\bot\rangle, \\ g(\langle \{q^a\}, \mathtt{a}, u_3\rangle, \langle \{q^a\}, \mathtt{a}, u_3\rangle) & \xrightarrow{1=2} & \langle \{q^{f(a,b)}\}, \mathtt{a}, u^\circ\rangle \\ g(\langle \{q^a\}, \mathtt{a}, u_3\rangle, \langle \{q^a\}, \mathtt{a}, u_3\rangle) & \xrightarrow{1\neq 2} & \langle \{q^{f(a,b)}\}, \mathtt{a}, u_\bot\rangle \\ g(\langle \{q^a\}, \mathtt{a}, u^\circ\rangle, \langle \{q^a\}, \mathtt{a}, u^\circ\rangle) & \xrightarrow{\top} & \langle \{q^{f(a,b)}\}, \mathtt{a}, u^\circ\rangle \\ g(\langle \{q^a\}, \mathtt{a}, u_1\rangle, \langle \{q^a\}, \mathtt{a}, u_4\rangle) & \xrightarrow{\top} & \langle \{q^{f(a,b)}\}, \mathtt{a}, u^\circ\rangle \end{array}\right\}$$

where $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$, $u_1, u_4 \in Q_{\mathrm{NF}}$, $u_3 \in Q^f_{\mathrm{NF}}$, and $u_1 = u_4$ at Step 2.

At $k = 1$, we produce the transition rules

$$\left\{ h(\langle \{q^a\}, \mathtt{a}, u_3\rangle, \langle \{q^a\}, \mathtt{a}, u_3\rangle) \xrightarrow{1=2} \langle \{q^{f(x,b)}\}, \mathtt{a}, u_\bot\rangle \right\}$$

where $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$ and $u_3 \in Q^f_{\mathrm{NF}}$, and $\{b \xrightarrow{\top} \langle \{q^a, q^b\}, \mathtt{x}, u_b\rangle\}$ at Step 3 where $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$.

The saturation step at $k = 2$, and we have $\Delta_* = \Delta_1$. TA $\mathcal{A}_* = \langle Q_*, Q^f_*, \Delta_*\rangle$ holds that $\mathcal{L}(\mathcal{A}_*) = \xhookrightarrow[\mathcal{R}]{}[\{f(a,b)\}]$.

A.2 Concrete Proofs of Lemma 42 and 43

A.2.1 Lemma 42

Proof: Similarly to the proof of Lemma 12, we give the proof for $m = 2$ by induction on $|t|$.

Let $t = f(t_1, \ldots, t_n)$. Then, each transition sequence is represented as $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P^j_1, \mathtt{x}^j_1, u_1\rangle, \ldots, \langle P^j_n, \mathtt{x}^j_n, u_n\rangle) \xrightarrow[\Delta_*]{} \langle P^j_r, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P^j, \mathtt{a}, u\rangle$ for $j \in \{1, 2\}$. From Lemma 40, we have $\langle P^1_r \cup P^2_r, \mathtt{a}, u\rangle \xrightarrow[\Delta_*]{*} \langle P^1 \cup P^2, \mathtt{a}, u\rangle$. Thus, we show that $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} \langle P^1_r \cup P^2_r, \mathtt{a}, u\rangle$.

From (1) of Step 3 of $\mathrm{P_{cs}}$, we have the transition rule $f(\langle P_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n\rangle) \xrightarrow{c'} \langle P, \mathtt{a}, u\rangle \in \Delta_*$ where

- $P_i = \begin{cases} P^j_i & \cdots \text{ if } \mathtt{x}^j_i = \mathtt{i} \text{ for some } j \in \{1, 2\} \text{ and} \\ P^1_i \cup P^2_i & \cdots \text{ if } \mathtt{x}^1_i = \mathtt{x}^2_i = \mathtt{a} \end{cases}$

- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \\ \mathtt{i} \cdots \text{otherwise} \end{cases}$
- $c' = c^1 \wedge c^2$.

Here we have $t_i \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{x}_i, u_i \rangle$ for $1 \leq i \leq n$ similarly to the proof of Lemma 12.

Moreover, since t satisfies both c^1 and c^2, t also satisfies c'. Thus, we have the transition $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow[\Delta_*]{} \langle P_r, \mathtt{a}, u \rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u \rangle$. □

A.2.2 Lemma 43

Proof: We show this lemma by induction on $|t|$. We can assume that the transition $t \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u \rangle$ is represented as $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow[\Delta_*]{} \langle P_\mathtt{r}, \mathtt{a}, u \rangle \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u \rangle$. From Lemma 41, there exists $P'_\mathtt{r} \subseteq P_\mathtt{r}$ such that $\langle P'_\mathtt{r}, \mathtt{a}, u \rangle \xrightarrow[\Delta_*]{*} \langle P', \mathtt{a}, u \rangle$ for all $P' \subseteq P$. Therefore, we show that we have $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow[\Delta_*]{} \langle P'_\mathtt{r}, \mathtt{a}, u \rangle$. Let $P'_\mathtt{r} = P_\mathtt{r} \backslash P''_\mathtt{r}$. We show the claim by the induction on $\Sigma_{i=1}^n |P_i| + |P_\mathtt{r}|$.

If $|P_\mathtt{r}| = 1$, then the claim holds trivially. If $|P_\mathtt{r}| > 1$, the transition rule $f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow{c} \langle P_\mathtt{r}, \mathtt{a}, u \rangle$ is produced from the transition rules $f(\langle P_1^j, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n^j, \mathtt{x}_n, u_n \rangle) \rightarrow \langle P_\mathtt{r}^j, \mathtt{a}, u \rangle$ where $j \in \{1, 2\}$ from (1) of Step 3 of $\mathrm{P_{cs}}$ and $P_\mathtt{r}$, P_i's, $\mathtt{x}_i$'s, and c' are represented as the follows:

- $P_\mathtt{r} = P_\mathtt{r}^1 \cup P_\mathtt{r}^2$,
- $P_i = \begin{cases} P_i^j & \cdots \text{if } \mathtt{x}_i^j = \mathtt{i} \text{ for some } j \in \{1, 2\} \text{ and} \\ P_i^1 \cup P_i^2 & \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \end{cases}$
- $\mathtt{x}_i = \begin{cases} \mathtt{a} \cdots \text{if } \mathtt{x}_i^1 = \mathtt{x}_i^2 = \mathtt{a} \\ \mathtt{i} \cdots \text{otherwise} \end{cases}$
- $c = c^1 \wedge c^2$

Here, we have $t \xrightarrow[\Delta_*]{*} \langle P_i^j, \mathtt{x}_i^j, u_i \rangle$ for $j \in \{1, 2\}$ and $1 \leq i \leq n$ similarly to the proof of Lemma 14. Since t satisfies $c = c^1 \wedge c^2$, t also satisfies both c^1 and c^2. Thus, we have $f(t_1, \ldots, t_n) \xrightarrow[\Delta_*]{*} f(\langle P_1^j, \mathtt{x}_1^j, u_1 \rangle, \ldots, \langle P_n^j, \mathtt{x}_n^j, u_n \rangle) \xrightarrow[\Delta_*]{} \langle P_\mathtt{r}^j, \mathtt{a}, u \rangle$ for both $j = 1$ and $j = 2$.

Thus, similarly to the proof of Lemma 14, we have $t \xrightarrow[\Delta_*]{*} \langle P_\mathtt{r}^1 \cup P_\mathtt{r}^2 \backslash P''_\mathtt{r}, \mathtt{a}, u \rangle$ from Lemma 42. □

A.3 Supplement of Proofs of Lemma 44 and 47

A.3.1 Lemma 44

(1) Here we show that if we have the transition $f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow[\Delta_*]{} \langle \{q\}, \mathtt{a}, u \rangle$ and the rewrite rule $f(l_1, \ldots, l_n) \rightarrow g(r_1, \ldots, r_m) \in R$, we have $f(l_1, \ldots, l_n)\sigma \xhookrightarrow[\mathcal{R}]{\text{in}} g(r_1, \ldots, r_m)\sigma$. and the transition rule $g(\langle P'_1, \mathtt{x}'_1, u'_1 \rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m \rangle) \xrightarrow{c'} \langle \{q\}, \mathtt{a}, v'' \rangle \in \Delta_*$ such as the definition of $\mathrm{P_{csin}}$.

For $i \in \mu(f)$, $l_i\sigma$ is a context-sensitive normal form and hence we have $u_j \in Q_{\mathrm{NF}}^f$ from Lemma 34. For i such that $i \notin \mu(f)$, we have $\mathtt{x}_i = \mathtt{i}$ or $u_j \in Q_{\mathrm{NF}}^f$ from Lemma 37.

Since $f(l_1, \ldots, l_n) \rightarrow g(r_1, \ldots, r_m) \in R$, $f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow{c} \langle \{q\}, \mathtt{a}, u'' \rangle \in \Delta_*$ where $u_i \in Q_{\mathrm{NF}}^f$ or $\mathtt{x}_i = \mathtt{i}$, and σ such that $f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_*]{*} f(\langle P_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle P_n, \mathtt{x}_n, u_n \rangle) \xrightarrow[\Delta_*]{*} \langle P, \mathtt{a}, u \rangle$, there exist transition rules $g(\langle P'_1, \mathtt{x}'_1, u'_1 \rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m \rangle) \xrightarrow{c'} \langle \{q\}, \mathtt{a}, v'' \rangle \in \Delta_*$ such as the definition of $\mathrm{P_{csin}}$

(2) Here, we show that $r_j\sigma \xrightarrow[\Delta_k]{*} \langle P'_j, \mathtt{x}'_j, u'_j \rangle$.

(a) For j such that $r_j \in X$ and there exists i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, we have $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P_i, \mathtt{i}, u_i \rangle = \langle P'_j, \mathtt{i}, u_i \rangle$. We can take $u'_j = u_i$ and we also have $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P'_j, \mathtt{a}, u'_j \rangle$ from Proposition 31.

(b) For j such that $r_j \in X$ and $\mathtt{x}_i = \mathtt{a}$ for all i such that $l_i = r_j$, then let $i_1, \ldots, i_k$ be all the numbers such that $l_{i_h} = r_j$ for $1 \leq h \leq k$. In this case, we have $l_{i_h}\sigma \xrightarrow[\Delta_*]{*} \langle P_{i_h}, \mathtt{x}_{i_h}, u_{i_h} \rangle$ for all i_h. Note that all u_{i_h}'s are equal from the determinacy of Δ_{NF}. Hence, we have $r_j\sigma \xrightarrow[\Delta_*]{*} \langle P_{i_1} \cup \cdots \cup P_{i_k}, \mathtt{a}, u'_j \rangle = \langle P'_j, \mathtt{x}'_j, u'_j \rangle$ where u'_j is equal to all u_{i_h}'s from Lemma 43.

(c) For j such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$ and $r_j\sigma = r_j$ since R is right-shallow so we can take the arbitrary state in Q_{NF} as u'_j. Since $r_j \xrightarrow[\Delta]{*} q^{r_j}$, we have $r_j \xrightarrow[\Delta_0]{*} \langle \{q^{r_j}\}, \mathtt{i}, v'' \rangle$ for some $v'' \in Q_{\mathrm{NF}}$ from Proposition 28. Moreover, since we also have $r_j \xrightarrow[\Delta_0]{*} \langle \{q^{r_j}\}, \mathtt{a}, v'' \rangle$ by Proposition 30, we obtain $r_j\sigma = r_j \xrightarrow[\Delta_*]{*} \langle \{q^{r_j}\}, \mathtt{x}'_j, v'' \rangle = \langle \{q'_j\}, \mathtt{x}'_j, u'_j \rangle$ where $u'_j = v''$.

Thus, we have $g(r_1, \ldots, r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1, u'_1 \rangle, \ldots, \langle P'_m, \mathtt{x}'_m, u'_m \rangle)$. Moreover, since $g(r_1,\ldots,r_m)\sigma$ satisfies c', we have $g(r_1,\ldots,r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1,\mathtt{x}'_1,u'_1\rangle,\ldots,\langle P'_m,\mathtt{x}'_m,u'_m\rangle) \xrightarrow[\Delta_{k+1}]{} \langle\{q\},\mathtt{a},u'\rangle$

□

A.3.2 Lemma 47

In the following, we show that we have substitution σ such that $g(r_1,\ldots,r_m)\sigma \xleftarrow[\mathcal{R}]{*} g(t_1,\ldots,t_m)$ and $\alpha' : g(r_1,\ldots,r_m)\sigma \xrightarrow[\Delta_*]{*} g(\langle P'_1, \mathtt{x}'_1, v'_1\rangle,\ldots,\langle P'_m,\mathtt{x}'_m,v'_m\rangle) \xrightarrow[\Delta_k]{} \langle\{q\},\mathtt{a},v'\rangle$, where $\alpha' \sqsubseteq \alpha$.

(1) For j such that $r_j \in X$, $j \notin \mu(g)$, and there exists i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, we have $t_j \xrightarrow[\Delta_*]{*} \langle P'_j,\mathtt{x}'_j,u'_j\rangle = \langle P_i,\mathtt{i},u_i\rangle$. Hence, we have $t_j \xrightarrow[\Delta_0]{*} \langle P'_j,\mathtt{x}'_j,u'_j\rangle$ from Proposition 29, and let $r_j\sigma = t_j$.

(2) For j such that $r_j \in X$, $j \in \mu(g)$, and there exists i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, we have $t_j \xrightarrow[\Delta_*]{*} \langle P'_j,\mathtt{x}'_j,u'_j\rangle = \langle P_i,\mathtt{a},u\rangle$ where u is an arbitrary state in Q_{NF}. Since P_i is of the form $\{q_i\}$ from Proposition 29, there exists some s_j such that $s_j \xleftarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j,\mathtt{x}'_j,v'_j\rangle$ for some v'_j from the induction hypothesis. Let s_j be $r_j\sigma$.

(3) For j such that $r_j \in X$ and there exists no i such that $l_i = r_j$ and $\mathtt{x}_i = \mathtt{i}$, we take $r_j\sigma = t_j$.

(4) For $j \notin \mu(g)$ such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$ and $x'_j = \mathtt{i}$, and u'_j is an arbitrary state in Q_{NF}. Since $t_j \xrightarrow[\Delta_0]{*} \langle\{q^{r_j}\},\mathtt{i},u'_j\rangle$ by Proposition 29, we have $t_j = r_j$ from Proposition 28 and the construction of Δ_{RS}.

(5) For $j \in \mu(g)$ such that $r_j \notin X$, we have $P'_j = \{q^{r_j}\}$ and $x'_j = \mathtt{i}$, and u'_j is an arbitrary state in Q_{NF}. Here, we have $s_j \xleftarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle P'_j,\mathtt{x}'_j,v'_j\rangle = \langle\{q^{r_j}\},\mathtt{a},v'_j\rangle$ for some $v'_j \in Q_{\mathrm{NF}}$ from the induction hypothesis. Since $s_j \xrightarrow[\Delta_0]{*} \langle\{q^{r_j}\},\mathtt{i},v'_j\rangle$ by Proposition 30, we have $s_j = r_j$ from Proposition 28 and the construction of Δ.

□

(Received February 14, 2011)
(Accepted May 20, 2011)

Yoshiharu Kojima was born in 1986. He was received his B.E. and M.E. degrees from Nagoya University. Now he is a student of the Graduate School of Information Science, Nagoya University, and a research fellow of the Japan Society for the Promotion of Science. His research interest is term rewriting systems.

Masahiko Sakai completed graduate course of Nagoya University in 1989 and became assistant professor, where he obtained a D.E. degree in 1992. From April 1993 to March 1997, he was an associate professor in JAIST, Hokuriku. In 1996 he stayed at SUNY at Stony Brook for six months as a visiting research professor. From April 1997, he was an associate professor in Nagoya University. Since December 2002, he has been a professor. He is interested in term rewriting system, verification of specification and software generation. He received the Best Paper Award from IEICE in 1992 and 2011. He is a member of IEICE and JSSST.

Naoki Nishida graduated with his D.E. degree from the Graduate School of Engineering at Nagoya University in 2004. He became a research associate in the Graduate School of Information Science at Nagoya University in 2004. From April 2007, he has been an assistant professor, and he was a visiting researcher in DSIC at Universitat Politècnica de València from July 2011 to December 2011. He received the Best Paper Award from IEICE in 2011. He is interested in program inversion, theorem proving, term rewriting, and program verification. He is a member of IEICE and JSSST.

Keiichirou Kusakari received his B.E. from Tokyo Institute of Technology in 1994, received M.E. and the Ph.D. degrees from Japan Advanced Institute of Science and Technology in 1996 and 2000. From 2000, he was a research associate at Tohoku University. He transferred to Nagoya University's Graduate School of Information Science in 2003 as an assistant professor and became an associate professor in 2006. His research interests include term rewriting systems, program theory, and automated theorem proving. He is a member of IEICE, IPSJ and JSSST.

Toshiki Sakabe was born in 1949. He received his B.E., M.E. and D.E. degrees from Nagoya University in 1972, 1974 and 1978, respectively. He was a research associate at Nagoya University during 1977–1985, and an associate professor at Mie University and Nagoya University during 1985–1987 and 1987–1993, respectively. He has been a professor of Nagoya University since 1993. His research interests are in the field of theoretical foundations of software including algebraic specifications, rewriting computation, program verification, model checking and so on. He is a member of IPSJ, IEICE, JSAI and JSSST.