Vol.2011-MPS-84 No.11
2011/7/18

# Construction of a Mathematical Model and Quantitative Assessments of Impression in Western Painting

URANO Sachi[†]

We present a new method to evaluate the impression of paintings, in terms of a mathematical modeling and quantitative assessments of impression in western painting. The present method is based on a detailed modeling of various factors and elements that consist of the composition of the painting. Impressions of 20 subject-people for 50 paintings are measured using the semantic differential method, which are compared to the impressions calculated with the present method. Their correlation is analyzed with Akaike's Information Criterion (AIC) standard. A strong correlation is obtained.

## 1. Introduction

A number of fascinating paintings, which gives us a deep sympathy have remained up to now. These works are not only superior in their painting materials and techniques but also in the construction of the motif and the arrangement of the layout. There is an approach to understand the layout of objects in the painting by categorizing the whole screen to several common parts, which is called the composition.

Because the composition strongly affects the impression of the painting, it has been studied by many researchers to find a good composition or the general rule for the good painting, which gives a specific impression. In many previous studies and textbooks, it has been a common method to draw a parting line on the painting to analyze the composition. However, there are few works that succeeded to evaluate the relation between the impression and the composition, explicitly. Ozawa[1)] analyzed the composition of *ukiyoe* paintings based on three-dimensional space geometry. Unfortunately it was difficult to expand the method to the western paintings. The research groups in Visual Perception and Aesthetics Lab[2)] and Christopher Tyler Lab[3)] tried to figure out the general rules for the composition in terms of an arrangement of the subject, using many photographs and paintings. While their approach was straightforward, the relation among the *composition elements* and their quantitative definition (*composition factors)* derived by their relation are not clearly expressed, since they focused on the intuitive and subjective appreciations of the viewers.

In this paper, we present a new mathematical approach to evaluate the relation between the composition and the impression, which is based on the analysis of a three-dimensional space geometry.

## 2. Description of the method

### 2.1 Terminology

The composition is classified into two major categories, which are the shape and the relation among the objects (*fundamental form*), and the size of the objects and their layout (*arrangement form*). We discuss the *arrangement form*, which can be analyzed mathematically. The following ten *composition factors* are prepared:

1) $\mu$ (*balance*), area fraction of elements on the left side of the screen after splitting the screen vertically in the middle.
2) $\varphi$ (*density*), area fraction of all the elements that are in the screen.
3) J (*jump ratio)*, area fraction of the smallest element to the biggest one inside the slit (magnification ratio).
4) X (*information value*), number of all elements.
5) $\sigma_s$ (*slit ratio*), area fraction of the smallest rectangle that includes all the elements in a picture (*slit*) to that of the whole paintings.
6) $p_+$ (*upper vertical proportion*), range between the horizon and the upper limit of the picture.
7) $p_-$ (*lower vertical proportion*), range between the horizon and the lower limit of the slit.
8) $\delta$ (*density parameter*), homothetic area of elements not to occlude each other.
9) $\lambda$ (*similarity ratio*), homothetic ratio of the fronting element to the backend element.
10) $\sigma_v$ (*printing domain vertical ratio*), slit height.

Objects in the painting are described with $e_i\,\{i = 1, 2, \ldots, n\}$, where $e_1$ and $e_n$ stand for the fronting and the backend elements, respectively.

[†]School of Informatics, University of Tsukuba, Tsukuba, Ibaraki, Japan

### 2.2 Mathematical modeling and definition

#### 2.2.1 Domains

We divided the screen into four *domains* to evaluate the *composition factors* and the object arrangement (Figure 1):

| | |
|---|---|
| $D_+ = [-1,1] \times [p_- + 2\sigma_v(1 + 2p_+), 2p_+]$ | : upper domain |
| $S = [-1,1] \times [-p_-, p_- + 2\sigma_v(1 + 2p_+)]$ | : printing slit |
| $D \subset S$ | : printing domain |
| $D_- = [-1,1] \times [-1, -p_-]$ | : lower domain |
| $D_+ \cup S \cup D_- = [-1,1] \times [-1, 2p_+]$ | : image plane. |

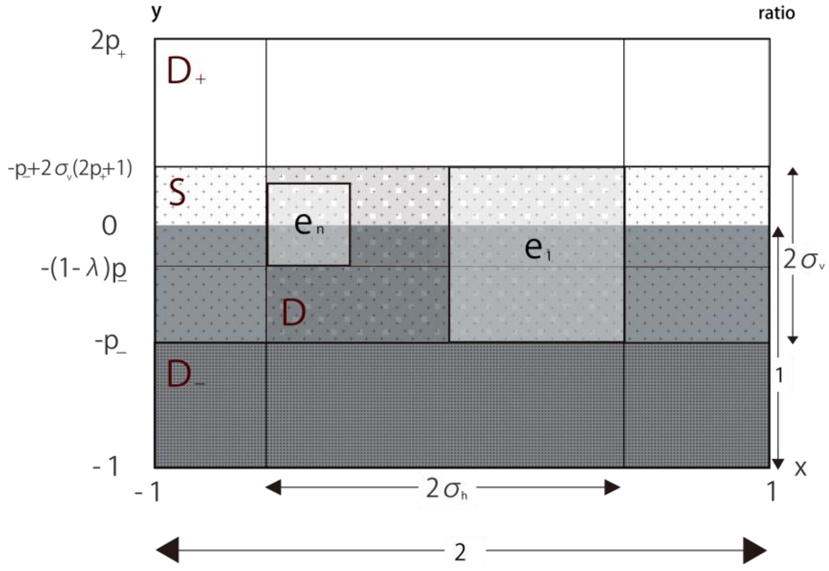


Figure 1: *Domains and parameters (* $D_+, S, D$ *and* $D_-$ *stand for upper domain, printing slit, printing domain and lower domain, respectively).* $\{p_+, p_-, \sigma_v, \lambda\} \in [0,1]$.

#### 2.2.2 Object size and composition factors

A viewer maps objects in the three-dimensional space into the two-dimensional image plane through a screen settled between the objects and the viewer. It is done by fixing the observing point, which was followed by changing the visual points, several times.

Geometry models of the three different visual points towards a group of objects $\{O_i: i = 1, \dots, n\}$ that are lined up over the horizon in a similar space interval are shown in Figure 2 (objects are trees, in this example). We see how the objects are projected over the screen. We assume the same heights for the closest and the farthest trees, $O_1$ and $O_n$ respectively, for the simplicity of the discussion. The heights of the visual points in the screen are set to: (a) the top of the trees, (b) between the top of the tree and the horizon and (c) the horizon, respectively. The image of the closest and the farthest trees $e_1$ and $e_n$ aligns on (a) the top, (b) middle and (c) root of the trees, respectively, in the two dimensional plane. It is understood that the size and layout of the images $\{e_n\}$ in two-dimensional picture plane are the function of the relative position of the horizon and the slit; range between the horizon and the upper limit of the picture ($p_+$), the lower limit of the slit ($p_-$) and the slit height ($\sigma_v$), respectively, when we observe the objects from the fixed visual point.

The area ratio of $e_1$ to $e_n$ is $\lambda^2$, where $\lambda$ is a homothetic ratio of $e_1$ to $e_n$. The distance between $O_1$ and $O_n$ is $1 - \lambda^{-1}$ when we assume the distance between the viewer and $O_1$ of 1. The *composition elements* are expressed as follows,

| | |
|---|---|
| $\{e_i \subset D: i = 1, 2, \dots, n\}$ | : texture elements |
| $e_1 \subset D$ | : fronting element |
| $e_n \subset D$ | : backend element, |

using the following *parameters*

| | |
|---|---|
| $\rho \in [0,1]$ | : principal and subsidiary element ratio |
| $\rho_p \in [0,1]$ | : principal elemental ratio |
| $\rho_s \in [0,1]$ | : subsidiary elemental ratio |
| $\delta \in [0,1]$ | : density parameter |
| $\varepsilon$ | : quantization unit of $e_n$. |

When we introduce the *secondary parameters*,

| | |
|---|---|
| $\sigma_h \in [0,1]$ | : printing domain horizontal ratio, |
| $n \in [1, 2, \dots, N]$ | : information number, |

we obtain,

$\sigma_s = \frac{2\sigma_h\sigma_v}{1+2p_+} \quad \in [0,1]$ : slit ratio,

$J = (1-\lambda)^2 \quad \in [0,1]$ : jump ratio (reciprocal),

$\chi = \frac{n}{N} \quad \in [0,1]$ : information value.

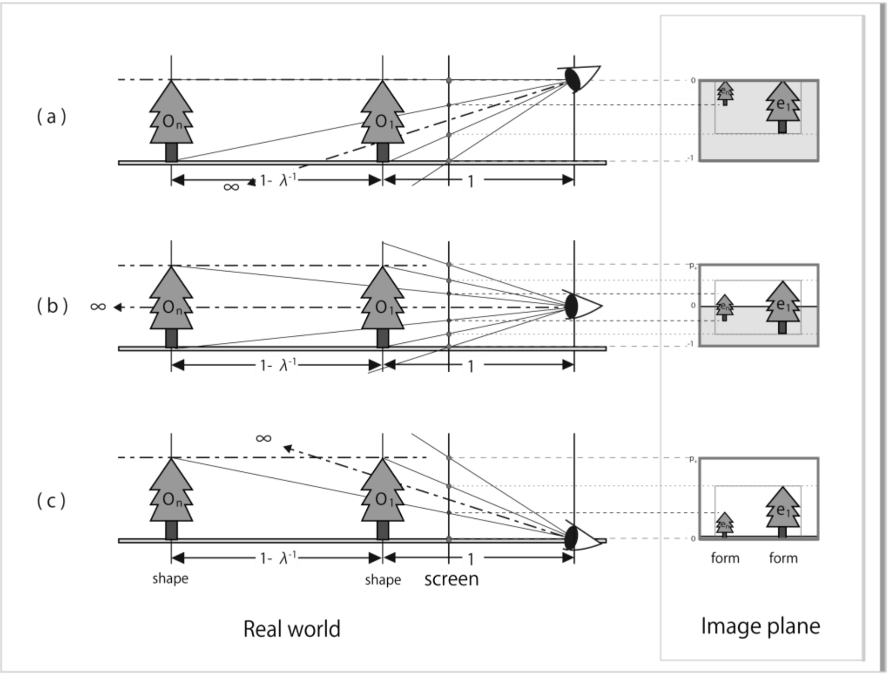


Figure 2: *Geometry models of the three different visual points towards a group of trees (a: top of the trees, b: between the top of the trees and the horizon, c: the horizon).*

### 2.2.3 Element layout rule

We consider a digitized picture or painting. The smallest element size $\varepsilon$ is $\min(\frac{2}{H}, \frac{1+2p_+}{V})$, when the screen has $H \times V$ pixels. Assuming the heights of the fronting and backend elements of $d$ and $(1-\lambda)d$ respectively, we have the following equations:

(Case 1) $p_+ \geq \frac{\varepsilon}{4\sigma_v(1-\lambda)} - \frac{1}{2}$, in case of $p_- \leq 2\sigma_v(1+2p_+)$, *(1)*

(Case 2) $p_+ \geq \frac{\varepsilon+\lambda p_-}{4\sigma_v} - \frac{1}{2}$, in case of $p_- > 2\sigma_v(1+2p_+)$. *(2)*

When we choose $p_+$ so as to satisfy the above mentioned conditions, the height $d$ is uniquely expressed with the four fundamental parameters $\{p_+, p_-, \sigma_v, \lambda\}$:

(Case 1) $d = 2\sigma_v(1+2p_+)$, *(3)*

(Case 2) $d = \frac{2\sigma_v(1+2p_+)+\lambda p_-}{(1-\lambda)}$. *(4)*

Figure 3 shows the two types of arrangement models. In *case 1*, the fronting element $e_1$ is the largest in the picture, since it is inscribed inside $S$, bordering upper and lower parts. In *case 2*, $e_1$ is not the largest element, since $e_1$ and $e_n$ border $S$ on the lower and upper sides, respectively, yet both are inscribed in $S$.

The height of the elements $\{e_i : i = 2, \ldots, n-1\}$ becomes $-\frac{d}{p_-}y$, when we assume the base coordinates of $y \in [-p_-, -(1-\lambda)p_-]$, because of the mutual similarity.

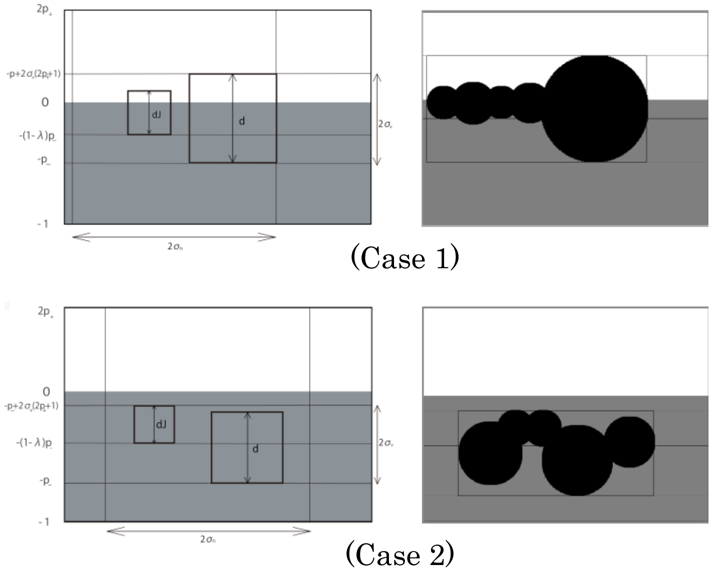


Figure 3: *Arrangement of the elements (a : Case 1, b : Case 2).*

We assume $\{e_1, e_n\}$ of the principal elements, and the proportion of the height and width of $1{:}\,4\rho_p$. The height ratio of the principal and subsidiary elements is $1{:}\,\rho$, so that the aspect ratio becomes $\rho{:}\,\rho\rho_s$. The elements $\{e_i{:}\, i = 2, \ldots, n-1\}$ can be principal or subsidiary (see Figure 4 for details).

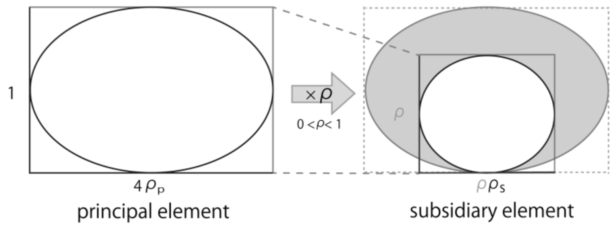


Figure 4: *Principal and subsidiary elements.*

All the elements that belong to the *printing slit* S must satisfy $e_i \subset S$. $\{e_1, e_n\}$ are arranged according to the *condition 1* explained below, to avoid occlusion $e_n \subset e_1$. Assuming the center of the element being fixed, we recursively arrange $e_i$ in the paintings using the image of $e_i$, $U_\delta(e_i)$, whose homothetic ratio is $1{:}\,\delta$ in order to fulfill both conditions below (see Figure 5 for details).

(Condition 1) $U_\delta(e_i) \cap (\cup_{j=1}^{i-1} e_j \cup e_n) = \emptyset$ for $\delta \in [0.5, 1.0]$, *(5)*
(Condition 2) $e_i \cap (\cup_{j=1}^{i-1} e_j \cup e_n) \neq \emptyset$ for $\delta \in [0, 0.5]$. *(6)*

The images can be reduced or magnified depending on the size of $\delta$:

reduced : $e_i \subset U_\delta(e_i)$ for $\delta \in [0.5, 1.0]$, *(7)*
magnified : $U_\delta(e_i) \subset e_i$ for $\delta \in [0, 0.5]$. *(8)*

If all the elements cannot be arranged, the number of the elements will be reduced to $i+1$, where $i(<n)$ represents the number of the elements, which completed the arrangement.

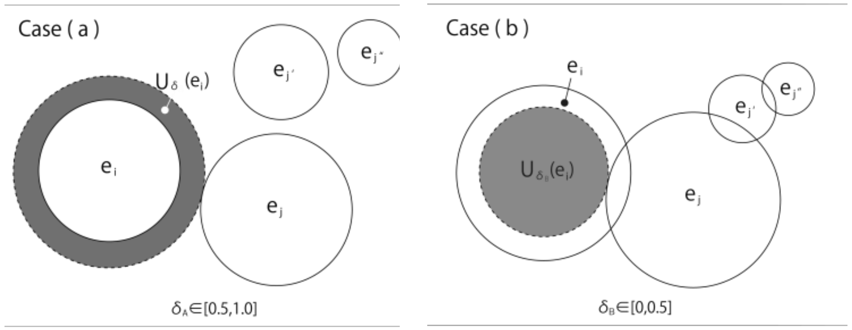


Figure 5: *Elements layout rules(a: $e_i \subset U_\delta(e_i)$, b: $U_\delta(e_i) \subset e_i$ ).*

## 3. Quantitative evaluation of impression

### 3.1 Sample preparation

An experimental confirmation about the present method has been carried out with an experiment. It is important to generate a geometric model that the objects in the paintings can be distinguished from the background, satisfying the conditions mentioned above, and that the picture is visually perspective with the horizon contained in it. We chose 50 western landscape paintings with people from the art books[4),5),6)] that fulfill all the conditions. 44% paintings were from between 15$^{th}$ and 20$^{th}$ centuries, and 26% paintings were in 18$^{th}$ and 19$^{th}$, which were categorized to the Romanticism, Neoclassicism, Realism and Impressionism. The paintings were processed into monochrome prior to the experiment. The maximum information number N was limited to ten.


Figure 6: *Example of Principal and subsidiary elements*.

In Figure 6, four people are pictured (N = 4). In this case, the objects $\{D, B\}$ or $\{e_1, e_4\}$ are the principal (fronting and the backend) elements. $\{e_2, e_3\}$ are the subsidiary elements with reduction ratios of $\rho = 0.6, \rho_s = 0.05, \rho_s = 0.38$, respectively. After arranging all the elements on the screen, we calculated the balance and the density, $\{\mu, \varphi\} \in [0,1]$[7)].

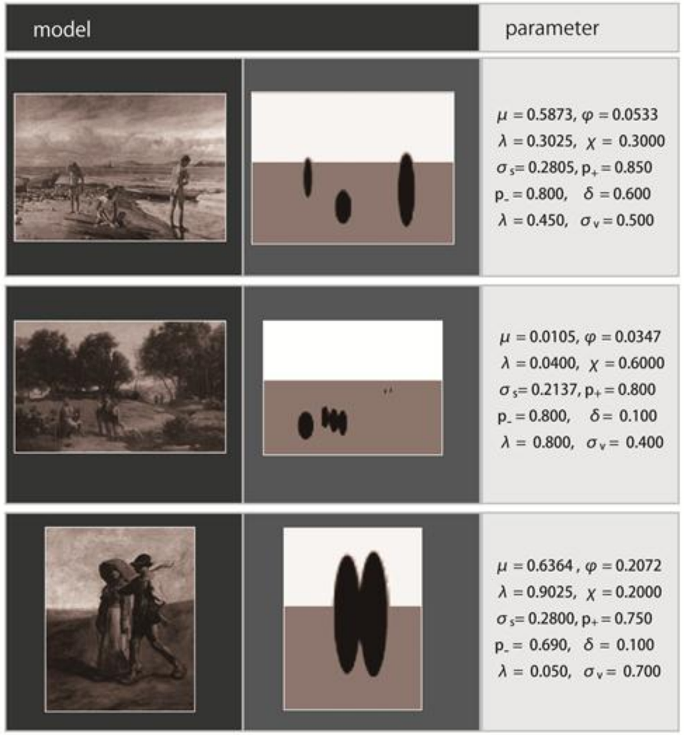


Figure 7: *Examples of the paintings and the analytical results with the characteristic quantities.*

### 3.2 Experimental

We evaluated the impressions of the paintings using the methods introduced by Ooyama[8)]. The adjectives were categorized into three factors; evaluation factor, activity factor and potency factor, respectively. A rating scale group {i=1,2,…,10}, which consisted of ten pairs of adjectives of preferably independent meanings dispersed each of the factors was prepared to measure the impression. We showed 50 different paintings {j=1,2,…,50} to the 20 people. The subject-people assessed their impressions using a scale of seven degrees (Semantic Differential Method).

We, then, calculated the mean impression $Y_{ij} \in [0,1]$ for each panting. The impression according to the *arrangement form* $Im_{ij\_e}$ was obtained,

$$\mathrm{Im}_{ij\_e} = \mathrm{Y}_{ij} - \alpha_{ij}, \qquad (9)$$

where, $|\alpha_{ij}| \in [0, 1]$ is the correction factor for the experiment uncertainty due to the impressions caused by the variously-shaped objects, eye direction and some patterns.

The impression calculated by the present method, $\mathrm{Im}_{ij\_c}$, can also be expressed using ten *composition factors*,

$$\mathrm{Im}_{ij\_c} = a_i^1\mu_{ij} + a_i^2\varphi_{ij} + a_i^3 J_{ij} + a_i^4\chi_{ij} + a_i^5\sigma_{s_{ij}} + a_i^6 p_{+_{ij}} + a_i^7 p_{-_{ij}} + a_i^8\delta_{ij} + a_i^9\lambda_{ij} + a_i^{10}\sigma_{v_{ij}}, \qquad (10)$$

where $\{|a_i^k| : k = 1,2,\dots,10\} \in [0,1]$ are the free coefficients. We calculated the AIC-value, ( $\mathrm{AIC} = -2\log(maximum\ likelihood\ function) - 2(Arity)$ ) to evluate the degree of correlation between the experimentally-obtained impression ($\mathrm{Im}_{ij\_e}$) and the calculated one by the composition factors ($\mathrm{Im}_{ij\_c}$). The multiple classification analysis was applied to evaluate the correlation. Table 1 summarizes the adjectives and the impression, $\mathrm{Im}_{ij}$, which gave the minimum AIC-values. According to the AIC standard, the smallest AIC-value represents the strongest correlation.

### 3.3 Discussion

$\mathrm{Im}_{ij}$ shows the impression in numerical form. Multiple parameters were required to evaluate the impression except for adjectives No.5 and No.10. Here, we can observe a correlation between the impression obtained by the 20 subject-people and that evaluated with the present method. We have found that people feel the painting noisy, when the impression was expressed with *information value* $\chi_{4j}$ and *printing domain vertical ratio* $\sigma_{v_{4j}}$. It has also been pointed out by the previous study carried out by the ref.[7].

The definition of the *composition factors* affect to the evaluation results. For example, if we define $\kappa = \{2\mu \text{ for } \mu \in [0, 0.5],\ -2\mu + 2 \text{ for } \mu \in [0.5, 1.0]\}$, and analyze

$$\mathrm{Im}'_{ij} = a_i^1\kappa_{ij} + a_i^2 J_{ij} + a_i^3\chi_{ij} + a_i^4\sigma_{s_{ij}} + a_i^5 p_{+_{ij}} + a_i^6 p_{-_{ij}} + a_i^7\sigma_{v_{ij}}, \qquad (11)$$

the potency factors will change (Table 2), and the AIC-values are even smaller.

We may have to introduce other *composition factors*, or appropriate *factors* other than the composition to improve the strength of the correlation, so that it well represents the impression of the people. The selection of the adjectives also affects the results. We will survey various adjectives that show strong correlation to the composition.

### 3.4 Conclusions

In summary, we were able to explicitly evaluate the impression of the people for the 50 different paintings, with the present method, which was based on the detailed evaluation of the *composition factors* of the paintings.

The results obtained with the present study supported that the impression can be quantitatively assessed. We will improveme the present method by introducing appropriate *composition factors* and other ones to find the strong correlation to the impression. The present method can be applied to the evaluation of the impression not only for the paintings, but also for the photographs. It may also be applied to the other scientific fields, such as Kansei Engineering.

**Acknowledgements** The author thank Prof. K. Sakai from the University of Tsukuba for his valuable comments. This work is supported in part by MEXT "Support project for students majoring in the field of math and science (University of Tsukuba)".

## References

1) Ozawa Kazumasa: Towards a Model of Painting – A Consideration, Information Processing Society of Japan, (1996).
2) Jonathan S. Gardner et al.: Exploring Aesthetic Principles of Spatial Composition Through Stock Photography, VSS Poster, (2008).
3) CHRISTOPHER W. TYLER: Some principles of spatial organization in art, Spatial Vision Vol.20 No.6, pp.509-530, (2007).
4) John Oliver Hand: National Gallery of Art Master Paintings from the Collection, ABRAMS, (2004).
5) Vincent Pomerede: 1001 Paintings at the Louvre, MUSEE DU LOUVRE EDITIONS, (2008).
6) Stephan Farthing: 1001 Paintings You must See Before You Die, UNIVERSE, (2007).
7) VISUAL DESIGN LABORATORY INC.: Basics of Composition(Japanese), VISUAL D.
8) Ooyama Tadasu et al.: Chapter 4, Means for Psychological Investigating(Japanese), SAIENSU-SHA, pp.65-75, (2005).