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Resultant-factorization Technique for Obtaining Solutions to Ordinary Differential Equations

Kinji Kimura †1 and Hiroshi Yoshida †2

We propose a technique for obtaining solutions to ordinary differential equations. A system of differential equations sometimes has multiple solutions with distinct features. Prime ideal decomposition can be used for extracting the desired solution from these solutions. Solutions to algebraic equations contain many parameters, and in such a case, prime ideal decomposition is less tractable. As an alternative, we propose a *resultant-factorization technique* for extracting the desired solution. We also demonstrate the implementation of this technique and show its timing data.

1. Introduction

In ordinary differential equations, we often need to analyze complicated polynomials with multiple irreducible affine varieties (solutions). It is therefore essential to perform prime ideal decomposition. To understand the importance of the decomposition, consider the following set of polynomials: $\{x^3 - x^2y - xy - 2x + y^2 + 2y, xy - x - y^2 + y\}$. Prime ideal decomposition shows that the above polynomials can be decomposed into two irreducible affine varieties, $\{x-y\}$ and $\{x^2-3, y-1\}$. While one cannot determine concrete values of x and y using the former, one can determine them, $x = \pm\sqrt{3}$ and $y = 1$, using the latter. Hence, to decide whether or not we can determine the values of variables, we must perform prime ideal decomposition of polynomials. In ordinary differential equations, however, there are many parameters and variables that describe observed data and systems, respectively. In general, large numbers of parameters and variables render prime ideal decomposition more difficult[4]. Especially, the method in[4] needs the computation of Gröbner basis in the beginning. Under large numbers of parameters and variables render, the computational cost of Gröbner basis increases. This leads us to propose a *resultant-factorization* technique[1] that provides the desired irreducible affine variety (solution). By using the resultant-factorization technique, we can reduce the computational cost of Gröbner basis.

†1 Kyoto University
†2 Kyushu University

2. Resultant-factorization technique

On the basis of[6], we propose an efficient technique to arrive at the targeted affine variety (solution); the technique is shown in Fig. 1(a).

Let $BP = \{BP_i | 1 \leq i \leq n\}$ be an original set of polynomials. Let F_1 be a set of $n-1$ resultants of polynomials BP_j $(1 \leq j \leq n, i \neq j)$ for some $BP_i (1 \leq i \leq n)$ in a variable, say, x_1. It is reasonable to select a variable x_1 such that BP_i and BP_j are low-degree polynomials in x_1. For instance, if one can factorize some element f in F_1 into mutually disjoint elements f_1, f_2, and f_3, $\langle F_1 \rangle$ can be decomposed into $\langle F_1, f_1 \rangle \cap \langle F_1, f_2 \rangle \cap \langle F_1, f_3 \rangle$. As a result of the factorization, the resultant F_{21} in x_2 can usually be described by a smaller set of polynomials. Furthermore, when one obtains a factorized term like $(y_1 + y_2^2)(z - y_3^2 + 3)^3$ with the rate constants y_1 and y_2, it is sufficient to consider only $(z - y_3^2 + 3)$ because the rate constants are guaranteed to be positive $(y_1 > 0 \wedge y_2 > 0 \Rightarrow y_1 + y_2^2 > 0)$ and the radical $\sqrt{\langle BP \rangle}$ suffices. We can also ignore a factor like $(y_1 - y_2)$ when we assume $y_1 \neq y_2$ because of unacceptable factors or mathematically trival factors. These simplifications allow us to prune the branches of the resultant-factorization series[⋆1]. When we arrive at an appropriate ideal ($\langle F_{32} \rangle$ in Fig. 1(a)), we can efficiently determine all the rate constants by using the ideal $\langle BP \rangle + \langle F_{32} \rangle$, as illustrated in Fig. 1(a).

3. Implementation

We have implemented our "Resultant-factorization technique," which is shown as "ScodeRFP.rr" on http://sites.google.com/site/codes86/. The main routine of "ScodeRFP.rr" is "automatic_decom," which calls the following seven procedures:

(1) Procedure 1: shortcut_speed(ideal I, list of polynomials lp) returns an ideal

⋆1 Recently, such a pruning procedure in algebraic approaches has been studied, e.g., "positive quantifier elimination"(QE)[5].

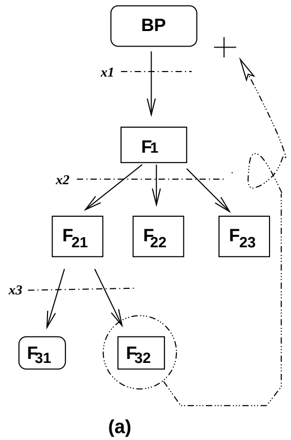

Fig. 1 Schematic illustration of the resultant-factorization technique

obtained by removing the factors in lp from ideal I. This procedure aims to remove unacceptable factors or mathematically trival factors.

(2) Procedure 2: idealclean(ideal I) returns an ideal obtained by removing redundant elements from ideal I.

(3) Procedure 3: coefficientcleaner(ideal I, poly p) returns an ideal obtained by removing common factors and p from ideal I.

(4) Procedure 4: constant_check(ideal I) checks whether or not ideal I has an element composed of only the parameters. If there is such an element , the function returns 0 to indicate the presence of an error in the top-level function.

(5) Procedure 5: idealfactorize returns a list of elements that can be factorized over $\mathbb{Q}$ in a given ideal; if there are no such elements in the ideal, then it returns the given ideal. This is based on the relation $\sqrt{< I, f * g >} = \sqrt{< I, f >} \cap \sqrt{< I, g >}$.

(6) Procedure 6: variable_choice returns a variable that is to be removed in the next procedure. There are two types of outputs. Let lp be a given list of polynomials.

(a) In lp, when there is a variable contained in only one polynomial, "variable_choice" returns the variable. In this case, it is not necessary to calculate resultants in the next procedure because the resultant of polynomials p and q in x is q^r, where q does not contain the variable x and r is the degree of x in p.

(b) Otherwise, we select a variable as follows:

(i) We calculate d_i– the maximum degree of variable $x_i (1 \leq i \leq n)$ by considering lp. If a single d_j is the minimum among $d_i (1 \leq i \leq n)$, x_j is returned.

(ii) If multiple d_i's have the same minimum value, let $y_1, y_2, \ldots, y_m$ be variables that provide this minimum. We calculate n_i, the number of polynomials that contain y_i. If a single n_j provides the minimum among $n_i (1 \leq i \leq m)$, y_j is returned.

(iii) If multiple n_i's provide the same minimum, let $z_1, z_2, \ldots, z_k$ be the variables that provide this minimum. We calculate t_i which means the number of terms in the polynomials that contain z_i. Variable z_j that provides the minimum and that is calculated first is returned.

In (i)-(iii), as an accompanying output, we return the polynomial that contains the returned variable and has the minimum number of terms.

(7) Procedure 7: idealresultant returns a set of resultants on the basis of the output of Procedure 6.

We perform Procedures 1,2,. . . ,7 for a given input ideal. Procedure 5 gives rise to branches of Procedures where the main routine is recursively called. Finally, the main routine returns a list of polynomials together with the input original ideal. Prime ideal decomposition can be rapidly performed for each of the polynomials in the list.

4. Model

In this section, we introduce the Painlevé VI equation[2)]

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \times \left(\frac{\alpha_1^2}{2} - \frac{\alpha_4^2}{2}\frac{t}{y^2} + \frac{\alpha_3^2}{2}\frac{(t-1)}{(y-1)^2} - \frac{\alpha_0^2 - 1}{2}\frac{t(t-1)}{(y-t)^2}\right), \quad (1)$$

which is an ordinary differential equation. Painlevé VI equation (1) has four parameters:$\alpha_0, \alpha_1, \alpha_3$, and α_4. We discuss typical solutions, which are called "rational solutions." All rational solutions for the equation have been obtained in[3]. We can use algebraic geometry for studying the equation (1). However, in general, we cannot use these techniques for all ordinary differential equations. Thus, we treat equation (1) in the condition that we do not know the mathematical structure of the equation (1). We assume rational solutions of the form $y(t) = (k_0 + k_1 t + k_2 t^2)/(l_0 + t)$ and substitute these solutions in the equation. We then obtain algebraic equations for the eight variables $k_0, k_1, k_2, l_0, \alpha_0, \alpha_1, \alpha_3$, and α_4. By using the resultant-factorization technique proposed in the previous section 2, we can determine the solutions.

We assume rational solutions of the form $y(t) = (k_0 + k_1 t + k_2 t^2)/(l_0 + t)$ and substitute these solutions in the equation (1). Then, we get the following ideal,

$$\begin{aligned} P = \{ & -k_0^4 \times (\alpha_1 k_0 + (-\alpha_1 - \alpha_3) l_0) \times (\alpha_1 k_0 + (-\alpha_1 + \alpha_3) l_0), \ldots, \\ & -k_2^4 \times (\alpha_1 k_2 - \alpha_1 + \alpha_0) \times (\alpha_1 k_2 - \alpha_1 - \alpha_0)\} = \{p_1, p_2, \ldots, p_{13}\} = 0. \end{aligned} \quad (2)$$

For convenience, we define five equations: $f_1 = \alpha_1 k_0 + (-\alpha_1 + \alpha_3) l_0$, $f_2 = \alpha_1 k_0 + (-\alpha_1 - \alpha_3) l_0$, $g_1 = \alpha_1 k_2 - \alpha_1 + \alpha_0$, $g_2 = \alpha_1 k_2 - \alpha_1 - \alpha_0$, and $h = k_0 - l_0 k_1 + l_0^2 k_2$. Hence, we can divide the original problem($P = 0$) to the six cases,

$$P_1 = \{f_1, p_2, \cdots, p_{12}, g_1\} = 0, P_2 = \{f_1, p_2, \cdots, p_{12}, g_2\} = 0,$$
$$P_3 = \{f_2, p_2, \cdots, p_{12}, g_1\} = 0, P_4 = \{f_2, p_2, \cdots, p_{12}, g_2\} = 0,$$
$$P_5 = \{k_0, p_2, \cdots, p_{12}, p_{13}\} = 0, P_6 = \{p_1, p_2, \cdots, p_{12}, k_2\} = 0.$$

We explain the resultant-factorization technique for the case of $P_1 = 0$ in detail. Thus, we consider the ideal, $< f_1, p_2, \cdots, p_{12}, g_1 >$.

We compute solutions in the case of $f_1 = 0$ and $g_1 = 0$ as follows,

(1) Step 1: we compute,

$$Q_1^{(0)} = \text{resultant}_{\alpha_0}(g_1, f_1), Q_2^{(0)} = \text{resultant}_{\alpha_0}(g_1, p_2), \ldots$$
$$Q_{12}^{(0)} = \text{resultant}_{\alpha_0}(g_1, p_{12}).$$

We factorize $Q_{12}^{(0)}$ and set $Q_1^{(1)}, \ldots, Q_{12}^{(1)}$,

$$Q_1^{(1)} = Q_1^{(0)}, \ldots, Q_{11}^{(1)} = Q_{11}^{(0)}, Q_{12}^{(0)} = (k_2 - 1) Q_{12}^{(1)}, Q_{12}^{(1)} = \frac{Q_{12}^{(0)}}{k_2 - 1}.$$

(2) Step 2: we compute,

$$R_2^{(0)} = \text{resultant}_{\alpha_3}(Q_1^{(1)}, Q_2^{(1)}), \ldots, R_{12}^{(0)} = \text{resultant}_{\alpha_3}(Q_1^{(1)}, Q_{12}^{(1)}).$$

We factorize $R_2^{(0)}$ and set $R_2^{(1)}, \ldots, R_{12}^{(1)}$,

$$R_2^{(0)} = l_0(k_0 - l_0) R_2^{(1)}, R_2^{(1)} = \frac{R_2^{(0)}}{l_0(k_0 - l_0)}, R_3^{(1)} = R_3^{(0)}, \ldots, R_{12}^{(1)} = R_{12}^{(0)}.$$

(3) Step 3: we compute,

$$S_2^{(0)} = \text{resultant}_{\alpha_4}(R_{12}^{(1)}, R_2^{(1)}), \ldots, S_{11}^{(0)} = \text{resultant}_{\alpha_4}(R_{12}^{(1)}, R_{11}^{(1)}).$$

We factorize $S_2^{(0)}, \cdots, S_{11}^{(0)}$ and set $S_2^{(1)}, \ldots, S_{11}^{(1)}$,

$$S_2^{(0)} = 4 l_0^4 \left(S_2^{(1)}\right)^2, S_2^{(1)} = \sqrt{\frac{S_2^{(0)}}{4 l_0^4}}, S_3^{(0)} = l_0^4 \left(S_3^{(1)}\right)^2, S_2^{(1)} = \sqrt{\frac{S_3^{(0)}}{l_0^4}}$$
$$\cdots$$
$$S_{10}^{(0)} = l_0^4 \left(S_{10}^{(1)}\right)^2, S_2^{(1)} = \sqrt{\frac{S_{10}^{(0)}}{l_0^4}}, S_{11}^{(0)} = l_0^4 (k_2 - 1)^2 \left(S_{11}^{(1)}\right)^2,$$
$$S_2^{(1)} = \sqrt{\frac{S_{11}^{(0)}}{l_0^4 (k_2 - 1)^2}}$$

(4) Step 4: we compute,

$$T_3^{(0)} = \text{resultant}_{\alpha_1}(S_2^{(1)}, S_3^{(1)}), \ldots, T_{11}^{(0)} = \text{resultant}_{\alpha_1}(S_2^{(1)}, S_{11}^{(1)}).$$

We factorize $T_2^{(0)}, \cdots, T_{11}^{(0)}$ and set $T_3^{(1)}, \ldots, T_{11}^{(1)}$,

$$T_3^{(0)} = l_0^6 (k_2 - 1)^2 (k_0 - l_0)^2 h^2 \left(T_3^{(1)}\right)^2, T_3^{(1)} = \sqrt{\frac{T_3^{(0)}}{l_0^6 (k_2 - 1)^2 (k_0 - l_0)^2 h^2}}$$
$$T_4^{(0)} = l_0^4 (k_2 - 1)^2 h^2 \left(T_4^{(1)}\right)^2, T_4^{(1)} = \sqrt{\frac{T_4^{(0)}}{l_0^4 (k_2 - 1)^2 h^2}}, \cdots,$$
$$T_{11}^{(0)} = l_0^4 (k_2 - 1)^2 h^2 \left(T_{11}^{(1)}\right)^2, T_{11}^{(1)} = \sqrt{\frac{T_{11}^{(0)}}{l_0^4 (k_2 - 1)^2 h^2}}.$$

(5) Step 5: from the process of the above-mentioned computation, we can assume the following conditions, $k_2 \neq 1, l_0 \neq 0, k_0 \neq l_0, h \neq 0, k_0 \neq 0$.

(6) Step 6: we compute the Gröbner basis(G_0)[4)] of the ideal, $< T_3^{(1)}, \cdots, T_{11}^{(1)} >$.
(7) Step 7: by using saturation technique[4)], we remove the component $k_2 - 1$ from the ideal(G_0), $G_1 =< G_0, 1 - u(k_2 - 1) >$.
(8) Step 8: by using saturation technique, we remove the component l_0 from the ideal(G_1), $G_2 =< G_1, 1 - u(l_0) >$.
(9) Step 9: by using saturation technique, we remove the component $k_0 - l_0$ from the ideal(G_2), $G_3 =< G_2, 1 - u(k_0 - l_0) >$.
(10) Step 10: by using saturation technique, we remove the component h from the ideal(G_3), $G_4 =< G_3, 1 - u(h) >$.
(11) Step 11: by using saturation technique, we remove the component k_0 from the ideal(G_3), $G_5 =< G_4, 1 - u(k_0) >$.
(12) Step 12: we can get the ideal(G_6) by using computation of the following ideal, $G_6 =< G_5, P_1 >$.
(13) Step 13: we get solutions(D_1) from G_6 by using prime ideal decomposition.

5. Timing data

If we do not use the resultant-factorization technique, we cannot obtain all the solutions for the ideal (2) within 48 h. However, we can compute solutions with Table 1 by using the resultant-factorization technique. Since the solutions of the equation (1) has a large number of irreducible affine varieties, the resultant-factorization technique functions effectively. Table 1 shows the computing times for the different cases. All the algorithms are implemented on Risa/Asir [⋆1] and measurements are performed on a PC with Intel Core i7 920 and 12 GB of main memory.

6. Solutions

We show $D_1, D_5, D_6, D_7, D_8, D_9$ and D_{10}.

⋆1 http://www.math.kobe-u.ac.jp/Asir/asir.html

Table 1 computing time for different cases

Cases	Time	Solutions
$f_1 = 0$ & $g_1 = 0$	4m34s	D_1
$f_1 = 0$ & $g_2 = 0$	4m46s	D_2
$f_2 = 0$ & $g_1 = 0$	4m35s	D_3
$f_2 = 0$ & $g_2 = 0$	4m41s	D_4

Cases	Time	Solutions
$k_0 = 0$	351m56s	D_5
$l_0 = 0$	45s	D_6
$k_2 = 0$	146m14s	D_7

Cases	Time	Solutions
$k_2 = 1$	7s	D_8
$k_0 = l_0$	78m57s	D_9
$h = 0$	3s	D_{10}

$$
\begin{aligned}
D_1 = \{&\{\alpha_1 + 2, k_1 + 2k_2 + \alpha_4, -k_1 + \alpha_3 + 1, 2k_2 - \alpha_0 - 2, -k_1 + 2l_0k_2 - l_0,\\
&-2k_0 + l_0k_1 + l_0, (-4k_2 + 2)k_0 + k_1^2 + k_1\}, \cdots\\
&\{k_1 + 2k_2 - 2, k_0 - k_2 - l_0 + 1, \alpha_4 + 2, \alpha_1 - \alpha_3 - \alpha_0 + 1,\\
&\alpha_3l_0 - \alpha_1 + \alpha_3 - 1, -\alpha_1k_2 + \alpha_3 - 1\}, \cdots\} = 0\\
D_5 = \{&\{l_0, k_2, -\alpha_3 + \alpha_4 + \alpha_1, \alpha_0 + 1, -\alpha_4 - k1\alpha_1\}, \cdots\\
&\{l_0, k1 + k_2 - 1, \alpha_4 + 1, \alpha_0 + \alpha_3 + \alpha_1, \alpha_3 + k_2\alpha_1\}, \cdots\\
&\{l_0, \alpha_1 + 1, \alpha_4 + k1 + k_2, \alpha_3 + k1 - 1, \alpha_0 + k_2 - 1\}, \cdots\\
&\{l_0, k_2, k1 - 1, \alpha_3\}, \{k_2, k1 - l_0 - 1, \alpha_1 + 1, -\alpha_0 + \alpha_3 + \alpha_4,\\
&(l_0 + 1)\alpha_3 + l_0\alpha_4\}, \cdots\\
&\{k_2, \alpha_4 + 1, \alpha_0 + \alpha_3 + \alpha_1 + 2, (l_0 + 1)\alpha_3 + \alpha_1 + l_0 + 1, k1\alpha_1 + l_0 + 1,\\
&k1\alpha_3 + k1 - 1\}, \cdots\\
&\{k_2, k1 - l_0 - 1, \alpha_1 - 1, -\alpha_0 + \alpha_3 + \alpha_4, (l_0 + 1)\alpha_3 + l_0\alpha_4\}, \cdots\\
&\{k_2, \alpha_4 + 1, -\alpha_0 - \alpha_3 + \alpha_1 - 2, (l_0 + 1)\alpha_3 - \alpha_1 + l_0 + 1, k1\alpha_1 - l_0 - 1,\\
&k1\alpha_3 + k1 - 1\}, \cdots\\
&\{k1 - l_0k_2, \alpha_3 + 1, \alpha_0 - \alpha_4 + \alpha_1, -\alpha_4 + k_2\alpha_1\}, \cdots\\
&\{k1 + k_2 - l_0 - 1, -\alpha_3 + \alpha_4 - \alpha_1 + 2, \alpha_0 + 1,
\end{aligned}
$$

$$-\alpha_4 - l_0\alpha_1 - 1, (k_2-1)\alpha_1 + 1, (-k_2+1)\alpha_4 - k_2 + l_0 + 1\}, \cdots$$
$$\{k1+1, \alpha_3+2, \alpha_0 - \alpha_4 + \alpha_1 + 1, l_0\alpha_4 + \alpha_1 + 1, -\alpha_4 + k_2\alpha_1 + 1\}, \cdots$$
$$\{k1 - l_0k_2, \alpha_3+1, -\alpha_0+\alpha_4+\alpha_1, -\alpha_4 - k_2\alpha_1\}, \cdots$$
$$\{k1 + k_2 - l_0 - 1, -\alpha_3 - \alpha_4 - \alpha_1 + 2, \alpha_0 - 1, \alpha_4 - l_0\alpha_1 - 1, (k_2-1)\alpha_1 + 1,$$
$$(k_2-1)\alpha_4 - k_2 + l_0 + 1\}, \cdots$$
$$\{k1+1, \alpha_3+2, -\alpha_0+\alpha_4+\alpha_1+1, -l_0\alpha_4+\alpha_1+1, -\alpha_4 - k_2\alpha_1 - 1\}, \cdots$$
$$\{k_2, k1 - l_0 - 1, \alpha_1+1, \alpha_0+\alpha_3+\alpha_4, (l_0+1)\alpha_3 + l_0\alpha_4\}, \cdots$$
$$\{k_2, \alpha_4+1, -\alpha_0+\alpha_3+\alpha_1+2, (l_0+1)\alpha_3+\alpha_1+l_0+1, k1\alpha_1+l_0+1,$$
$$k1\alpha_3 + k1 - 1\}, \cdots$$
$$\{k_2, k1 - l_0 - 1, \alpha_1 - 1, \alpha_0+\alpha_3+\alpha_4, (l_0+1)\alpha_3 + l_0\alpha_4\}, \cdots$$
$$\{k_2, \alpha_4+1, \alpha_0 - \alpha_3 + \alpha_1 - 2, (l_0+1)\alpha_3 - \alpha_1 + l_0 + 1, k1\alpha_1 - l_0 - 1,$$
$$k1\alpha_3 + k1 - 1\}, \cdots, \{k_2, k1, \alpha_4\},$$
$$\{k1 - l_0k_2, \alpha_3+1, \alpha_0+\alpha_4+\alpha_1, \alpha_4 + k_2\alpha_1\}, \cdots$$
$$\{k1 + k_2 - l_0 - 1, -\alpha_3 - \alpha_4 - \alpha_1 + 2, \alpha_0 + 1, \alpha_4 - l_0\alpha_1 - 1, (k_2-1)\alpha_1 + 1,$$
$$(k_2-1)\alpha_4 - k_2 + l_0 + 1\}, \cdots$$
$$\{k1+1, \alpha_3+2, \alpha_0+\alpha_4+\alpha_1+1, -l_0\alpha_4+\alpha_1+1, \alpha_4 + k_2\alpha_1 + 1\}, \cdots$$
$$\{k1 - l_0k_2, \alpha_3+1, -\alpha_0-\alpha_4+\alpha_1, \alpha_4 - k_2\alpha_1\}, \cdots$$
$$\{k1 + k_2 - l_0 - 1, -\alpha_3 + \alpha_4 - \alpha_1 + 2, \alpha_0 - 1, -\alpha_4 - l_0\alpha_1 - 1, (k_2-1)\alpha_1 + 1,$$
$$(-k_2+1)\alpha_4 - k_2 + l_0 + 1\}, \cdots, \{k_2 - 1, k1 - l_0, \alpha_0\},$$
$$\{k1+1, \alpha_3+2, -\alpha_0-\alpha_4+\alpha_1+1, l_0\alpha_4+\alpha_1+1, \alpha_4 - k_2\alpha_1 - 1\}, \cdots\} = 0$$

$$D_6 = \{\{k_2, k_0+k_1-1, \alpha_1, \alpha_3+1, \alpha_4+\alpha_0+2, (\alpha_4+1)k_1 - \alpha_4\}, \cdots$$
$$\{k_1 + 2k_2 - 2, k_0 - k_2 + 1, \alpha_1, \alpha_4+2, \alpha_3+1, \alpha_0\}, \cdots$$
$$\{k_1 + k_2 - 1, k_0, \alpha_4+1, \alpha_1+\alpha_3+\alpha_0, \alpha_1k_2+\alpha_3\}, \cdots$$
$$\{k_2, k_0, \alpha_1+\alpha_4-\alpha_3, \alpha_0+1, -\alpha_1k_1 - \alpha_4\}, \cdots$$
$$\{k_0, \alpha_1+1, k_1+k_2+\alpha_4, k_1 - \alpha_3 - 1, k_2 + \alpha_0 - 1\}, \cdots$$
$$\{k_1 + k_2 - 1, k_0, \alpha_4+1, \alpha_1+\alpha_3-\alpha_0, -\alpha_1k_2 - \alpha_3\}, \cdots$$
$$\{k_1, k_0, \alpha_3+1, \alpha_1+\alpha_4+\alpha_0, \alpha_1k_2+\alpha_4\}, \cdots$$
$$\{k_1, k_0, \alpha_3+1, \alpha_1+\alpha_4-\alpha_0, -\alpha_1k_2-\alpha_4\}, \cdots$$
$$\{k_2, k_1, \alpha_1, \alpha_4, \alpha_3+1, \alpha_0+2\}, \cdots, \{k_2, k_1-1, k_0, \alpha_3\}, \{k_2, k_1, k_0, \alpha_4\},$$
$$\{k_2 - 1, k_1, k_0, \alpha_0\}\} = 0$$

$$D_7 = \{\{k_0, \alpha_4+1, \alpha_1+\alpha_3+\alpha_0+2, (\alpha_3+1)l_0+\alpha_1+\alpha_3+1, \alpha_1k_1+l_0+1,$$
$$(\alpha_3+1)k_1 - 1\}, \cdots$$
$$\{k_1 - l_0 - 1, k_0, \alpha_1+1, \alpha_4-\alpha_3+\alpha_0, (\alpha_4-\alpha_3)l_0 - \alpha_3\}, \cdots$$
$$\{k_0, \alpha_4+1, \alpha_1-\alpha_3+\alpha_0+2, (\alpha_3-1)l_0-\alpha_1+\alpha_3-1, \alpha_1k_1+l_0+1,$$
$$(\alpha_3-1)k_1 + 1\}, \cdots$$
$$\{k_1 - l_0 - 1, k_0, \alpha_1+1, \alpha_4+\alpha_3+\alpha_0, (\alpha_4+\alpha_3)l_0 + \alpha_3\}, \cdots$$
$$\{k_1, k_0, \alpha_4\}, \{k_0 - l_0k_1, \alpha_1-\alpha_4+\alpha_3, \alpha_0+1, \alpha_1k_1 - \alpha_4\}, \cdots$$
$$\{k_0 + k_1 - l_0 - 1, \alpha_3 - 1, \alpha_1 - \alpha_4 + \alpha_0 + 2, (\alpha_4-1)l_0 + \alpha_1,$$
$$\alpha_1k_1 + l_0 - \alpha_1, (\alpha_4-1)k_1 - \alpha_4\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1+\alpha_4-\alpha_3-1, \alpha_0+2, (-\alpha_1-1)l_0 - \alpha_4,$$
$$-\alpha_1k_0 + (\alpha_4-1)l_0, (-l_0-\alpha_4)k_0 + (-\alpha_4+1)l_0^2\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1+\alpha_4-\alpha_3+1, \alpha_0+2, (-\alpha_1+1)l_0 - \alpha_4,$$
$$-\alpha_1k_0 + (\alpha_4+1)l_0, (l_0-\alpha_4)k_0 + (-\alpha_4-1)l_0^2\}, \cdots$$
$$\{k_0 - l_0k_1, \alpha_1+\alpha_4+\alpha_3, \alpha_0+1, \alpha_1k_1+\alpha_4\}, \cdots$$
$$\{k_0 + k_1 - l_0 - 1, \alpha_3 - 1, \alpha_1+\alpha_4+\alpha_0+2, (\alpha_4+1)l_0 - \alpha_1, \alpha_1k_1 + l_0 - \alpha_1,$$
$$(\alpha_4+1)k_1 - \alpha_4\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1-\alpha_4-\alpha_3-1, \alpha_0+2, (-\alpha_1-1)l_0 + \alpha_4, -\alpha_1k_0 + (-\alpha_4-1)l_0,$$
$$(-l_0+\alpha_4)k_0 + (\alpha_4+1)l_0^2\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1-\alpha_4-\alpha_3+1, \alpha_0+2,$$
$$(-\alpha_1+1)l_0 + \alpha_4, -\alpha_1k_0 + (-\alpha_4+1)l_0,$$
$$(l_0+\alpha_4)k_0 + (\alpha_4-1)l_0^2\}, \cdots$$
$$\{k_0 - l_0k_1, \alpha_1+\alpha_4-\alpha_3, \alpha_0+1, -\alpha_1k_1 - \alpha_4\}, \cdots$$
$$\{k_0 + k_1 - l_0 - 1, \alpha_3 + 1, \alpha_1+\alpha_4+\alpha_0+2, (\alpha_4+1)l_0 - \alpha_1, \alpha_1k_1 + l_0 - \alpha_1,$$
$$(\alpha_4+1)k_1 - \alpha_4\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1-\alpha_4+\alpha_3-1, \alpha_0+2, (\alpha_1+1)l_0 - \alpha_4, \alpha_1k_0 + (\alpha_4+1)l_0,$$
$$(-l_0+\alpha_4)k_0 + (\alpha_4+1)l_0^2\}, \cdots$$
$$\{k_1 + l_0, -\alpha_1-\alpha_4+\alpha_3+1, \alpha_0+2, (\alpha_1-1)l_0 - \alpha_4, \alpha_1k_0 + (\alpha_4-1)l_0,$$
$$(l_0+\alpha_4)k_0 + (\alpha_4-1)l_0^2\}, \cdots$$
$$\{k_0 - l_0k_1, \alpha_1-\alpha_4-\alpha_3, \alpha_0+1, -\alpha_1k_1 + \alpha_4\}, \cdots$$
$$\{k_0 + k_1 - l_0 - 1, \alpha_3 + 1, \alpha_1-\alpha_4+\alpha_0+2, (\alpha_4-1)l_0 + \alpha_1, \alpha_1k_1 + l_0 - \alpha_1,$$
$$(\alpha_4-1)k_1 - \alpha_4\}, \cdots$$
$$\{k_1 - 1, k_0 - l_0, \alpha_3\}, \{k_1 + l_0, -\alpha_1+\alpha_4+\alpha_3-1, \alpha_0+2,$$
$$(\alpha_1+1)l_0 + \alpha_4, \alpha_1k_0 + (-\alpha_4+1)l_0,$$
$$(-l_0-\alpha_4)k_0 + (-\alpha_4+1)l_0^2\}, \cdots\} = 0$$

$$
\begin{aligned}
D_8 = \{ & \{k_1+1, k_0, \alpha_1+1, \alpha_4, \alpha_3+2, \alpha_0\}, \cdots, \{k_1-l_0, k_0, \alpha_0\}, \\
& \{k_0-l_0k_1+l_0^2, \alpha_1+1, k_1-l_0+\alpha_4+1, k_1-l_0+\alpha_3-1, \alpha_0\}, \cdots \\
& \{k_1-l_0, 2k_0-l_0^2-l_0, \alpha_1+2, l_0+\alpha_4+2, l_0+\alpha_3-1, \alpha_0\}, \cdots \\
& \{k_0-l_0k_1+l_0^2, \alpha_1+1, k_1-l_0+\alpha_4+1, k_1-l_0-\alpha_3-1, \alpha_0\}, \cdots \\
& \{k_1-l_0, 2k_0-l_0^2-l_0, \alpha_1+2, l_0+\alpha_4+2, l_0-\alpha_3-1, \alpha_0\}, \cdots \\
& \{k_1, k_0-l_0, \alpha_1+1, \alpha_4+2, \alpha_3, \alpha_0\}, \cdots \} = 0
\end{aligned}
$$

$$
\begin{aligned}
D_9 = \{ & \{l_0, k_1, \alpha_3-1, \alpha_1-\alpha_4-\alpha_0, \alpha_1k_2-\alpha_4\}, \cdots \{l_0, k_2-1, k_1, \alpha_0\}, \cdots \\
& \{2l_0-1, 2k_2-1, k_1-1, \alpha_1-2, \alpha_4+2, \alpha_3-2, \alpha_0-1\}, \cdots \\
& \{2l_0-1, 2k_2+1, k_1+1, \alpha_1-2, \alpha_4+2, \alpha_3-2, \alpha_0-3\}, \cdots \\
& \{k_2-1, k_1-l_0, \alpha_1l_0-\alpha_1+\alpha_4, \alpha_0, l_0^2-l_0+1, \alpha_1^2l_0-\alpha_1^2+\alpha_3^2+1\}, \\
& \{k_2-l_0, k_1-l_0+1, \alpha_1l_0-\alpha_4, \alpha_3-1, \alpha_1l_0-\alpha_1+\alpha_0, l_0^2-l_0+1\}, \cdots \\
& \{k_2-1, k_1-l_0, \alpha_0, l_0^2-l_0+1, \alpha_1^2l_0-\alpha_1^2+\alpha_3^2+1\}, \\
& \{k_2-1, k_1-l_0, \alpha_0, l_0^2-l_0+1\}\} = 0
\end{aligned}
$$

$$
\begin{aligned}
D_{10} = \{ & \{k_2, k_1, \alpha_4\}, \{k_1-l_0k_2, \alpha_3+1, \alpha_1+\alpha_4+\alpha_0, \alpha_1k_2+\alpha_4\}, \cdots \\
& \{k_2-1, k_1-l_0, \alpha_0\}, \{k_2, \alpha_1+\alpha_4+\alpha_3, \alpha_0+1, \alpha_1k_1+\alpha_4\}, \cdots \\
& \{k_1+(-l_0+1)k_2-1, \alpha_4+1, \alpha_1+\alpha_3+\alpha_0, \alpha_1k_2+\alpha_3\}, \cdots \\
& \{\alpha_1+1, k_1+(-l_0+1)k_2+\alpha_4, k_1-l_0k_2+\alpha_3-1, k_2+\alpha_0-1\}, \cdots \\
& \{k_1+(-l_0+1)k_2-1, \alpha_4+1, \alpha_1+\alpha_3-\alpha_0, -\alpha_1k_2-\alpha_3\}, \cdots \\
& \{\alpha_1+1, k_1+(-l_0+1)k_2+\alpha_4, k_1-l_0k_2+\alpha_3-1, k_2-\alpha_0-1\}, \cdots \\
& \{k_2, \alpha_1+\alpha_4-\alpha_3, \alpha_0+1, -\alpha_1k_1-\alpha_4\}, \cdots, \{k_2, k_1-1, \alpha_3\}, \\
& \{k_1+(-l_0+1)k_2-1, \alpha_4+1, \alpha_1-\alpha_3+\alpha_0, \alpha_1k_2-\alpha_3\}, \cdots \\
& \{\alpha_1+1, k_1+(-l_0+1)k_2+\alpha_4, k_1-l_0k_2-\alpha_3-1, k_2+\alpha_0-1\}, \cdots \\
& \{k_1+(-l_0+1)k_2-1, \alpha_4+1, \alpha_1-\alpha_3-\alpha_0, -\alpha_1k_2+\alpha_3\}, \cdots \\
& \{\alpha_1+1, k_1+(-l_0+1)k_2+\alpha_4, k_1-l_0k_2-\alpha_3-1, \\
& k_2-\alpha_0-1\}, \cdots \} = 0
\end{aligned}
$$

7. Conclusion

We proposed the technique for obtaining solutions to ordinary differential equations. And, we also demonstrated the implementation of this technique and showed its timing data. If we do not use the resultant-factorization technique, we cannot obtain all the solutions for the ideal (2) within 48 h. However, we can compute solutions with Table 1 by using resultant-factorization technique.

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