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# 均衡型 $(C_5, C_{16})$-Foil デザインと関連デザイン

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グラフ理論において、グラフの分解問題は主要な研究テーマである。$C_5$ を５点を通るサイクル、$C_{16}$ を１６点を通るサイクルとする。１点を共有する辺素な $t$ 個の $C_5$ と $t$ 個の $C_{16}$ からなるグラフを $(C_5, C_{16})$-$2t$-foil という。本研究では、完全グラフ $K_n$ を 均衡的に $(C_5, C_{16})$-$2t$-foil 部分グラフに分解する均衡型 $(C_5, C_{16})$-$2t$-foil デザインについて述べる。さらに、均衡型 $C_{21}$-$t$-foil デザイン、均衡型 $(C_{10}, C_{32})$-$2t$-foil デザイン、均衡型 $C_{42}$-$t$-foil デザインについて述べる。

# Balanced $(C_5, C_{16})$-Foil Designs and Related Designs

Kazuhiko Ushio

In graph theory, the decomposition problem of graphs is a very important topic. Various type of decompositions of many graphs can be seen in the literature of graph theory. This paper gives balanced $(C_5, C_{16})$-$2t$-foil designs, balanced $C_{21}$-$t$-foil designs, balanced $(C_{10}, C_{32})$-$2t$-foil designs, and balanced $C_{42}$-$t$-foil designs.

## 1. Balanced $(C_5, C_{16})$-$2t$-Foil Designs

Let $K_n$ denote *the complete graph* of $n$ vertices. Let $C_5$ and $C_{16}$ be *the* 5-*cycle* and *the* 16-*cycle*, respectively. *The* $(C_5, C_{16})$-$2t$-*foil* is a graph of $t$ edge-disjoint $C_5$'s and $t$ edge-disjoint $C_{16}$'s with a common vertex and the common vertex is called *the center of the* $(C_5, C_{16})$-$2t$-*foil*. When $K_n$ is decomposed into edge-disjoint sum of $(C_5, C_{16})$-$2t$-foils, we say that $K_n$ *has a* $(C_5, C_{16})$-$2t$-*foil decomposition*. Moreover, when every vertex of $K_n$ appears in the same number of $(C_5, C_{16})$-$2t$-foils, we say that $K_n$ *has a balanced* $(C_5, C_{16})$-$2t$-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* $(C_5, C_{16})$-$2t$-*foil design*.

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**Theorem 1.** $K_n$ has a balanced $(C_5, C_{16})$-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{42t}$.

**Proof.** **(Necessity)** Suppose that $K_n$ has a balanced $(C_5, C_{16})$-$2t$-foil decomposition. Let $b$ be the number of $(C_5, C_{16})$-$2t$-foils and $r$ be the replication number. Then $b = n(n-1)/42t$ and $r = (19t+1)(n-1)/42t$. Among $r$ $(C_5, C_{16})$-$2t$-foils having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $(C_5, C_{16})$-$2t$-foils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/42t$ and $r_2 = 19(n-1)/42$. Therefore, $n \equiv 1 \pmod{42t}$ is necessary.
**(Sufficiency)** Put $n = 42st+1$ and $T = st$. Then $n = 42T+1$. Construct a $(C_5, C_{16})$-$2T$-foil as follows:
$\{(42T+1, 1, 18T+2, 38T+2, 18T), (42T+1, 3T+1, 4T+2, 15T+2, 22T+3, 7T+2, 13T+3, 25T+3, 11T+3, 26T+3, 20T+3, 16T+2, 32T+3, 24T+2, 11T+2, 2T+1)\}$
$\cup$
$\{(42T+1, 2, 18T+4, 38T+3, 18T-1), (42T+1, 3T+2, 4T+4, 15T+3, 22T+5, 7T+3, 13T+5, 25T+4, 11T+5, 26T+4, 20T+5, 16T+3, 32T+5, 24T+3, 11T+4, 2T+2)\}$
$\cup$
$\{(42T+1, 3, 18T+6, 38T+4, 18T-2), (42T+1, 3T+3, 4T+6, 15T+4, 22T+7, 7T+4, 13T+7, 25T+5, 11T+7, 26T+5, 20T+7, 16T+4, 32T+7, 24T+4, 11T+6, 2T+3)\}$
$\cup$
... $\cup$
$\{(42T+1, T-1, 20T-2, 39T, 17T+2), (42T+1, 4T-1, 6T-2, 16T, 24T-1, 8T, 15T-1, 26T+1, 13T-1, 27T+1, 22T-1, 17T, 34T-1, 25T, 13T-2, 3T-1)\}$ $\cup$
$\{(42T+1, T, 20T, 39T+1, 17T+1), (42T+1, 4T, 6T, 16T+1, 24T+1, 8T+1, 15T+1, 26T+2, 13T+1, 27T+2, 22T+2, 39T+2, 34T+1, 25T+1, 13T, 3T)\}$.
($21T$ edges, $21T$ all lengths)

Decompose the $(C_5, C_{16})$-$2T$-foil into $s$ $(C_5, C_{16})$-$2t$-foils. Then these starters comprise a balanced $(C_5, C_{16})$-$2t$-foil decomposition of $K_n$.

**Example 1.1. Balanced $(C_5, C_{16})$-2-foil decomposition of $K_{43}$.**
$\{(43, 1, 20, 40, 18), (43, 4, 6, 17, 25, 9, 16, 28, 14, 29, 24, 41, 35, 26, 13, 3)\}$.
(21 edges, 21 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-2-foil decomposition of $K_{43}$.

**Example 1.2. Balanced $(C_5, C_{16})$-4-foil decomposition of $K_{85}$.**
$\{(85, 1, 38, 78, 36), (85, 7, 10, 32, 47, 16, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5)\} \cup$
$\{(85, 2, 40, 79, 35), (85, 8, 12, 33, 49, 17, 31, 54, 27, 56, 46, 80, 69, 51, 26, 6)\}$.
(42 edges, 42 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-4-foil decomposition of $K_{85}$.

**Example 1.3. Balanced $(C_5, C_{16})$-6-foil decomposition of $K_{127}$.**
$\{(127, 1, 56, 116, 54), (127, 10, 14, 47, 69, 23, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7)\} \cup$
$\{(127, 2, 58, 117, 53), (127, 11, 16, 48, 71, 24, 44, 79, 38, 82, 65, 51, 101, 75, 37, 8)\} \cup$
$\{(127, 3, 60, 118, 52), (127, 12, 18, 49, 73, 25, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)\}$.
(63 edges, 63 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-6-foil decomposition of $K_{127}$.

**Example 1.4. Balanced $(C_5, C_{16})$-8-foil decomposition of $K_{169}$.**
$\{(169, 1, 74, 154, 72), (169, 13, 18, 62, 91, 30, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9)\} \cup$
$\{(169, 2, 76, 155, 71), (169, 14, 20, 63, 93, 31, 57, 104, 49, 108, 85, 67, 133, 99, 48, 10)\} \cup$
$\{(169, 3, 78, 156, 70), (169, 15, 22, 64, 95, 32, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11)\} \cup$
$\{(169, 4, 80, 157, 69), (169, 16, 24, 65, 97, 33, 61, 106, 53, 110, 90, 158, 137, 101, 52, 12)\}$.
(84 edges, 84 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-8-foil decomposition of $K_{169}$.

**Example 1.5. Balanced $(C_5, C_{16})$-10-foil decomposition of $K_{211}$.**
$\{(211, 1, 92, 192, 90), (211, 16, 22, 77, 113, 37, 68, 128, 58, 133, 103, 82, 163, 122, 57, 11)\} \cup$
$\{(211, 2, 94, 193, 89), (211, 17, 24, 78, 115, 38, 70, 129, 60, 134, 105, 83, 165, 123, 59, 12)\} \cup$
$\{(211, 3, 96, 194, 88), (211, 18, 26, 79, 117, 39, 72, 130, 62, 135, 107, 84, 167, 124, 61, 13)\} \cup$
$\{(211, 4, 98, 195, 87), (211, 19, 28, 80, 119, 40, 74, 131, 64, 136, 109, 85, 169, 125, 63, 14)\} \cup$
$\{(211, 5, 100, 196, 86), (211, 20, 30, 81, 121, 41, 76, 132, 66, 137, 112, 197, 171, 126, 65, 15)\}$.
(105 edges, 105 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-10-foil decomposition of $K_{211}$.

**Example 1.6. Balanced $(C_5, C_{16})$-12-foil decomposition of $K_{253}$.**
$\{(253, 1, 110, 230, 108), (253, 19, 26, 92, 135, 44, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13)\} \cup$
$\{(253, 2, 112, 231, 107), (253, 20, 28, 93, 137, 45, 83, 154, 71, 160, 125, 99, 197, 147, 70, 14)\} \cup$
$\{(253, 3, 114, 232, 106), (253, 21, 30, 94, 139, 46, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15)\}$
$\cup$
$\{(253, 4, 116, 233, 105), (253, 22, 32, 95, 141, 47, 87, 156, 75, 162, 129, 101, 201, 149, 74, 16)\}$
$\cup$
$\{(253, 5, 118, 234, 104), (253, 23, 34, 96, 143, 48, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17)\}$
$\cup$
$\{(253, 6, 120, 235, 103), (253, 24, 36, 97, 145, 49, 91, 158, 79, 164, 134, 236, 205, 151, 78, 18)\}$.
(126 edges, 126 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-12-foil decomposition of $K_{253}$.

**Example 1.7. Balanced $(C_5, C_{16})$-14-foil decomposition of $K_{295}$.**
$\{(295, 1, 128, 268, 126), (295, 22, 30, 107, 157, 51, 94, 178, 80, 185, 143, 114, 227, 170, 79, 15)\}$
$\cup$
$\{(295, 2, 130, 269, 125), (295, 23, 32, 108, 159, 52, 96, 179, 82, 186, 145, 115, 229, 171, 81, 16)\}$
$\cup$
$\{(295, 3, 132, 270, 124), (295, 24, 34, 109, 161, 53, 98, 180, 84, 187, 147, 116, 231, 172, 83, 17)\}$
$\cup$
$\{(295, 4, 134, 271, 123), (295, 25, 36, 110, 163, 54, 100, 181, 86, 188, 149, 117, 233, 173, 85, 18)\}$
$\cup$
$\{(295, 5, 136, 272, 122), (295, 26, 38, 111, 165, 55, 102, 182, 88, 189, 151, 118, 235, 174, 87, 19)\}$
$\cup$

$\{(295, 6, 138, 273, 121), (295, 27, 40, 112, 167, 56, 104, 183, 90, 190, 153, 119, 237, 175, 89, 20)\}$
$\cup$
$\{(295, 7, 140, 274, 120), (295, 28, 42, 113, 169, 57, 106, 184, 92, 191, 156, 275, 239, 176, 91, 21)\}$.
(147 edges, 147 all lengths)
This starter comprises a balanced $(C_5, C_{16})$-14-foil decomposition of $K_{295}$.

## 2. Balanced $C_{21}$-Foil Designs

Let $K_n$ denote *the complete graph* of $n$ vertices. Let $C_{21}$ be *the* 21-*cycle*. *The* $C_{21}$-$t$-*foil* is a graph of $t$ edge-disjoint $C_{21}$'s with a common vertex and the common vertex is called *the center of the* $C_{21}$-$t$-*foil*. When $K_n$ is decomposed into edge-disjoint sum of $C_{21}$-$t$-foils, it is called that $K_n$ *has a* $C_{21}$-$t$-*foil decomposition*. Moreover, when every vertex of $K_n$ appears in the same number of $C_{21}$-$t$-foils, it is called that $K_n$ *has a balanced* $C_{21}$-$t$-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* $C_{21}$-$t$-*foil design*.

**Theorem 2.** $K_n$ has a balanced $C_{21}$-$t$-foil decomposition if and only if $n \equiv 1 \pmod{42t}$.

**Proof. (Necessity)** Suppose that $K_n$ has a balanced $C_{21}$-$t$-foil decomposition. Let $b$ be the number of $C_{21}$-$t$-foils and $r$ be the replication number. Then $b = n(n-1)/42t$ and $r = (20t+1)(n-1)/42t$. Among $r$ $C_{21}$-$t$-foils having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $C_{21}$-$t$-foils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $2tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/42t$ and $r_2 = 20(n-1)/42$. Therefore, $n \equiv 1 \pmod{42t}$ is necessary.
**(Sufficiency)** Put $n = 42st + 1, T = st$. Then $n = 42T + 1$. Construct a $C_{21}$-$T$-foil as follows:
$\{$ $(42T+1, T, 20T, 39T+1, 17T+1, 20T+2, 3T+1, 4T+2, 15T+2, 22T+3, 7T+2, 13T+3, 25T+3, 11T+3, 26T+3, 20T+3, 16T+2, 32T+3, 24T+2, 11T+2, 2T+1)$,
$(42T+1, T-1, 20T-2, 39T, 17T+2, 20T+4, 3T+2, 4T+4, 15T+3, 22T+5, 7T+3, 13T+5, 25T+4, 11T+5, 26T+4, 20T+5, 16T+3, 32T+5, 24T+3, 11T+4, 2T+2)$,
$(42T+1, T-2, 20T-4, 39T-1, 17T+3, 20T+6, 3T+3, 4T+6, 15T+4, 22T+7, 7T+4, 13T+7, 25T+5, 11T+7, 26T+5, 20T+7, 16T+4, 32T+7, 24T+4, 11T+6, 2T+3)$,
...,
$(42T+1, 2, 18T+4, 38T+3, 18T-1, 22T-2, 4T-1, 6T-2, 16T, 24T-1, 8T, 15T-1, 26T+1, 13T-1, 27T+1, 22T-1, 17T, 34T-1, 25T, 13T-2, 3T-1)$,
$(42T+1, 1, 18T+2, 38T+2, 18T, 22T, 4T, 6T, 16T+1, 24T+1, 8T+1, 15T+1, 26T+2, 13T+1, 27T+2, 22T+2, 39T+2, 34T+1, 25T+1, 13T, 3T)$ $\}$.
($21T$ edges, $21T$ all lengths)
Decompose this $C_{21}$-$T$-foil into $s$ $C_{21}$-$t$-foils. Then these starters comprise a balanced $C_{21}$-$t$-foil decomposition of $K_n$.

**Example 2.1. Balanced $C_{21}$-decomposition of $K_{43}$.**
$\{(43, 1, 20, 40, 18, 22, 4, 6, 17, 25, 9, 16, 28, 14, 29, 24, 41, 35, 26, 13, 3)\}$.
(21 edges, 21 all lengths)
This stater comprises a balanced $C_{21}$-decomposition of $K_{43}$.

**Example 2.2. Balanced $C_{21}$-2-foil decomposition of $K_{85}$.**
$\{(85, 2, 40, 79, 35, 42, 7, 10, 32, 47, 16, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5)$,
$(85, 1, 38, 78, 36, 44, 8, 12, 33, 49, 17, 31, 54, 27, 56, 46, 80, 69, 51, 26, 6)\}$.
(42 edges, 42 all lengths)
This stater comprises a balanced $C_{21}$-2-foil decomposition of $K_{85}$.

**Example 2.3. Balanced $C_{21}$-3-foil decomposition of $K_{127}$.**
$\{(127, 3, 60, 118, 52, 62, 10, 14, 47, 69, 23, 42, 78, 36, 81, 63, 50, 99, 74, 35, 7)$,
$(127, 2, 58, 117, 53, 64, 11, 16, 48, 71, 24, 44, 79, 38, 82, 65, 51, 101, 75, 37, 8)$,
$(127, 1, 56, 116, 54, 66, 12, 18, 49, 73, 25, 46, 80, 40, 83, 68, 119, 103, 76, 39, 9)\}$.
(63 edges, 63 all lengths)
This stater comprises a balanced $C_{21}$-3-foil decomposition of $K_{127}$.

**Example 2.4. Balanced $C_{21}$-4-foil decomposition of $K_{169}$.**

$\{(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 30, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9),$
$(169, 3, 78, 156, 70, 84, 14, 20, 63, 93, 31, 57, 104, 49, 108, 85, 67, 133, 99, 48, 10),$
$(169, 2, 76, 155, 71, 86, 15, 22, 64, 95, 32, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11),$
$(169, 1, 74, 154, 72, 88, 16, 24, 65, 97, 33, 61, 106, 53, 110, 90, 158, 137, 101, 52, 12)\}.$
(84 edges, 84 all lengths)
This stater comprises a balanced $C_{21}$-4-foil decomposition of $K_{169}$.

**Example 2.5. Balanced $C_{21}$-5-foil decomposition of $K_{211}$.**
$\{(211, 5, 100, 196, 86, 102, 16, 22, 77, 113, 37, 68, 128, 58, 133, 103, 82, 163, 122, 57, 11),$
$(211, 4, 98, 195, 87, 104, 17, 24, 78, 115, 38, 70, 129, 60, 134, 105, 83, 165, 123, 59, 12),$
$(211, 3, 96, 194, 88, 106, 18, 26, 79, 117, 39, 72, 130, 62, 135, 107, 84, 167, 124, 61, 13),$
$(211, 2, 94, 193, 89, 108, 19, 28, 80, 119, 40, 74, 131, 64, 136, 109, 85, 169, 125, 63, 14),$
$(211, 1, 92, 192, 90, 110, 20, 30, 81, 121, 41, 76, 132, 66, 137, 112, 197, 171, 126, 65, 15)\}.$
(105 edges, 105 all lengths)
This stater comprises a balanced $C_{21}$-5-foil decomposition of $K_{211}$.

**Example 2.6. Balanced $C_{21}$-6-foil decomposition of $K_{253}$.**
$\{(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 44, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13),$
$(253, 5, 118, 234, 104, 124, 20, 28, 93, 137, 45, 83, 154, 71, 160, 125, 99, 197, 147, 70, 14),$
$(253, 4, 116, 233, 105, 126, 21, 30, 94, 139, 46, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15),$
$(253, 3, 114, 232, 106, 128, 22, 32, 95, 141, 47, 87, 156, 75, 162, 129, 101, 201, 149, 74, 16),$
$(253, 2, 112, 231, 107, 130, 23, 34, 96, 143, 48, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17),$
$(253, 1, 110, 230, 108, 132, 24, 36, 97, 145, 49, 91, 158, 79, 164, 134, 236, 205, 151, 78, 18)\}.$
(126 edges, 126 all lengths)
This stater comprises a balanced $C_{21}$-6-foil decomposition of $K_{253}$.

## 3. Balanced $(C_{10}, C_{32})$-Foil Designs

Let $K_n$ denote *the complete graph* of $n$ vertices. Let $C_{10}$ and $C_{32}$ be *the* 10-*cycle* and *the* 32-*cycle*, respectively. *The* $(C_{10}, C_{32})$-$2t$-*foil* is a graph of $t$ edge-disjoint $C_{10}$'s and $t$ edge-disjoint $C_{32}$'s with a common vertex and the common vertex is called *the center of the* $(C_{10}, C_{32})$-$2t$-*foil*. When $K_n$ is decomposed into edge-disjoint sum of $(C_{10}, C_{32})$-$2t$-foils, we say that $K_n$ *has a* $(C_{10}, C_{32})$-$2t$-*foil decomposition*. Moreover, when every vertex of $K_n$ appears in the same number of $(C_{10}, C_{32})$-$2t$-foils, we say that $K_n$ *has a balanced* $(C_{10}, C_{32})$-$2t$-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* $(C_{10}, C_{32})$-$2t$-*foil design*.

**Theorem 3.** $K_n$ has a balanced $(C_{10}, C_{32})$-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{84t}$.

**Proof. (Necessity)** Suppose that $K_n$ has a balanced $(C_{10}, C_{32})$-$2t$-foil decomposition. Let $b$ be the number of $(C_{10}, C_{32})$-$2t$-foils and $r$ be the replication number. Then $b = n(n-1)/84t$ and $r = (40t+1)(n-1)/84t$. Among $r$ $(C_{10}, C_{32})$-$2t$-foils having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $(C_{10}, C_{32})$-$2t$-foils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/84t$ and $r_2 = 40(n-1)/84$. Therefore, $n \equiv 1 \pmod{84t}$ is necessary.
**(Sufficiency)** Put $n = 84st+1$ and $T = st$. Then $n = 84T+1$. Construct a $(C_{10}, C_{32})$-$2T$-foil as follows:
$\{(84T+1, 1, 36T+2, 76T+2, 36T, 72T-1, 36T-1, 76T+3, 36T+4, 2),$
$(84T+1, 6T+1, 8T+2, 30T+2, 44T+3, 14T+2, 26T+3, 50T+3, 22T+3, 52T+3, 40T+3, 32T+2, 64T+3, 48T+2, 22T+2, 4T+1, 8T+3, 4T+2, 22T+4, 48T+3, 64T+5, 32T+3, 40T+5, 52T+4, 22T+5, 50T+4, 26T+5, 14T+3, 44T+5, 30T+3, 8T+4, 6T+2)\}$
$\cup$
$\{(84T+1, 3, 36T+6, 76T+4, 36T-2, 72T-5, 36T-3, 76T+5, 36T+8, 4),$
$(84T+1, 6T+3, 8T+6, 30T+4, 44T+7, 14T+4, 26T+7, 50T+5, 22T+7, 52T+5, 40T+7, 32T+4, 64T+7, 48T+4, 22T+6, 4T+3, 8T+7, 4T+4, 22T+8, 48T+5, 64T+9, 32T+5, 40T+9, 52T+6, 22T+9, 50T+6, 26T+9, 14T+5, 44T+9, 30T+5, 8T+8, 6T+4)\}$
$\cup$
$\{(84T+1, 5, 36T+10, 76T+6, 36T-4, 72T-9, 36T-5, 76T+7, 36T+12, 6),$
$(84T+1, 6T+5, 8T+10, 30T+6, 44T+11, 14T+6, 26T+11, 50T+7, 22T+11, 52T+7, 40T+11, 32T+6, 64T+11, 48T+6, 22T+10, 4T+5, 8T+11, 4T+6, 22T+12, 48T+$

$7, 64T + 13, 32T + 7, 40T + 13, 52T + 8, 22T + 13, 50T + 8, 26T + 13, 14T + 7, 44T + 13, 30T + 7, 8T + 12, 6T + 6)\} \cup$
$... \cup$
$\{(84T + 1, 2T - 1, 40T - 2, 78T, 34T + 2, 68T + 3, 34T + 1, 78T + 1, 40T, 2T),$
$(84T + 1, 8T - 1, 12T - 2, 32T, 48T - 1, 16T, 30T - 1, 52T + 1, 26T - 1, 54T + 1, 44T - 1, 34T, 68T - 1, 50T, 26T - 2, 6T - 1, 12T - 1, 6T, 26T, 50T + 1, 68T + 1, 78T + 2, 44T + 2, 54T + 2, 26T + 1, 52T + 2, 30T + 1, 16T + 1, 48T + 1, 32T + 1, 12T, 8T)\}.$
($42T$ edges, $42T$ all lengths)
Decompose the $(C_{10}, C_{32})$-$2T$-foil into $s$ $(C_{10}, C_{32})$-$2t$-foils. Then these starters comprise a balanced $(C_{10}, C_{32})$-$2t$-foil decomposition of $K_n$.

**Example 3.1. Balanced $(C_{10}, C_{32})$-2-foil decomposition of $K_{85}$.**
$\{(85, 1, 38, 78, 36, 71, 35, 79, 40, 2),$
$(85, 7, 10, 32, 47, 16, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5, 11, 6, 26, 51, 69, 80, 46, 56, 27, 54, 31, 17, 49, 33, 12, 8)\}.$
(42 edges, 42 all lengths)
This starter comprises a balanced $(C_{10}, C_{32})$-2-foil decomposition of $K_{85}$.

**Example 3.2. Balanced $(C_{10}, C_{32})$-4-foil decomposition of $K_{169}$.**
$\{(169, 1, 74, 154, 72, 143, 71, 155, 76, 2),$
$(169, 3, 78, 156, 70, 139, 69, 157, 80, 4)\}$
$\cup$
$\{(169, 13, 18, 62, 91, 30, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9, 19, 10, 48, 99, 133, 67, 85, 108, 49, 104, 57, 31, 93, 63, 20, 14),$
$(169, 15, 22, 64, 95, 32, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11, 23, 12, 52, 101, 137, 158, 90, 110, 53, 106, 61, 33, 97, 65, 24, 16)\}.$
(84 edges, 84 all lengths)
This starter comprises a balanced $(C_{10}, C_{32})$-4-foil decomposition of $K_{169}$.

**Example 3.3. Balanced $(C_{10}, C_{32})$-6-foil decomposition of $K_{253}$.**
$\{(253, 1, 110, 230, 108, 215, 107, 231, 112, 2),$
$(253, 3, 114, 232, 106, 211, 105, 233, 116, 4),$
$(253, 5, 118, 234, 104, 207, 103, 235, 120, 6)\}$
$\cup$
$\{(253, 19, 26, 92, 135, 44, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13, 27, 14, 70, 147, 197, 99, 125, 160, 71, 154, 83, 45, 137, 93, 28, 20),$
$(253, 21, 30, 94, 139, 46, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15, 31, 16, 74, 149, 201, 101, 129, 162, 75, 156, 87, 47, 141, 95, 32, 22),$
$(253, 23, 34, 96, 143, 48, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17, 35, 18, 78, 151, 205, 236, 134, 164, 79, 158, 91, 49, 145, 97, 36, 24)\}.$
(126 edges, 126 all lengths)
This starter comprises a balanced $(C_{10}, C_{32})$-6-foil decomposition of $K_{253}$.

**Example 3.4. Balanced $(C_{10}, C_{32})$-8-foil decomposition of $K_{337}$.**
$\{(337, 1, 146, 306, 144, 287, 143, 307, 148, 2),$
$(337, 3, 150, 308, 142, 283, 141, 309, 152, 4),$
$(337, 5, 154, 310, 140, 279, 139, 311, 156, 6),$
$(337, 7, 158, 312, 138, 275, 137, 313, 160, 8)\}$
$\cup$
$\{(337, 25, 34, 122, 179, 58, 107, 203, 91, 211, 163, 130, 259, 194, 90, 17, 35, 18, 92, 195, 261, 131, 165, 212, 93, 204, 109, 59, 181, 123, 36, 26),$
$(337, 27, 38, 124, 183, 60, 111, 205, 95, 213, 167, 132, 263, 196, 94, 19, 39, 20, 96, 197, 265, 133, 169, 214, 97, 206, 113, 61, 185, 125, 40, 28),$
$(337, 29, 42, 126, 187, 62, 115, 207, 99, 215, 171, 134, 267, 198, 98, 21, 43, 22, 100, 199, 269, 135, 173, 216, 101, 208, 117, 63, 189, 127, 44, 30),$
$(337, 31, 46, 128, 191, 64, 119, 209, 103, 217, 175, 136, 271, 200, 102, 23, 47, 24, 104, 201, 273, 314, 178, 218, 105, 210, 121, 65, 193, 129, 48, 32)\}.$
(168 edges, 168 all lengths)
This starter comprises a balanced $(C_{10}, C_{32})$-8-foil decomposition of $K_{337}$.

**Example 3.5. Balanced $(C_{10}, C_{32})$-10-foil decomposition of $K_{421}$.**
$\{(421, 1, 182, 382, 180, 359, 179, 383, 184, 2),$

$(421, 3, 186, 384, 178, 355, 177, 385, 188, 4)$,
$(421, 5, 190, 386, 176, 351, 175, 387, 192, 6)$,
$(421, 7, 194, 388, 174, 347, 173, 389, 196, 8)$,
$(421, 9, 198, 390, 172, 343, 171, 391, 200, 10)\}$
$\cup$
$\{(421, 31, 42, 152, 223, 72, 133, 253, 113, 263, 203, 162, 323, 242, 112, 21, 43, 22, 114, 243, 325, 163, 205, 264, 115, 254, 135, 73, 225, 153, 44, 32)$,
$(421, 33, 46, 154, 227, 74, 137, 255, 117, 265, 207, 164, 327, 244, 116, 23, 47, 24, 118, 245, 329, 165, 209, 266, 119, 256, 139, 75, 229, 155, 48, 34)$,
$(421, 35, 50, 156, 231, 76, 141, 257, 121, 267, 211, 166, 331, 246, 120, 25, 51, 26, 122, 247, 333, 167, 213, 268, 123, 258, 143, 77, 233, 157, 52, 36)$,
$(421, 37, 54, 158, 235, 78, 145, 259, 125, 269, 215, 168, 335, 248, 124, 27, 55, 28, 126, 249, 337, 169, 217, 270, 127, 260, 147, 79, 237, 159, 56, 38)$,
$(421, 39, 58, 160, 239, 80, 149, 261, 129, 271, 219, 170, 339, 250, 128, 29, 59, 30, 130, 251, 341, 392, 222, 272, 131, 262, 151, 81, 241, 161, 60, 40)\}$.
(210 edges, 210 all lengths)
This starter comprises a balanced $(C_{10}, C_{32})$-10-foil decomposition of $K_{421}$.

## 4. Balanced $C_{42}$-Foil Designs

Let $K_n$ denote *the complete graph* of $n$ vertices. Let $C_{42}$ be *the 42-cycle*. *The $C_{42}$-$t$-foil* is a graph of $t$ edge-disjoint $C_{42}$'s with a common vertex and the common vertex is called *the center of the $C_{42}$-$t$-foil*. When $K_n$ is decomposed into edge-disjoint sum of $C_{42}$-$t$-foils, it is called that *$K_n$ has a $C_{42}$-$t$-foil decomposition*. Moreover, when every vertex of $K_n$ appears in the same number of $C_{42}$-$t$-foils, it is called that *$K_n$ has a balanced $C_{42}$-$t$-foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced $C_{42}$-$t$-foil design*.

**Theorem 4.** $K_n$ has a balanced $C_{42}$-$t$-foil decomposition if and only if $n \equiv 1 \pmod{84t}$.

**Proof.** **(Necessity)** Suppose that $K_n$ has a balanced $C_{42}$-$t$-foil decomposition. Let $b$ be the number of $C_{42}$-$t$-foils and $r$ be the replication number. Then $b = n(n-1)/84t$ and $r = (41t+1)(n-1)/84t$. Among $r$ $C_{42}$-$t$-foils having a vertex $v$ of $K_n$, let $r_1$ and $r_2$ be the numbers of $C_{42}$-$t$-foils in which $v$ is the center and $v$ is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to $v$, $2tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/84t$ and $r_2 = 41(n-1)/84$. Therefore, $n \equiv 1 \pmod{84t}$ is necessary.
**(Sufficiency)** Put $n = 84st+1$, $T = st$. Then $n = 84T+1$. Construct a $C_{42}$-$T$-foil as follows:
$\{\ (84T+1, 2T, 40T, 78T+1, 34T+1, 40T+2, 6T+1, 8T+2, 30T+2, 44T+3, 14T+2, 26T+3, 50T+3, 22T+3, 52T+3, 40T+3, 32T+2, 64T+3, 48T+2, 22T+2, 4T+1, 8T+3, 4T+2, 22T+4, 48T+3, 64T+5, 32T+3, 40T+5, 52T+4, 22T+5, 50T+4, 26T+5, 14T+3, 44T+5, 30T+3, 8T+4, 6T+2, 40T+4, 34T+2, 78T, 40T-2, 2T-1)$,
$(84T+1, 2T-2, 40T-4, 78T-1, 34T+3, 40T+6, 6T+3, 8T+6, 30T+4, 44T+7, 14T+4, 26T+7, 50T+5, 22T+7, 52T+5, 40T+7, 32T+4, 64T+7, 48T+4, 22T+6, 4T+3, 8T+7, 4T+4, 22T+8, 48T+5, 64T+9, 32T+5, 40T+9, 52T+6, 22T+9, 50T+6, 26T+9, 14T+5, 44T+9, 30T+5, 8T+8, 6T+4, 40T+8, 34T+4, 78T-2, 40T-6, 2T-3)$,
$(84T+1, 2T-4, 40T-8, 78T-3, 34T+5, 40T+10, 6T+5, 8T+10, 30T+6, 44T+11, 14T+6, 26T+11, 50T+7, 22T+11, 52T+7, 40T+11, 32T+6, 64T+11, 48T+6, 22T+10, 4T+5, 8T+11, 4T+6, 22T+12, 48T+7, 64T+13, 32T+7, 40T+13, 52T+8, 22T+13, 50T+8, 26T+13, 14T+7, 44T+13, 30T+7, 8T+12, 6T+6, 40T+12, 34T+6, 78T-4, 40T-10, 2T-5)$,
...,
$(84T+1, 2, 36T+4, 76T+3, 36T-1, 44T-2, 8T-1, 12T-2, 32T, 48T-1, 16T, 30T-1, 52T+1, 26T-1, 54T+1, 44T-1, 34T, 68T-1, 50T, 26T-2, 6T-1, 12T-1, 6T, 26T, 50T+1, 68T+1, 78T+2, 44T+2, 54T+2, 26T+1, 52T+2, 30T+1, 16T+1, 48T+1, 32T+1, 12T, 8T, 44T, 36T, 76T+2, 36T+2, 1)\ \}$.
($42T$ edges, $42T$ all lengths)
Decompose this $C_{42}$-$T$-foil into $s$ $C_{42}$-$t$-foils. Then these starters comprise a balanced $C_{42}$-$t$-foil decomposition of $K_n$.

**Example 4.1. Balanced $C_{42}$-decomposition of $K_{85}$.**
$\{(85, 2, 40, 79, 35, 42, 7, 10, 32, 47, 16, 29, 53, 25, 55, 43, 34, 67, 50, 24, 5, 11, 6, 26, 51, 69, 80, 46, 56, 27, 54, 31, 17, 49, 33, 12, 8, 44, 36, 78, 38, 1)\}$.
(42 edges, 42 all lengths)
This starter comprises a balanced $C_{42}$-decomposition of $K_{85}$.

**Example 4.2. Balanced $C_{42}$-2-foil decomposition of $K_{169}$.**
$\{(169, 4, 80, 157, 69, 82, 13, 18, 62, 91, 30, 55, 103, 47, 107, 83, 66, 131, 98, 46, 9, 19, 10, 48, 99, 133, 67, 85, 108, 49, 104, 57, 31, 93, 63, 20, 14, 84, 70, 156, 78, 3)$,
$(169, 2, 76, 155, 71, 86, 15, 22, 64, 95, 32, 59, 105, 51, 109, 87, 68, 135, 100, 50, 11, 23, 12, 52, 101, 137, 158, 90, 110, 53, 106, 61, 33, 97, 65, 24, 16, 88, 72, 154, 74, 1)\}$.
(84 edges, 84 all lengths)
This starter comprises a balanced $C_{42}$-2-foil decomposition of $K_{169}$.

**Example 4.3. Balanced $C_{42}$-3-foil decomposition of $K_{253}$.**
$\{(253, 6, 120, 235, 103, 122, 19, 26, 92, 135, 44, 81, 153, 69, 159, 123, 98, 195, 146, 68, 13, 27, 14, 70, 147, 197, 99, 125, 160, 71, 154, 83, 45, 137, 93, 28, 20, 124, 104, 234, 118, 5)$,
$(253, 4, 116, 233, 105, 126, 21, 30, 94, 139, 46, 85, 155, 73, 161, 127, 100, 199, 148, 72, 15, 31, 16, 74, 149, 201, 101, 129, 162, 75, 156, 87, 47, 141, 95, 32, 22, 128, 106, 232, 114, 3)$,
$(253, 2, 112, 231, 107, 130, 23, 34, 96, 143, 48, 89, 157, 77, 163, 131, 102, 203, 150, 76, 17, 35, 18, 78, 151, 205, 236, 134, 164, 79, 158, 91, 49, 145, 97, 36, 24, 132, 108, 230, 110, 1)\}$.
(126 edges, 126 all lengths)
This starter comprises a balanced $C_{42}$-3-foil decomposition of $K_{253}$.

**Example 4.4. Balanced $C_{42}$-4-foil decomposition of $K_{337}$.**
$\{(337, 8, 160, 313, 137, 162, 25, 34, 122, 179, 58, 107, 203, 91, 211, 163, 130, 259, 194, 90, 17, 35, 18, 92, 195, 261, 131, 165, 212, 93, 204, 109, 59, 181, 123, 36, 26, 164, 138, 312, 158, 7)$,
$(337, 6, 156, 311, 139, 166, 27, 38, 124, 183, 60, 111, 205, 95, 213, 167, 132, 263, 196, 94, 19, 39, 20, 96, 197, 265, 133, 169, 214, 97, 206, 113, 61, 185, 125, 40, 28, 168, 140, 310, 154, 5)$,
$(337, 4, 152, 309, 141, 170, 29, 42, 126, 187, 62, 115, 207, 99, 215, 171, 134, 267, 198, 98, 21, 43, 22, 100, 199, 269, 135, 173, 216, 101, 208, 117, 63, 189, 127, 44, 30, 172, 142, 308, 150, 3)$,
$(337, 2, 148, 307, 143, 174, 31, 46, 128, 191, 64, 119, 209, 103, 217, 175, 136, 271, 200, 102, 23, 47, 24, 104, 201, 273, 314, 178, 218, 105, 210, 121, 65, 193, 129, 48, 32, 176, 144, 306, 146, 1)\}$.
(168 edges, 168 all lengths)
This starter comprises a balanced $C_{42}$-4-foil decomposition of $K_{337}$.

**Example 4.5. Balanced $C_{42}$-5-foil decomposition of $K_{421}$.**
$\{(421, 10, 200, 391, 171, 202, 31, 42, 152, 223, 72, 133, 253, 113, 263, 203, 162, 323, 242, 112, 21, 43, 22, 114, 243, 325, 163, 205, 264, 115, 254, 135, 73, 225, 153, 44, 32, 204, 172, 390, 198, 9)$,
$(421, 8, 196, 389, 173, 206, 33, 46, 154, 227, 74, 137, 255, 117, 265, 207, 164, 327, 244, 116, 23, 47, 24, 118, 245, 329, 165, 209, 266, 119, 256, 139, 75, 229, 155, 48, 34, 208, 174, 388, 194, 7)$,
$(421, 6, 192, 387, 175, 210, 35, 50, 156, 231, 76, 141, 257, 121, 267, 211, 166, 331, 246, 120, 25, 51, 26, 122, 247, 333, 167, 213, 268, 123, 258, 143, 77, 253, 157, 52, 36, 212, 176, 386, 190, 5)$,
$(421, 4, 188, 385, 177, 214, 37, 54, 158, 235, 78, 145, 259, 125, 269, 215, 168, 335, 248, 124, 27, 55, 28, 126, 249, 337, 169, 217, 270, 127, 260, 147, 79, 237, 159, 56, 38, 216, 178, 384, 186, 3)$,
$(421, 2, 184, 383, 179, 218, 39, 58, 160, 239, 80, 149, 261, 129, 271, 219, 170, 339, 250, 128, 29, 59, 30, 130, 251, 341, 392, 222, 272, 131, 262, 151, 81, 241, 161, 60, 40, 220, 180, 382, 182, 1)\}$.
(210 edges, 210 all lengths)
This starter comprises a balanced $C_{42}$-5-foil decomposition of $K_{421}$.

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