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均衡型 (C_5, C_{14})-Foil デザインと関連デザイン

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グラフ理論において、グラフの分解問題は主要な研究テーマである。C_5 を５点を通るサイクル、C_{14} を１４点を通るサイクルとする。１点を共有する辺素な t 個の C_5 と t 個の C_{14} からなるグラフを (C_5, C_{14})-$2t$-foil という。本研究では、完全グラフ K_n を 均衡的に (C_5, C_{14})-$2t$-foil 部分グラフに分解する均衡型 (C_5, C_{14})-$2t$-foil デザインについて述べる。さらに、均衡型 C_{19}-t-foil デザイン、均衡型 (C_{10}, C_{28})-$2t$-foil デザイン、均衡型 C_{38}-t-foil デザインについて述べる。

Balanced (C_5, C_{14})-Foil Designs and Related Designs

Kazuhiko Ushio

In graph theory, the decomposition problem of graphs is a very important topic. Various type of decompositions of many graphs can be seen in the literature of graph theory. This paper gives balanced (C_5, C_{14})-$2t$-foil designs, balanced C_{19}-t-foil designs, balanced (C_{10}, C_{28})-$2t$-foil designs, and balanced C_{38}-t-foil designs.

1. Balanced (C_5, C_{14})-$2t$-Foil Designs

Let K_n denote *the complete graph* of n vertices. Let C_5 and C_{14} be *the* 5-*cycle* and *the* 14-*cycle*, respectively. *The* (C_5, C_{14})-$2t$-*foil* is a graph of t edge-disjoint C_5's and t edge-disjoint C_{14}'s with a common vertex and the common vertex is called *the center of the* (C_5, C_{14})-$2t$-*foil*. When K_n is decomposed into edge-disjoint sum of (C_5, C_{14})-$2t$-foils, we say that K_n *has a* (C_5, C_{14})-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_5, C_{14})-$2t$-foils, we say that K_n *has a balanced* (C_5, C_{14})-$2t$-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* (C_5, C_{14})-$2t$-*foil design*.

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Theorem 1. K_n has a balanced (C_5, C_{14})-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{38t}$.

Proof. **(Necessity)** Suppose that K_n has a balanced (C_5, C_{14})-$2t$-foil decomposition. Let b be the number of (C_5, C_{14})-$2t$-foils and r be the replication number. Then $b = n(n-1)/38t$ and $r = (17t+1)(n-1)/38t$. Among r (C_5, C_{14})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_5, C_{14})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/38t$ and $r_2 = 17(n-1)/38$. Therefore, $n \equiv 1 \pmod{38t}$ is necessary.
(Sufficiency) Put $n = 38st + 1$ and $T = st$. Then $n = 38T + 1$.
Case 1. n = 39. (Example 1. Balanced (C_5, C_{14})**-2-foil decomposition of** K_{39}**.)**
Case 2. $\mathbf{n = 38T + 1, T \geq 2}$. Construct a (C_5, C_{14})-$2T$-foil as follows:
$\{(38T+1, 1, 14T+2, 35T+2, 17T), (38T+1, 10T+1, 11T+2, 17T+2, 21T+3, 29T+3, 6T+3, 18T+3, 14T+3, 5T+2, 30T+3, 24T+2, 21T+2, 13T+1)\} \cup$
$\{(38T+1, 2, 14T+4, 35T+3, 17T-1), (38T+1, 10T+2, 11T+4, 17T+3, 21T+5, 29T+4, 6T+5, 18T+4, 14T+5, 5T+3, 30T+5, 24T+3, 21T+4, 13T+2)\} \cup$
$\{(38T+1, 3, 14T+6, 35T+4, 17T-2), (38T+1, 10T+3, 11T+6, 17T+4, 21T+7, 29T+5, 6T+7, 18T+5, 14T+7, 5T+4, 30T+7, 24T+4, 21T+6, 13T+3)\} \cup$
... $\cup$
$\{(38T+1, T-1, 16T-2, 36T, 16T+2), (38T+1, 11T-1, 13T-2, 18T, 23T-1, 30T+1, 8T-1, 19T+1, 16T-1, 6T, 32T-1, 25T, 23T-2, 14T-1)\} \cup$
$\{(38T+1, T, 16T, 36T+1, 16T+1), (38T+1, 11T, 13T, 18T+1, 23T+1, 30T+2, 8T+1, 19T+2, 9T+2, 6T+1, 32T+1, 25T+1, 23T, 14T)\}$.
($19T$ edges, $19T$ all lengths)
Decompose the (C_5, C_{14})-$2T$-foil into s (C_5, C_{14})-$2t$-foils. Then these starters comprise a balanced (C_5, C_{14})-$2t$-foil decomposition of K_n.

Example 1.1. Balanced (C_5, C_{14})-2-foil decomposition of K_{39}.
$\{(39, 1, 16, 37, 17), (39, 2, 13, 19, 24, 32, 9, 21, 11, 7, 33, 26, 23, 14)\}$.
(19 edges, 19 all lengths)
This starter comprises a balanced (C_5, C_{14})-2-foil decomposition of K_{39}.

Example 1.2. Balanced (C_5, C_{14})-4-foil decomposition of K_{77}.
$\{(77, 1, 30, 72, 34), (77, 21, 24, 36, 45, 61, 15, 39, 31, 12, 63, 50, 44, 27)\} \cup$
$\{(77, 2, 32, 73, 33), (77, 22, 26, 37, 47, 62, 17, 40, 20, 13, 65, 51, 46, 28)\}$.
(38 edges, 38 all lengths)
This starter comprises a balanced (C_5, C_{14})-4-foil decomposition of K_{77}.

Example 1.3. Balanced (C_5, C_{14})-6-foil decomposition of K_{115}.
$\{(115, 1, 44, 107, 51), (115, 31, 35, 53, 66, 90, 21, 57, 45, 17, 93, 74, 65, 40)\} \cup$
$\{(115, 2, 46, 108, 50), (115, 32, 37, 54, 68, 91, 23, 58, 47, 18, 95, 75, 67, 41)\} \cup$
$\{(115, 3, 48, 109, 49), (115, 33, 39, 55, 70, 92, 25, 59, 29, 19, 97, 76, 69, 42)\}$.
(51 edges, 51 all lengths)
This starter comprises a balanced (C_5, C_{14})-6-foil decomposition of K_{115}.

Example 1.4. Balanced (C_5, C_{14})-8-foil decomposition of K_{153}.
$\{(153, 1, 58, 142, 68), (153, 41, 46, 70, 87, 119, 27, 75, 59, 22, 123, 98, 86, 53)\} \cup$
$\{(153, 2, 60, 143, 67), (153, 42, 48, 71, 89, 120, 29, 76, 61, 23, 125, 99, 88, 54)\} \cup$
$\{(153, 3, 62, 144, 66), (153, 43, 50, 72, 91, 121, 31, 77, 63, 24, 127, 100, 90, 55)\} \cup$
$\{(153, 4, 64, 145, 65), (153, 44, 52, 73, 93, 122, 33, 78, 38, 25, 129, 101, 92, 56)\}$.
(76 edges, 76 all lengths)
This starter comprises a balanced (C_5, C_{14})-8-foil decomposition of K_{153}.

Example 1.5. Balanced (C_5, C_{14})-10-foil decomposition of K_{191}.
$\{(191, 1, 72, 177, 85), (191, 51, 57, 87, 108, 148, 33, 93, 73, 27, 153, 122, 107, 66)\} \cup$
$\{(191, 2, 74, 178, 84), (191, 52, 59, 88, 110, 149, 35, 94, 75, 28, 155, 123, 109, 67)\} \cup$
$\{(191, 3, 76, 179, 83), (191, 53, 61, 89, 112, 150, 37, 95, 77, 29, 157, 124, 111, 68)\} \cup$
$\{(191, 4, 78, 180, 82), (191, 54, 63, 90, 114, 151, 39, 96, 79, 30, 159, 125, 113, 69)\} \cup$
$\{(191, 5, 80, 181, 81), (191, 55, 65, 91, 116, 152, 41, 97, 47, 31, 161, 126, 115, 70)\}$.
(95 edges, 95 all lengths)
This starter comprises a balanced (C_5, C_{14})-10-foil decomposition of K_{191}.

Example 1.6. Balanced (C_5, C_{14})-12-foil decomposition of K_{229}.
$\{(229, 1, 86, 212, 102), (229, 61, 68, 104, 129, 177, 39, 111, 87, 32, 183, 146, 128, 79)\} \cup$
$\{(229, 2, 88, 213, 101), (229, 62, 70, 105, 131, 178, 41, 112, 89, 33, 185, 147, 130, 80)\} \cup$
$\{(229, 3, 90, 214, 100), (229, 63, 72, 106, 133, 179, 43, 113, 91, 34, 187, 148, 132, 81)\} \cup$
$\{(229, 4, 92, 215, 99), (229, 64, 74, 107, 135, 180, 45, 114, 93, 35, 189, 149, 134, 82)\} \cup$
$\{(229, 5, 94, 216, 98), (229, 65, 76, 108, 137, 181, 47, 115, 95, 36, 191, 150, 136, 83)\} \cup$
$\{(229, 6, 96, 217, 97), (229, 66, 78, 109, 139, 182, 49, 116, 56, 37, 193, 151, 138, 84)\}$.
(114 edges, 114 all lengths)
This starter comprises a balanced (C_5, C_{14})-12-foil decomposition of K_{229}.

Example 1.7. Balanced (C_5, C_{14})-14-foil decomposition of K_{267}.
$\{(267, 1, 100, 247, 119), (267, 71, 79, 121, 150, 206, 45, 129, 101, 37, 213, 170, 149, 92)\} \cup$
$\{(267, 2, 102, 248, 118), (267, 72, 81, 122, 152, 207, 47, 130, 103, 38, 215, 171, 151, 93)\} \cup$
$\{(267, 3, 104, 249, 117), (267, 73, 83, 123, 154, 208, 49, 131, 105, 39, 217, 172, 153, 94)\} \cup$
$\{(267, 4, 106, 250, 116), (267, 74, 85, 124, 156, 209, 51, 132, 107, 40, 219, 173, 155, 95)\} \cup$
$\{(267, 5, 108, 251, 115), (267, 75, 87, 125, 158, 210, 53, 133, 109, 41, 221, 174, 157, 96)\} \cup$
$\{(267, 6, 110, 252, 114), (267, 76, 89, 126, 160, 211, 55, 134, 111, 42, 223, 175, 159, 97)\} \cup$
$\{(267, 7, 112, 253, 113), (267, 77, 91, 127, 162, 212, 57, 135, 65, 43, 225, 176, 161, 98)\}$.
(133 edges, 133 all lengths)
This starter comprises a balanced (C_5, C_{14})-14-foil decomposition of K_{267}.

2. Balanced C_{19}-Foil Designs

Let K_n denote *the complete graph* of n vertices. Let C_{19} be *the* 19*-cycle*. *The* C_{19}*-t-foil* is a graph of t edge-disjoint C_{19}'s with a common vertex and the common vertex is called *the center of the* C_{19}*-t-foil*. In particular, the C_{19}-2-foil and the C_{19}-3-foil are called *the*

C_{19}-*bowtie* and *the* C_{19}-*trefoil*, respectively. When K_n is decomposed into edge-disjoint sum of C_{19}-t-foils, it is called that K_n *has a* C_{19}-t-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_{19}-t-foils, it is called that K_n *has a balanced* C_{19}-t-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* C_{19}-t-*foil design*.

Theorem 2. K_n has a balanced C_{19}-t-foil decomposition if and only if $n \equiv 1 \pmod{38t}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{19}-t-foil decomposition. Let b be the number of C_{19}-t-foils and r be the replication number. Then $b = n(n-1)/38t$ and $r = (18t+1)(n-1)/38t$. Among r C_{19}-t-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{19}-t-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/38t$ and $r_2 = 18(n-1)/38$. Therefore, $n \equiv 1 \pmod{38t}$ is necessary.
(Sufficiency) Put $n = 38st+1, T = st$. Then $n = 38T+1$.
Case 1. n = 39. (Example 1. Balanced C_{19}-decomposition of K_{39}.)
Case 2. $\mathbf{n = 38T+1, T \geq 2}$. Construct a C_{19}-T-foil as follows:
$\{\ (38T+1, T, 16T, 36T+1, 16T+1, 26T+2, 10T+1, 11T+2, 17T+2, 21T+3, 29T+3, 6T+3, 18T+3, 14T+3, 5T+2, 30T+3, 24T+2, 21T+2, 13T+1),$
$(38T+1, T-1, 16T-2, 36T, 16T+2, 26T+4, 10T+2, 11T+4, 17T+3, 21T+5, 29T+4, 6T+5, 18T+4, 14T+5, 5T+3, 30T+5, 24T+3, 21T+4, 13T+2),$
$(38T+1, T-2, 16T-4, 36T-1, 16T+3, 26T+6, 10T+3, 11T+6, 17T+4, 21T+7, 29T+5, 6T+7, 18T+5, 14T+7, 5T+4, 30T+7, 24T+4, 21T+6, 13T+3),$
...,
$(38T+1, 2, 14T+4, 35T+3, 17T-1, 28T-2, 11T-1, 13T-2, 18T, 23T-1, 30T+1, 8T-1, 19T+1, 16T-1, 6T, 32T-1, 25T, 23T-2, 14T-1),$
$(38T+1, 1, 14T+2, 35T+2, 17T, 28T, 11T, 13T, 18T+1, 23T+1, 30T+2, 8T+1, 19T+2, 9T+2, 6T+1, 32T+1, 25T+1, 23T, 14T)\ \}.$
($19T$ edges, $19T$ all lengths)
Decompose this C_{19}-T-foil into s C_{19}-t-foils. Then these starters comprise a balanced C_{19}-t-foil decomposition of K_n.

Example 2.1. Balanced C_{19}-decomposition of K_{39}.
$\{(39, 1, 16, 37, 17, 19, 2, 13, 18, 24, 32, 9, 21, 11, 7, 23, 26, 23, 14)\}$.
(19 edges, 19 all lengths)
This stater comprises a balanced C_{19}-decomposition of K_{39}.

Example 2.2. Balanced C_{19}-2-foil decomposition of K_{77}.
$\{(77, 2, 32, 73, 33, 54, 21, 24, 36, 45, 61, 15, 39, 31, 12, 63, 50, 44, 27),$
$(77, 1, 30, 72, 34, 56, 22, 26, 37, 47, 62, 17, 40, 20, 13, 65, 51, 46, 28)\}$.
(38 edges, 38 all lengths)
This stater comprises a balanced C_{19}-2-foil decomposition of K_{77}.

Example 2.3. Balanced C_{19}-3-foil decomposition of K_{115}.
$\{(115, 3, 48, 109, 49, 80, 31, 35, 53, 66, 90, 21, 57, 45, 17, 93, 74, 65, 40),$
$(115, 2, 46, 108, 50, 82, 32, 37, 54, 68, 91, 23, 58, 47, 18, 95, 75, 67, 41),$
$(115, 1, 44, 107, 51, 84, 33, 39, 55, 70, 92, 25, 59, 29, 19, 97, 76, 69, 42)\}$.
(57 edges, 57 all lengths)
This stater comprises a balanced C_{19}-3-foil decomposition of K_{115}.

Example 2.4. Balanced C_{19}-4-foil decomposition of K_{153}.
$\{(153, 4, 64, 145, 65, 106, 41, 46, 70, 87, 119, 27, 75, 59, 22, 123, 98, 86, 53),$
$(153, 3, 62, 144, 66, 108, 42, 48, 71, 89, 120, 29, 76, 61, 23, 125, 99, 88, 54),$
$(153, 2, 60, 143, 67, 110, 43, 50, 72, 91, 121, 31, 77, 63, 24, 127, 100, 90, 55),$
$(153, 1, 58, 142, 68, 112, 44, 52, 73, 93, 122, 33, 78, 38, 25, 129, 101, 92, 56)\}$.
(76 edges, 76 all lengths)
This stater comprises a balanced C_{19}-4-foil decomposition of K_{153}.

Example 2.5. Balanced C_{19}-5-foil decomposition of K_{191}.
$\{(191, 5, 80, 181, 81, 132, 51, 57, 87, 108, 148, 33, 93, 73, 27, 153, 122, 107, 66),$

$(191, 4, 78, 180, 82, 134, 52, 59, 88, 110, 149, 35, 94, 75, 28, 155, 123, 109, 67)$,
$(191, 3, 76, 179, 83, 136, 53, 61, 89, 112, 150, 37, 95, 77, 29, 157, 124, 111, 68)$,
$(191, 2, 74, 178, 84, 138, 54, 63, 90, 114, 151, 39, 96, 79, 30, 159, 125, 113, 69)$,
$(191, 1, 72, 177, 85, 140, 55, 65, 91, 116, 152, 41, 97, 47, 31, 161, 126, 115, 70)\}$.
(95 edges, 95 all lengths)
This stater comprises a balanced C_{19}-5-foil decomposition of K_{191}.

Example 2.6. Balanced C_{19}-6-foil decomposition of K_{229}.
$\{(229, 6, 96, 217, 97, 158, 61, 68, 104, 129, 177, 39, 111, 87, 32, 183, 146, 128, 79)$,
$(229, 5, 94, 216, 98, 160, 62, 70, 105, 131, 178, 41, 112, 89, 33, 185, 147, 130, 80)$,
$(229, 4, 92, 215, 99, 162, 63, 72, 106, 133, 179, 43, 113, 91, 34, 187, 148, 132, 81)$,
$(229, 3, 90, 214, 100, 164, 64, 74, 107, 135, 180, 45, 114, 93, 35, 189, 149, 134, 82)$,
$(229, 2, 88, 213, 101, 166, 65, 76, 108, 137, 181, 47, 115, 95, 36, 191, 150, 136, 83)$,
$(229, 1, 86, 212, 102, 168, 66, 78, 109, 139, 182, 49, 116, 56, 37, 193, 151, 138, 84)\}$.
(114 edges, 114 all lengths)
This stater comprises a balanced C_{19}-6-foil decomposition of K_{229}.

3. Balanced (C_{10}, C_{28})-Foil Designs

Let K_n denote *the complete graph* of n vertices. Let C_{10} and C_{28} be *the* 10-*cycle* and *the* 28-*cycle*, respectively. *The* (C_{10}, C_{28})-$2t$-*foil* is a graph of t edge-disjoint C_{10}'s and t edge-disjoint C_{28}'s with a common vertex and the common vertex is called *the center of the* (C_{10}, C_{28})-$2t$-*foil*. In particular, the (C_{10}, C_{28})-2-foil is called *the* (C_{10}, C_{28})-*bowtie*. When K_n is decomposed into edge-disjoint sum of (C_{10}, C_{28})-$2t$-foils, we say that K_n *has a* (C_{10}, C_{28})-$2t$-*foil decomposition*. Moreover, when every vertex of K_n appears in the same number of (C_{10}, C_{28})-$2t$-foils, we say that K_n *has a balanced* (C_{10}, C_{28})-$2t$-*foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* (C_{10}, C_{28})-$2t$-*foil design*.

Theorem 3. K_n has a balanced (C_{10}, C_{28})-$2t$-foil decomposition if and only if $n \equiv 1 \pmod{76t}$.

Proof. (Necessity) Suppose that K_n has a balanced (C_{10}, C_{28})-$2t$-foil decomposition. Let b be the number of (C_{10}, C_{28})-$2t$-foils and r be the replication number. Then $b = n(n-1)/76t$ and $r = (36t+1)(n-1)/76t$. Among r (C_{10}, C_{28})-$2t$-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of (C_{10}, C_{28})-$2t$-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $4tr_1 + 2r_2 = n-1$. From these relations, $r_1 = (n-1)/76t$ and $r_2 = 36(n-1)/76$. Therefore, $n \equiv 1 \pmod{76t}$ is necessary.
(Sufficiency) Put $n = 76st + 1$ and $T = st$. Then $n = 76T + 1$.
Construct a (C_{10}, C_{28})-$2T$-foil as follows:
$\{(76T+1, 1, 28T+2, 70T+2, 34T, 68T-1, 34T-1, 70T+3, 28T+4, 2)$,
$(76T+1, 20T+1, 22T+2, 34T+2, 42T+3, 58T+3, 12T+3, 36T+3, 28T+3, 10T+2, 60T+3, 48T+2, 42T+2, 26T+1, 52T+3, 26T+2, 42T+4, 48T+3, 60T+5, 10T+3, 28T+5, 36T+4, 12T+5, 58T+4, 42T+5, 34T+3, 22T+4, 20T+2)\} \cup$
$\{(76T+1, 3, 28T+6, 70T+4, 34T-2, 68T-5, 34T-3, 70T+5, 28T+8, 4)$,
$(76T+1, 20T+3, 22T+6, 34T+4, 42T+7, 58T+5, 12T+7, 36T+5, 28T+7, 10T+4, 60T+7, 48T+4, 42T+6, 26T+3, 52T+7, 26T+4, 42T+8, 48T+5, 60T+9, 10T+5, 28T+9, 36T+6, 12T+9, 58T+6, 42T+9, 34T+5, 22T+8, 20T+4)\} \cup$
$\{(76T+1, 5, 28T+10, 70T+6, 34T-4, 68T-9, 34T-5, 70T+7, 28T+12, 6)$,
$(76T+1, 20T+5, 22T+10, 34T+6, 42T+11, 58T+7, 12T+11, 36T+7, 28T+11, 10T+6, 60T+11, 48T+6, 42T+10, 26T+5, 52T+11, 26T+6, 42T+12, 48T+7, 60T+13, 10T+7, 28T+13, 36T+8, 12T+13, 58T+8, 42T+13, 34T+7, 22T+12, 20T+6)\} \cup$
... $\cup$
$\{(76T+1, 2T-1, 32T-2, 72T, 32T+2, 64T+3, 32T+1, 72T+1, 32T, 2T)$,
$(76T+1, 22T-1, 26T-2, 36T, 46T-1, 60T+1, 16T-1, 38T+1, 32T-1, 12T, 64T-1, 50T, 46T-2, 28T-1, 56T-1, 28T, 46T, 50T+1, 64T+1, 12T+1, 18T+2, 38T+2, 16T+1, 60T+2, 46T+1, 36T+1, 26T, 22T)\}$.
($38T$ edges, $38T$ all lengths)
Decompose the (C_{10}, C_{28})-$2T$-foil into s (C_{10}, C_{28})-$2t$-foils. Then these starters comprise a balanced (C_{10}, C_{28})-$2t$-foil decomposition of K_n.

Example 3.1. Balanced (C_{10}, C_{28})-2-foil decomposition of K_{77}.
$\{(77, 1, 30, 72, 34, 67, 33, 73, 32, 2),$
$(77, 21, 24, 36, 45, 61, 15, 39, 31, 12, 63, 50, 44, 27, 55, 28, 46, 51, 65, 13, 20, 40, 17, 62, 47,$
$37, 26, 22)\}.$
(38 edges, 38 all lengths)
This starter comprises a balanced (C_{10}, C_{28})-2-foil decomposition of K_{77}.

Example 3.2. Balanced (C_{10}, C_{28})-4-foil decomposition of K_{153}.
$\{(153, 1, 58, 142, 68, 135, 67, 143, 60, 2),$
$(153, 3, 62, 144, 66, 131, 65, 145, 64, 4)\}$
$\cup$
$\{(153, 41, 46, 70, 87, 119, 27, 75, 59, 22, 123, 98, 86, 53, 107, 54, 88, 99, 125, 23, 61, 76, 29,$
$120, 89, 71, 48, 42),$
$(153, 43, 50, 72, 91, 121, 31, 77, 63, 24, 127, 100, 90, 55, 111, 56, 92, 101, 129, 25, 38, 78, 33,$
$122, 93, 73, 52, 44)\}.$
(76 edges, 76 all lengths)
This starter comprises a balanced (C_{10}, C_{28})-4-foil decomposition of K_{153}.

Example 3.3. Balanced (C_{10}, C_{28})-6-foil decomposition of K_{229}.
$\{(229, 1, 86, 212, 102, 203, 101, 213, 88, 2),$
$(229, 3, 90, 214, 100, 199, 99, 215, 92, 4),$
$(229, 5, 94, 216, 98, 195, 97, 217, 96, 6)\}$
$\cup$
$\{(229, 61, 68, 104, 129, 177, 39, 111, 87, 32, 183, 146, 128, 79, 159, 80, 130, 147, 185, 33, 89,$
$112, 41, 178, 131, 105, 70, 62),$
$(229, 63, 72, 106, 133, 179, 43, 113, 91, 34, 187, 148, 132, 81, 163, 82, 134, 149, 189, 35, 93,$
$114, 45, 180, 135, 107, 74, 64),$
$(229, 65, 76, 108, 137, 181, 47, 115, 95, 36, 191, 150, 136, 83, 167, 84, 138, 151, 193, 37, 56,$
$116, 49, 182, 139, 109, 78, 66)\}.$
(114 edges, 114 all lengths)
This starter comprises a balanced (C_{10}, C_{28})-6-foil decomposition of K_{229}.

Example 3.4. Balanced (C_{10}, C_{28})-8-foil decomposition of K_{305}.
$\{(305, 1, 114, 282, 136, 271, 135, 283, 116, 2),$
$(305, 3, 118, 284, 134, 267, 133, 285, 120, 4),$
$(305, 5, 122, 286, 132, 263, 131, 287, 124, 6),$
$(305, 7, 126, 288, 130, 259, 129, 289, 128, 8)\}$
$\cup$
$\{(305, 81, 90, 138, 171, 235, 51, 147, 115, 42, 243, 194, 170, 105, 211, 106, 172, 195, 245, 43,$
$117, 148, 53, 236, 173, 139, 92, 82),$
$(305, 83, 94, 140, 175, 237, 55, 149, 119, 44, 247, 196, 174, 107, 215, 108, 176, 197, 249, 45,$
$121, 150, 57, 238, 177, 141, 96, 84),$
$(305, 85, 98, 142, 179, 239, 59, 151, 123, 46, 251, 198, 178, 109, 219, 110, 180, 199, 253, 47,$
$125, 152, 61, 240, 181, 143, 100, 86),$
$(305, 87, 102, 144, 183, 241, 63, 153, 127, 48, 255, 200, 182, 111, 223, 112, 184, 201, 257, 49,$
$74, 154, 65, 242, 185, 145, 104, 88)\}.$
(152 edges, 152 all lengths)
This starter comprises a balanced (C_{10}, C_{28})-8-foil decomposition of K_{305}.

Example 3.5. Balanced (C_{10}, C_{28})-10-foil decomposition of K_{381}.
$\{(381, 1, 142, 352, 170, 339, 169, 353, 144, 2),$
$(381, 3, 146, 354, 168, 335, 167, 355, 148, 4),$
$(381, 5, 150, 356, 166, 331, 165, 357, 152, 6),$
$(381, 7, 154, 358, 164, 327, 163, 359, 156, 8),$
$(381, 9, 158, 360, 162, 323, 161, 361, 160, 10)\}$
$\cup$
$\{(381, 101, 112, 172, 213, 293, 63, 183, 143, 52, 303, 242, 212, 131, 263, 132, 214, 243, 305, 53,$
$145, 184, 65, 294, 215, 173, 114, 102),$
$(381, 103, 116, 174, 217, 295, 67, 185, 147, 54, 307, 244, 216, 133, 267, 134, 218, 245, 309, 55,$
$149, 186, 69, 296, 219, 175, 118, 104),$
$(381, 105, 120, 176, 221, 297, 71, 187, 151, 56, 311, 246, 220, 135, 271, 136, 222, 247, 313, 57,$
$153, 188, 73, 298, 223, 177, 122, 106),$

$(381, 107, 124, 178, 225, 299, 75, 189, 155, 58, 315, 248, 224, 137, 275, 138, 226, 249, 317, 59, 157, 190, 77, 300, 227, 179, 126, 108)$,
$(381, 109, 128, 180, 229, 301, 79, 191, 159, 60, 319, 250, 228, 139, 279, 140, 230, 251, 321, 61, 92, 192, 81, 302, 231, 181, 130, 110)\}$.
(190 edges, 190 all lengths)
This starter comprises a balanced (C_{10}, C_{28})-10-foil decomposition of K_{381}.

4. Balanced C_{38}-Foil Designs

Let K_n denote *the complete graph* of n vertices. Let C_{38} be *the* 38*-cycle*. *The* C_{38}*-t-foil* is a graph of t edge-disjoint C_{38}'s with a common vertex and the common vertex is called *the center of the* C_{38}*-t-foil*. In particular, the C_{38}-2-foil and the C_{38}-3-foil are called *the* C_{38}*-bowtie* and *the* C_{38}*-trefoil*, respectively. When K_n is decomposed into edge-disjoint sum of C_{38}-t-foils, it is called that K_n *has a* C_{38}*-t-foil decomposition*. Moreover, when every vertex of K_n appears in the same number of C_{38}-t-foils, it is called that K_n *has a balanced* C_{38}*-t-foil decomposition* and this number is called *the replication number*. This decomposition is to be known as *a balanced* C_{38}*-t-foil design*.

Theorem 4. K_n has a balanced C_{38}-t-foil decomposition if and only if $n \equiv 1 \pmod{76t}$.

Proof. (Necessity) Suppose that K_n has a balanced C_{38}-t-foil decomposition. Let b be the number of C_{38}-t-foils and r be the replication number. Then $b = n(n-1)/76t$ and $r = (37t+1)(n-1)/76t$. Among r C_{38}-t-foils having a vertex v of K_n, let r_1 and r_2 be the numbers of C_{38}-t-foils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $2tr_1 + 2r_2 = n - 1$. From these relations, $r_1 = (n-1)/76t$ and $r_2 = 37(n-1)/76$. Therefore, $n \equiv 1 \pmod{76t}$ is necessary.
(Sufficiency) Put $n = 76st + 1, T = st$. Then $n = 76T + 1$. Construct a C_{38}-T-foil as follows:
$\{\ (76T+1, 2T, 32T, 72T+1, 32T+1, 52T+2, 20T+1, 22T+2, 34T+2, 42T+3, 58T+3, 12T+3, 36T+3, 28T+3, 10T+2, 60T+3, 48T+2, 42T+2, 26T+1, 52T+3, 26T+2, 42T+4, 48T+3, 60T+5, 10T+3, 28T+5, 36T+4, 12T+5, 58T+4, 42T+5, 34T+3, 22T+4, 20T+2, 52T+4, 32T+2, 72T, 32T-2, 2T-1)$,
$(76T+1, 2T-2, 32T-4, 72T-1, 32T+3, 52T+6, 20T+3, 22T+6, 34T+4, 42T+7, 58T+5, 12T+7, 36T+5, 28T+7, 10T+4, 60T+7, 48T+4, 42T+6, 26T+3, 52T+7, 26T+4, 42T+8, 48T+5, 60T+9, 10T+5, 28T+9, 36T+6, 12T+9, 58T+6, 42T+9, 34T+5, 22T+8, 20T+4, 52T+8, 32T+4, 72T-2, 32T-6, 2T-3)$,
$(76T+1, 2T-4, 32T-8, 72T-3, 32T+5, 52T+10, 20T+5, 22T+10, 34T+6, 42T+11, 58T+7, 12T+11, 36T+7, 28T+11, 10T+6, 60T+11, 48T+6, 42T+10, 26T+5, 52T+11, 26T+6, 42T+12, 48T+7, 60T+13, 10T+7, 28T+13, 36T+8, 12T+13, 58T+8, 42T+13, 34T+7, 22T+12, 20T+6, 52T+12, 32T+6, 72T-4, 32T-10, 2T-5)$,
...,
$(76T+1, 2, 28T+4, 70T+3, 34T-1, 56T-2, 22T-1, 26T-2, 36T, 46T-1, 60T+1, 16T-1, 38T+1, 32T-1, 12T, 64T-1, 50T, 46T-2, 28T-1, 56T-1, 28T, 46T, 50T+1, 64T+1, 12T+1, 18T+2, 38T+2, 16T+1, 60T+2, 46T+1, 36T+1, 26T, 22T, 56T, 34T, 70T+2, 28T+2, 1)\ \}$.
($38T$ edges, $38T$ all lengths)
Decompose this C_{38}-T-foil into s C_{38}-t-foils. Then these starters comprise a balanced C_{38}-t-foil decomposition of K_n.

Example 4.1. Balanced C_{38}-decomposition of K_{77}.
$\{(77, 2, 32, 73, 33, 54, 21, 24, 36, 45, 61, 15, 39, 31, 12, 63, 50, 44, 27, 55, 28, 46, 51, 65, 13, 20, 40, 17, 62, 47, 37, 26, 22, 56, 34, 72, 30, 1)\}$.
(38 edges, 38 all lengths)
This stater comprises a balanced C_{38}-decomposition of K_{77}.

Example 4.2. Balanced C_{38}-2-foil decomposition of K_{153}.
$\{(153, 4, 64, 145, 65, 106, 41, 46, 70, 87, 119, 27, 75, 59, 22, 123, 98, 86, 53, 107, 54, 88, 99, 125, 23, 61, 76, 29, 120, 89, 71, 48, 42, 108, 66, 144, 62, 3)$,
$(153, 2, 60, 143, 67, 110, 43, 50, 72, 91, 121, 31, 77, 63, 24, 127, 100, 90, 55, 111, 56, 92, 101, 129, 25, 38, 78, 33, 122, 93, 73, 52, 44, 112, 68, 142, 58, 1)\}$.

(76 edges, 76 all lengths)

This stater comprises a balanced C_{38}-2-foil decomposition of K_{153}.

Example 4.3. Balanced C_{38}-3-foil decomposition of K_{229}.

$\{(229, 6, 96, 217, 97, 158, 61, 68, 104, 129, 177, 39, 111, 87, 32, 183, 146, 128, 79, 159, 80, 130, 147, 185, 33, 89, 112, 41, 178, 131, 105, 70, 62, 160, 98, 216, 94, 5),$
$(229, 4, 92, 215, 99, 162, 63, 72, 106, 133, 179, 43, 113, 91, 34, 187, 148, 132, 81, 163, 82, 134, 149, 189, 35, 93, 114, 45, 180, 135, 107, 74, 64, 164, 100, 214, 90, 3),$
$(229, 2, 88, 213, 101, 166, 65, 76, 108, 137, 181, 47, 115, 95, 36, 191, 150, 136, 83, 167, 84, 138, 151, 193, 37, 56, 116, 49, 182, 139, 109, 78, 66, 168, 102, 212, 86, 1)\}.$

(114 edges, 114 all lengths)

This stater comprises a balanced C_{38}-3-foil decomposition of K_{229}.

Example 4.4. Balanced C_{38}-4-foil decomposition of K_{305}.

$\{(305, 8, 128, 289, 129, 210, 81, 90, 138, 171, 235, 51, 147, 115, 42, 243, 194, 170, 105, 211, 106, 172, 195, 245, 43, 117, 148, 53, 236, 173, 139, 92, 82, 212, 130, 288, 126, 7),$
$(305, 6, 124, 287, 131, 214, 83, 94, 140, 175, 237, 55, 149, 119, 44, 247, 196, 174, 107, 215, 108, 176, 197, 249, 45, 121, 150, 57, 238, 177, 141, 96, 84, 216, 132, 286, 122, 5),$
$(305, 4, 120, 285, 133, 218, 85, 98, 142, 179, 239, 59, 151, 123, 46, 251, 198, 178, 109, 219, 110, 180, 199, 253, 47, 125, 152, 61, 240, 181, 143, 100, 86, 220, 134, 284, 118, 3),$
$(305, 2, 116, 283, 135, 222, 87, 102, 144, 183, 241, 63, 153, 127, 48, 255, 200, 182, 111, 223, 112, 184, 201, 257, 49, 74, 154, 65, 242, 185, 145, 104, 88, 224, 136, 282, 114, 1)\}.$

(152 edges, 152 all lengths)

This stater comprises a balanced C_{38}-4-foil decomposition of K_{305}.

Example 4.5. Balanced C_{38}-5-foil decomposition of K_{381}.

$\{(381, 10, 160, 361, 161, 262, 101, 112, 172, 213, 293, 63, 183, 143, 52, 303, 242, 212, 131, 263, 132, 214, 243, 305, 53, 145, 184, 65, 294, 215, 173, 114, 102, 264, 162, 360, 158, 9),$
$(381, 8, 156, 359, 163, 266, 103, 116, 174, 217, 295, 67, 185, 147, 54, 307, 244, 216, 133, 267, 134, 218, 245, 309, 55, 149, 186, 69, 296, 219, 175, 118, 104, 268, 164, 358, 154, 7),$
$(381, 6, 152, 357, 165, 270, 105, 120, 176, 221, 297, 71, 187, 151, 56, 311, 246, 220, 135, 271, 136, 222, 247, 313, 57, 153, 188, 73, 298, 223, 177, 122, 106, 272, 166, 356, 150, 5),$
$(381, 4, 148, 355, 167, 274, 107, 124, 178, 225, 299, 75, 189, 155, 58, 315, 248, 224, 137, 275, 138, 226, 249, 317, 59, 157, 190, 77, 300, 227, 179, 126, 108, 276, 168, 354, 146, 3),$
$(381, 2, 144, 353, 169, 278, 109, 128, 180, 229, 301, 79, 191, 159, 60, 319, 250, 228, 139, 279, 140, 230, 251, 321, 61, 92, 192, 81, 302, 231, 181, 130, 110, 280, 170, 352, 142, 1)\}.$

(190 edges, 190 all lengths)

This stater comprises a balanced C_{38}-5-foil decomposition of K_{381}.

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