# 木の $(p,q)$-全ラベリング問題

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グラフ $G$ の $k$-$(p,q)$-全ラベリングとは，$G$ の節点と辺への 0 から $k$ までの整数値の割当てであり，節点とそれに接続する辺の間では少なくとも $p$，隣接する 2 節点間または 2 辺間では少なくとも $q$ の差があるもののことをいう．$G$ の $(p,q)$-全ラベリング数 $\lambda^T_{p,q}(G)$ は，$k$ がとりうる値の最小値として定義される．$G$ の $(p,q)$-全ラベリング問題とは，ラベリング数 $\lambda^T_{p,q}(G)$ を求める問題である．本研究では，いくつかのグラフクラスのグラフの $(p,q)$-全ラベリング数に対して新しい上界と下界を与える．とくに，グラフが木の場合にはタイトな上界と下界を与える．また，グラフが木で $p \leq 3q/2$ の場合に $(p,q)$-全ラベリング問題を解く線形時間アルゴリズムを与え，さらに最大次数が 4 以上という仮定が加わった場合，$\lambda^T_{p,q}(T)$ を実現する木 $T$ を完全に特徴づけられることも示す．この結果は，$(p,q)$-全ラベリング問題の一般化である $L(p,q)$-ラベリング問題が $q$ が $p$ の約数でないすべての $(p,q)$ の組に対し NP 困難であることと対照的である．

# The $(p,q)$-total labeling problem for trees

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A $(p,q)$-total labeling of a graph $G$ is an assignment $f$ from the vertex set $V(G)$ and the edge set $E(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ is a vertex and $y$ is an edge incident to $x$, and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are a pair of adjacent vertices or a pair of adjacent edges, for all $x$ and $y$ in $V(G) \cup E(G)$. A $k$-$(p,q)$-total labeling is a $(p,q)$-total labeling $f : V(G) \cup E(G) \to \{0,\ldots,k\}$, and the $(p,q)$-total labeling problem asks the minimum $k$, which we denote by $\lambda^T_{p,q}(G)$, among all possible assignments. In this paper, we first give new upper and lower bounds on $\lambda^T_{p,q}(G)$ for some classes of graphs $G$, in particular, tight bounds on $\lambda^T_{p,q}(T)$ for trees $T$. We then show that if $p \leq 3q/2$, the problem for trees $T$ is linearly solvable, and give a complete characterization of trees achieving $\lambda^T_{p,q}(T)$ if in addition $\Delta \geq 4$ holds, where $\Delta$ is the maximum degree of $T$. It is contrasting to the fact that the $L(p,q)$-labeling problem, which is a generalization of the $(p,q)$-total labeling problem, is NP-hard for any two positive integers $p$ and $q$ such that $q$ is not a divisor of $p$.

## 1. Introduction

In the channel/frequency assignment problems, we need to assign different frequencies to 'close' transmitters so that they can avoid interference. The $L(p,q)$-labelings of a graph have been extensively studied as one of important graph theoretical models of this problem. An $L(p,q)$-*labeling* of a graph $G$ is an assignment $f$ from the vertex set $V(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ and $y$ are adjacent and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are at distance 2, for all $x$ and $y$ in $V(G)$. A $k$-$L(p,q)$-labeling is an $L(p,q)$-labeling $f : V(G) \to \{0,\ldots,k\}$, and the $L(p,q)$-*labeling problem* asks the minimum $k$, which we denote by $\lambda_{p,q}(G)$, among all possible assignments. Notice that we can use $k+1$ different labels when $\lambda_{p,q}(G) = k$ since we can use 0 as a label for conventional reasons. We can find a lot of related results on $L(p,q)$-labelings in comprehensive surveys by Calamoneri[3] and by Yeh[25]. From the applicational point of view, we assume that $p \geq q \geq 1$ unless otherwise stated. Also, we assume that $p$ and $q$ are relatively prime, since otherwise, an $L(p,q)$-labeling is equivalent to an $L(p/\ell, q/\ell)$-labeling, where $\ell = \gcd(p,q)$.

**$(p,q)$-total labeling and a conjecture**

In[24], Whittlesey et al. studied the $L(2,1)$-labeling number of incidence graphs, where the *incidence graph* of a graph $G$ is the graph obtained from $G$ by replacing each edge $(v_i, v_j)$ with two edges $(v_i, v_{ij})$ and $(v_{ij}, v_j)$ after introducing one new vertex $v_{ij}$. Observe that an $L(p,q)$-labeling of the incidence graph of a given graph $G$ can be regarded as an assignment $f$ from $V(G) \cup E(G)$ to the set of nonnegative integers such that $|f(x) - f(y)| \geq p$ if $x$ is a vertex and $y$ is an edge incident to $x$, and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are a pair of adjacent vertices or a pair of adjacent edges, for all $x$ and $y$ in $V(G) \cup E(G)$. Such a labeling of $G$ is called a $(p,q)$-*total labeling* of $G$, while the case of $q = 1$ was first introduced as a $(p,1)$-total labeling by Havet and Yu[15),16]. In particular, a $k$-$(p,q)$-total labeling is a $(p,q)$-total labeling $f : V(G) \cup E(G) \to \{0,\ldots,k\}$, and the $(p,q)$-*total labeling problem* asks the minimum $k$ among all possible assignments. We call this invariant, the minimum $k$, the $(p,q)$-*total labeling number*, which is denoted by $\lambda^T_{p,q}(G)$.

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We notice that a $(1,1)$-total labeling of $G$ is equivalent to a total coloring of $G$. Generalizing the Total Coloring Conjecture[2),22)], Havet and Yu[15),16)] conjectured that

$$\lambda^T_{p,1}(G) \leq \Delta + 2p - 1 \qquad (1)$$

holds for any graph $G$, where $\Delta$ denotes the maximum degree of a vertex in $G$. They also investigated bounds on $\lambda^T_{p,1}(G)$ under various assumptions and some of their results are described as follows: (i) $\lambda^T_{p,1}(G) \geq \Delta+p-1$, (ii) $\lambda^T_{p,1}(G) \geq \Delta+p$ if $p \geq \Delta$, (iii) $\lambda^T_{p,1}(G) \leq \min\{2\Delta+p-1, \chi(G)+\chi'(G)+p-2\}$ for any graph $G$ where $\chi(G)$ and $\chi'(G)$ denote the chromatic number and the chromatic index of $G$, respectively, and (iv) $\lambda^T_{p,1}(G) \leq n+2p-2$ if $G$ is the complete graph where $n = |V(G)|$. In particular, it follows by (iii) that if $G$ is bipartite, then $\lambda^T_{p,1}(G) \leq \Delta+p$ holds (by $\chi(G) \leq 2$ and König's theorem), and if in addition, $p \geq \Delta$, then $\lambda^T_{p,1}(G) = \Delta+p$ by (ii)[1),15),16)]. Also, Bazzaro et al.[1)] showed that $\lambda^T_{p,1}(G) \leq \Delta+p+s$ for any $s$-degenerated graph (by $\chi(G) \leq s+1$ and $\chi'(G) \leq \Delta+1$), where an *$s$-degenerated graph $G$* is a graph which can be reduced to a trivial graph by successive removal of vertices with degree at most $s$, that $\lambda^T_{p,1}(G) \leq \Delta+p+3$ for any planar graph (by the Four-Color Theorem), and that $\lambda^T_{p,1}(G) \leq \Delta+p+1$ for any outerplanar graph other than an odd cycle (since any outerplanar graph is 2-degenerated, and any outerplanar graph other than an odd cycle satisfies $\chi'(G) = \Delta$[9)]). Also, there are many related works about bounds on $\lambda^T_{p,1}(G)$[6),14),19),20)]. From the algorithmic point of view, Havet and Thomassé[17)] recently showed that for bipartite graphs, if (i) $p \geq \Delta$ or (ii) $\Delta = 3$ and $p = 2$, then the $(p,1)$-total labeling problem is polynomially solvable and otherwise it is NP-hard.

In[7),18),21)], the $(r,s,t)$-coloring problem which is a generalization of the $(p,q)$-total labeling problem was studied, while results in the cases corresponding to the $(p,q)$-total labeling problem (actually, the cases of $t \geq r = s$) are limited to paths, cycles, stars or the complete graph with some $p$ and $q$. To our best knowledge, there are quite few studies about $(p,q)$-total labelings other than these. In this paper, we focus on the $(p,q)$-total labeling problem for some classes of graphs, especially for trees.

**$L(p,q)$-labelings and $(p,q)$-total labelings of trees**

Let $T$ be a tree. As for the $L(2,1)$-labeling problem, Griggs and Yeh[12)] showed that $\lambda_{2,1}(T) \in \{\Delta+1, \Delta+2\}$. Moreover, Chang and Kuo[5)] showed that $\lambda_{2,1}(T)$ can be computed in polynomial time, and recently Hasunuma et al.[13)] gave a linear time algorithm for this problem. However, a characterization of trees $T$ achieving $\lambda_{2,1}(T)$ is still open. Also, it was shown in[4)] that $\lambda_{p,1}(T) \leq \min\{\Delta+2p-2, 2\Delta+p-2\}$ and that $L(p,1)$-labeling problem for trees can be solved in $O((p+\Delta)^{5.5}n) = O(\lambda_{p,1}(T)^{5.5}n)$ time by extending the algorithm in[5)], where $n = |V(G)|$. On the other hand, Fiala et al. [8)] showed that the $L(p,q)$-labeling problem for trees is NP-hard for any two positive integers $p$ and $q$ such that $q$ is not a divisor of $p$. Concerning bounds on $\lambda_{p,q}(T)$, Georges and Mauro[11)] gave the exact value of $\lambda_{p,q}(T)$ for the infinite regular trees $T$, which gives tight upper bounds on $\lambda_{p,q}(T)$; following their results, we have $\lambda_{p,q}(T) \leq p+(2\Delta-2)q$.

As for the $(p,q)$-total labeling problem, the above algorithms can also be applied to the case of $q = 1$, since the incidence graph of a tree is also a tree. Moreover, due to the structure of the incidence graph of a tree, it can be observed that $\lambda^T_{p,1}(T)$ becomes much smaller than $\lambda_{p,1}(T)$. By bounds for bipartite graphs in[1),15),16)], it follows that $\lambda^T_{p,1}(T) \in \{p+\Delta-1, p+\Delta\}$, and that if $p \geq \Delta$, then $\lambda^T_{p,1}(T) = p+\Delta$. Recently, Wang and Chen[23)] gave a characterization of trees $T$ achieving $\lambda^T_{2,1}(T)$ in the case of $\Delta = 3$.

**Our contributions**

In this paper, we mainly focus on $(p,q)$-total labeling problem for trees for general $p$ and $q$, and obtain the following results:

- (Upper bounds on $\lambda^T_{p,q}(T)$) If $p = q+r$ for $r \in \{0,1,\ldots,q-1\}$ and $\Delta > 1$ (resp., $\Delta = 1$), then $\lambda^T_{p,q}(T) \leq p+(\Delta-1)q+r$ holds and this bound is tight (resp., $\lambda^T_{p,q}(T) = p+q$). If $p \geq 2q$, then $\lambda^T_{p,q}(T) \leq p+\Delta q$ holds and this bound is tight. In particular, if $p \geq \Delta q$, then $\lambda^T_{p,q}(T) = p+\Delta q$.
- (Lower bounds on $\lambda^T_{p,q}(T)$) If $q \leq p < (\Delta-1)q$, then $\lambda^T_{p,q}(T) \geq p+(\Delta-1)q$ holds and this bound is tight. If $p = (\Delta-1)q+r$ for $r \in \{0,1,\ldots,q-1\}$, then $\lambda^T_{p,q}(T) \geq p+(\Delta-1)q+r$ holds and this bound is tight. If $p \geq \Delta q$, then $\lambda^T_{p,q}(T) = p+\Delta q$.
- The $(p,q)$-total labeling problem with $p \leq 3q/2$ for trees can be solved in linear time. In particular, if $\Delta \geq 2$, we have $\lambda^T_{p,q}(T) \in \{p+(\Delta-1)q, p+(\Delta-1)q+r\}$. If $p > q$ and $\Delta \geq 4$, then $\lambda^T_{p,q}(T) = p+(\Delta-1)q$ holds if and only if no two vertices with degree $\Delta$ are adjacent.
- In the case of $p = 2q$, the condition that no two vertices with degree $\Delta$ are adjacent is sufficient for $\lambda^T_{p,q}(T) = p+(\Delta-1)q$, while in the case of $p > 3q/2$ and $p \neq 2q$, this condition is not sufficient.
- For any two nonnegative integers $p$ and $q$, the $L(p,q)$-labeling problem for trees can be solved in polynomial time if $\Delta = O(\log^{1/3}|I|)$ where $|I| = \max\{|V(T)|, \log p\}$. Particularly, if $\Delta$ is a fixed constant, it is solved in linear time.

The first and second results provide tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for all pairs $(p,q)$ with $p \geq q$. The first statement in the third result indicates that as for the $(p,q)$-total labeling

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problem for trees, there exists a tractable case even if $q$ is not a divisor of $p$, in contrast to the NP-hardness of the $L(p, q)$-labeling problem. The second and third statements in the third result completely characterize trees $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $p \leq 3q/2$ and $\Delta \geq 4$ (note that if $p = q$, we have $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ by the first and second results). This is also contrasting to the fact that no simple characterization of trees $T$ achieving $\lambda_{2,1}(T)$ is known even for the $L(2, 1)$-labeling problem.

**Organization of the paper**

The rest of this paper is organized as follows. In Section 2, after giving some basic definitions, we show several properties about bounds on $\lambda^T_{p,q}(G)$. In Sections 3 and 4, we focus on the cases where a given graph is a tree. Section 3 provides tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for trees $T$. In Section 4, we propose a linear time algorithm for solving the $(p, q)$-total labeling problem with $p \leq 3q/2$ for trees, and give a characterization of trees $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $p > q$ and $\Delta \geq 4$. Also, we discuss the case of $p > 3q/2$. Finally, we give concluding remarks in Section 5. Some parts of the detailed analyses are omitted due to space limitation.

## 2. Bounds on $\lambda^T_{p,q}(G)$

In this section, we investigate several properties on $(p, q)$-total labelings of a graph $G$. For this, we first define some terminology. A graph $G$ is an ordered set of its vertex set $V(G)$ and edge set $E(G)$ and is denoted by $G = (V(G), E(G))$. We assume throughout this paper that a graph is undirected, simple and connected unless otherwise stated. Therefore, an edge $e \in E(G)$ is an unordered pair of vertices $u$ and $v$, which are *end vertices* of $e$, and we often denote it by $e = (u, v)$. Let $N_G(v)$ denote the set of neighbors of a vertex $v$ in $G$; $N_G(v) = \{u \in V \mid (u, v) \in E(G)\}$. The *degree* of a vertex $v$ is $|N_G(v)|$, and is denoted by $d_G(v)$. A vertex $v$ with $d_G(v) = k$ is called a *k-vertex*. We use $\Delta(G)$ (resp., $\delta(G)$) to denote the maximum (resp., minimum) degree of a vertex in a graph $G$. A $\Delta(G)$-vertex is called *major*. We often drop $G$ in these notations if there are no confusions. For a $(p, q)$-total labeling $f : V(G) \cup E(G) \to \{0, 1, \ldots, k\}$ of $G$ and an edge $e = (u, v) \in E(G)$, we may denote $f(e)$ by $f(u, v)$. Let $\overline{f}$ denote the labeling such that $\overline{f}(z) = k - f(z)$ for each $z \in V(G) \cup E(G)$. Note that $\overline{f}$ is also a $(p, q)$-total labeling of $G$.

We have the following lemmas about upper and lower bounds on $\lambda^T_{p,q}(G)$, some of which are extensions of those discussed in the case of $q = 1$[16].

**Lemma 1** (*i*) $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q$.
(*ii*) If $G$ has a major vertex whose neighbors are all major, then $\lambda^T_{p,q}(G) \geq p + \Delta q$ holds for $p \geq 2q$, and $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q + r$ holds for $p = q + r$ $(r = 0, 1, \ldots, q - 1)$.
(*iii*) $\lambda^T_{p,q}(G) \geq p + \min\{p, \Delta q\}$. Hence, $\lambda^T_{p,q}(G) \geq p + (\Delta - 1)q + r$ holds where $r = q$ if $p \geq \Delta q$ and $r = p - (\Delta - 1)q$ otherwise. □

**Lemma 2** (*i*) $\lambda^T_{p,q}(G) \leq p + q(\chi(G) + \chi'(G) - 2)$.
(*ii*) $\lambda^T_{p,q}(G) \leq p + (2\Delta - 1)q$.
(*iii*) Let $G$ be the complete graph. Then, $\lambda^T_{p,q}(G) \leq \min\{p + (2\Delta - 1)q, 2p + \Delta q - 1\}$. In particular, $\lambda^T_{p,q}(G) = p + (2\Delta - 1)q$ if $p \geq 2\Delta q + 1$ and $|V(G)| \geq 3$. □

In the case where $G$ is a path (resp., cycle), the incidence graph of $G$ is also a path (resp., cycle). The following lemma is obtained directly from Georges and Mauro's results about $L(p, q)$-labeling of paths or cycles[10].

**Lemma 3** (*i*) Let $G$ be a path. We have $\lambda^T_{p,q}(G) = p + q$ (resp., $p + 2q$, resp., $2p$) if $|V| = 2$ (resp., $|V| \geq 3$ and $p \geq 2q$, resp., $|V| \geq 3$ and $p \leq 2q$).
(*ii*) Let $G$ be a cycle. We have $\lambda^T_{p,q}(G) = p + 2q$ if (*a*) $|V|$ is even and $p \geq 2q$ or (*b*) $2|V| \neq 0 \bmod 3$ and $p \leq 2q$, $\lambda^T_{p,q}(G) = p + 3q$ if $|V|$ is odd and $p \geq 3q$, and $\lambda^T_{p,q}(G) = 2p$ otherwise. □

Similarly to the arguments in Section 1, we can see that the following properties hold, where a graph is called *a series-parallel graph* or *a partial 2-tree* if it contains no subgraph isomorphic to a subdivision of the complete graph with four vertices. Notice that an outerplanar graph is series-parallel.

**Corollary 4** (*i*) If $G$ is bipartite, then $\lambda^T_{p,q}(G) \leq p + \Delta q$. In particular, if $p \geq \Delta q$, then $\lambda^T_{p,q}(G) = p + \Delta q$.
(*ii*) If $G$ is $s$-degenerated, then $\lambda^T_{p,q}(G) \leq p + (\Delta + s)q$.
(*iii*) If $G$ is planar, then $\lambda^T_{p,q}(G) \leq p + (\Delta + 3)q$.
(*iv*) If $G$ is series-parallel, then $\lambda^T_{p,q}(G) \leq p + (\Delta + 1)q$. □

## 3. Tight bounds on $\lambda^T_{p,q}(G)$ for trees

In this section, we show the following properties about tight upper and lower bounds on $\lambda^T_{p,q}(T)$ for trees $T$.

**Theorem 5** Let $T$ be a tree. Then the following properties hold.
(*i*) If $p \geq \Delta q$, then $\lambda^T_{p,q}(T) = p + \Delta q$.
(*ii*) If $p = (\Delta - 1)q + r$ $(r = 0, 1, \ldots, q - 1)$, then $\lambda^T_{p,q}(T) \geq p + (\Delta - 1)q + r$ and this bound is tight.

(*iii*) If $p \geq 2q$, then $\lambda^T_{p,q}(T) \leq p + \Delta q$ and this bound is tight.
(*iv*) If $p = q + r$ ($r = 0, 1, \ldots, q-1$) and $\Delta > 1$ (resp., $\Delta = 1$), then $\lambda^T_{p,q}(T) \leq p + (\Delta-1)q + r$ and this bound is tight (resp., $\lambda^T_{p,q}(T) = p + q$).

Since trees are bipartite, the statement (i) and the former part of the statement (iii) follow from Corollary 4 (i). The former part of the statement (ii) follows from Lemma 1 (iii), and it is not difficult to see that a star $T$ achieves $\lambda^T_{p,q}(T) = p + (\Delta-1)q + r$. Lemma 1 (ii) indicates that a tree $T$ which has a major vertex whose neighbors are all major achieves $\lambda^T_{p,q}(T) = p + \Delta q$ (resp., $p + (\Delta-1)q + r$) if $p \geq 2q$ (resp., if $p = q + r$ ($< 2q$), $\Delta > 1$, and the former part of the statement (iv) is true). The case of $\Delta = 1$ in the statement (iv) follows from Lemma 3.

In the rest of this section, we give a proof of the former part of the case of $\Delta > 1$ in the statement (iv) to complete the proof of this theorem. For this, we assume that $\Delta \geq 2$ and give an algorithm for finding a $(p+(\Delta-1)q+r)$-$(p,q)$-total labeling of $T$ if $p = q+r$ for $r \in [0, q-1]$. For simplicity of description, assume that $T = (V, E)$ is a tree such that all non-leaves are major.

From[10] [Lemma 2.1], it follows that there exists a $\lambda^T_{p,q}(T)$-$(p,q)$-total labeling of $T$ which consists of labels with form $\alpha p + \beta q$ where $\alpha, \beta \in \mathbb{Z}_+$. Here we can assume that $\lambda^T_{p,q}(T) \geq p + (\Delta-1)q + r$ by Lemma 1(ii) and the assumption on $T$. Considering these two properties, we will seek a $(p + (\Delta-1)q + r)$-$(p,q)$-total labeling of $T$ with form $\alpha p + \beta q$. Then, it is not difficult to see that the candidates of labels of such a form to be assigned for each non-leaf vertex (i.e., major vertex) are 0, $p + (\Delta-1)q + r$ ($= 2p + (\Delta-2)q$), or $p + iq$ for some $i \in [0, \Delta-2]$. In particular, if a major vertex $v$ has label $p + iq$ for $i \in [0, \Delta-2]$, then the set of labels for edges incident to $v$ is $\{jq \mid j \in [0, i]\} \cup \{p + jq + r \mid j \in [i+1, \Delta-1]\}$. Based on these observations, we regard $T$ as a rooted tree by choosing a major vertex $v_r$ as the root, and assign labels with form $\alpha p + \beta q$ to $V \cup E$ from the root $v_r$ in the breadth-first-search order, as shown in Algorithm $(p,q)$-LABEL. Actually, we use labels 0, $p$, $(\Delta-1)q + r$ ($= p + (\Delta-2)q$), and $p + (\Delta-1)q + r$ for vertices, and repeat applying essentially four types of labelings to each scanned vertex, its incident edges, and its children. In the description of the algorithm, $p(v)$ denotes the parent of $v$ (if exists) and $C(v)$ denotes the set of children of $v$ for each vertex $v$.

**Algorithm** $(p,q)$-LABEL

**Input:** A tree $T = (V, E)$ with $\Delta \geq 2$ such that all non-leaves are major, and two positive integers $p$ and $q$ with $p = q + r$ and $r \in [0, q-1]$.

**Output:** A $(p,q)$-total labeling $f : V \cup E \to \{0, 1, \ldots, p + (\Delta-1)q + r\}$ of $T$.

Assign label 0 to the root $v_r$; let $f(v_r) := 0$. For each $i \in [0, \Delta-2]$, let $f(v_r, c_i(v_r)) := p + iq$ and $f(c_i(v_r)) := p + (\Delta-1)q + r$, where $C(v_r) = \{c_i(v_r) \mid i = 0, 1, \ldots, \Delta-1\}$ (i.e., assign labels $p + iq$ and $p + (\Delta-1)q + r$ to the edge $(v_r, c_i(v_r))$ and the child $c_i(v_r)$ of $v_r$, respectively). Let $f(v_r, c_{\Delta-1}(v_r)) := p + (\Delta-1)q + r$ and $f(c_{\Delta-1}(v_r)) := (\Delta-1)q + r$.
**while** there exists a non-leaf $v \in V - \{v_r\}$ such that $f(v)$ has been determined but no label is assigned to any child of $v$ where $C(v) = \{c_i(v) \mid i = 0, 1, \ldots, \Delta-2\}$ **do**
  **if** (Case-1) $f(p(v), v) \in \{p + iq \mid i \in [0, \Delta-2]\}$ and $f(v) = p + (\Delta-1)q + r$ **then**
    Let $f(c_i(v)) := 0$ for each $i \in [0, \Delta-3]$, $f(c_{\Delta-2}(v)) := p$, and $f(v, c_{\Delta-2}(v)) := 0$. Assign labels in $\{p+iq \mid i \in [0, \Delta-2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta-3]\}$.
  **else if** (Case-2) $f(p(v), v) = p + (\Delta-1)q + r$ and $f(v) = (\Delta-1)q + r$ **then**
    Let $f(c_i(v)) := p + (\Delta-1)q + r$ and $f(v, c_i(v)) := iq$ for each $i \in [0, \Delta-2]$.
  **else if** (Case-3) $f(p(v), v) \in \{iq \mid i \in [0, \Delta-2]\}$ and $f(v) = p + (\Delta-1)q + r$ **then**
    Let $f(c_i(v)) := (\Delta-1)q + r$ for each $i \in [0, \Delta-3]$, $f(c_{\Delta-2}(v)) := 0$, and $f(v, c_{\Delta-2}(v)) := (\Delta-1)q + r$. Assign labels in $\{iq \mid i \in [0, \Delta-2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta-3]\}$.
  **else if** (Case-4) $f(p(v), v) \in \{iq \mid i \in [0, \Delta-2]\}$ and $f(v) = (\Delta-1)q + r$ **then**
    Let $f(c_i(v)) := p + (\Delta-1)q + r$ for each $i \in [0, \Delta-3]$, $f(c_{\Delta-2}(v)) := 0$, and $f(v, c_{\Delta-2}(v)) := p + (\Delta-1)q + r$. Assign labels in $\{iq \mid i \in [0, \Delta-2]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta-3]\}$.
  **else if** (Case-$j'$) $\overline{f}(p(v), v))$ and $\overline{f}(v)$ satisfy the conditions of Case-$j$ for $j \in [1, 4]$ **then**
    After determining labels for $f(c_i(v))$ and $f(v, c_i(v))$ according to the above (Case-$j$) based on $\overline{f}(p(v), v))$ and $\overline{f}(v)$, let $f(c_i(v)) := p + (\Delta-1)q + r - f(c_i(v))$ and $f(v, c_i(v)) := p + (\Delta-1)q + r - f(v, c_i(v))$ for each $i \in [0, \Delta-2]$.
  **end if**
**end while**
Output $f$ as a $(p,q)$-total labeling of $T$.

We prove the correctness of Algorithm $(p,q)$-LABEL. Note that whenever a vertex $v$ is chosen in line 2, $f(p(v), v)$ has also been already determined. Also note that the labels assigned in each step do not violate the feasibility. Hence, it suffices to show that as a result of line 1 (resp., each iteration of the while loop in lines 2–14), each $c_i(v_r) \in C(v_r)$ (resp., $c_i(v) \in C(v)$) satisfies the conditions of Case-$j$ or Case-$j'$ in lines 2–14 for some $j \in \{1, 2, 3, 4\}$. As for the children of $v_r$, $c_i(v_r)$ for $i \in [0, \Delta - 2]$ satisfies the conditions of Case-1 and $c_{\Delta-1}(v_r)$ satisfies those of Case-2. Also as for the children of $v$ in each case of lines 2–14, we can prove this as follows, where Case-$j'$, $j' \in \{1, 2, 3, 4\}$ is omitted by symmetry of labelings:

(Case-1) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-1′ and $c_{\Delta-2}(v)$ satisfies those of Case-2′.
(Case-2) $c_i(v)$, $i \in [0, \Delta - 2]$ satisfies the conditions of Case-3.
(Case-3) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-4 and $c_{\Delta-2}(v)$ satisfies those of Case-1′.
(Case-4) $c_i(v)$, $i \in [0, \Delta - 3]$ satisfies the conditions of Case-3 and $c_{\Delta-2}(v)$ satisfies those of Case-3′.

Notice that in Case-1, $c_i(v)$ for $i \in [0, \Delta - 3]$ satisfies the conditions of Case-1′ because $\{p + iq \mid i \in [0, \Delta - 2]\} = \{p + (\Delta - 1)q + r - (p + iq) \mid i \in [0, \Delta - 2]\}$ by $p = q + r$. Consequently, the correctness of the algorithm is proved and hence the proof of Theorem 5 is completed.

Also, we remark that Algorithm $(p, q)$-LABEL can be implemented to run in linear time.

## 4. Algorithms for $(p, q)$-total labelings of trees

In this section, we consider an algorithm for finding an optimal $(p, q)$-total labeling (i.e., a $\lambda^T_{p,q}(T)$-$(p, q)$-total labeling) of trees $T$. Here, we focus on the cases of $\Delta \geq 3$ and $p > q$ since the case of $\Delta \leq 2$ has been shown as Lemma 3, and in the case of $\Delta \geq 3$ and $p = q$, we have $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ by Lemma 1 (i) and Theorem 5 and such a labeling can be found in linear time by Algorithm $(p, q)$-LABEL. We discuss the case of $p \leq 3q/2$ in Subsection 4.1 and other cases in Subsection 4.2.

### 4.1 Case: $p \leq 3q/2$

Assume that $p \leq 3q/2$. We show that the problem can be solved in linear time, and we give a complete characterization of trees $T$ with $\Delta \geq 4$ achieving $\lambda^T_{p,q}(T)$; namely, we have the following theorem.

**Theorem 6** Let $T$ be a tree with $p \leq 3q/2$.
($i$) An optimal $(p, q)$-total labeling (i.e., a $\lambda^T_{p,q}(T)$-$(p, q)$-total labeling) of $T$ can be found in linear time.
($ii$) In the case of $\Delta \geq 2$, we have $\lambda^T_{p,q}(T) \in \{p + (\Delta - 1)q, p + (\Delta - 1)q + r\}$ where $r = p - q$.
($iii$) In the case of $p > q$ and $\Delta \geq 4$, $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ if and only if
no two major vertices are adjacent in $T$. (2)

First we consider the case of $\Delta \geq 4$. In this case, for proving Theorem 6, it suffices to show the following two lemmas. Note that the first lemma holds for an arbitrary graph.

**Lemma 7** Let $G$ be a graph. If $p \leq 3q/2$ and $\lambda^T_{p,q}(G) < p + (\Delta - 1)q + r$ where $r = p - q$, then the condition (2) is satisfied. □

**Lemma 8** If $p \leq 3q/2$, the condition (2) is satisfied, and $\Delta \geq 4$, then $\lambda^T_{p,q}(G) = p + (\Delta - 1)q$ holds, and such a labeling can be found in linear time.

Recall that by Lemma 1 (i) and Theorem 5, we have $p + (\Delta - 1)q \leq \lambda^T_{p,q}(T) \leq p + (\Delta - 1)q + r$ where $r = p - q$. Hence, Lemmas 7 and 8 indicate that either $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$ or $\lambda^T_{p,q}(T) = p + (\Delta - 1)q + r$ holds, and that the former case is characterized by the condition (2). Furthermore, in both cases, an optimal labeling can be found in linear time by Lemma 8 and Algorithm $(p, q)$-LABEL. Thus, these two lemmas show Theorem 6 in the case of $\Delta \geq 4$.

On the other hand, in the case of $\Delta = 3$, there exist instances $T$ with $\lambda^T_{p,q}(T) > p + 2q$ even if the condition (2) holds. For example, consider a tree $T$ which contains the configuration (a) in Fig. 1 in which each major vertex is drawn by a black circle, and assume for contradiction that $T$ admits a $(p + 2q)$-$(p, q)$-total labeling $f$. Without loss of generality, let $f(u) = 0$ and $f(u, v) = p$ (note that the set of labels for edges incident to $u$ is $\{p, p + q, p + 2q\}$). By the feasibility of $f$, we have $f(v) \in [2p, p + 2q]$. Since $w$ is major, it follows that $f(w) = 0$, however, we cannot assign any label to the edge $(v, w)$. Similarly, we can observe that any tree which contains the configuration (b) in Fig. 1 cannot admit a $(p+2q)$-$(p, q)$-total labeling. We can observe that there are many other such instances, and it seems difficult to characterize instances $T$ achieving $\lambda^T_{p,q}(T)$ in the case of $\Delta = 3$.

Nevertheless, we can prove that the case of $\Delta = 3$ is linearly solvable and $\lambda^T_{p,q}(T) \in \{p + 2q, p + 2q + r\}$ in another way. The following lemma shows Theorem 6 in the case of $\Delta = 3$.

**Lemma 9** If $\Delta = 3$, then $\lambda^T_{p,q}(G) \in \{p + 2q, p + 2q + r\}$ holds, and such a labeling can be found in linear time. □

The latter part of Lemma 9 can be proved in a more general setting as the following theorem.

**Theorem 10** Let $|I| = \max\{|V(T)|, \log p\}$. For any nonnegative integers $p, q$, the $L(p, q)$-

labeling problem for trees (hence, the $(p, q)$-total labeling problem for trees also) can be solved in polynomial time, if $\Delta = O(\log^{1/3}|I|)$ for general $p$, or if $\Delta = O(\log^{1/2}|I|)$ for $p = \Omega(\Delta q)$. In particular, it can be solved in linear time, if $\Delta$ is bounded by a constant. □

Due to space limitation, we omit the proofs of Lemmas 7 and 9 and Theorem 10. We here give a proof of Lemma 8.

**Proof of Lemma 8.** Assume that a tree $T = (V, E)$ satisfies the condition (2) and $\Delta \geq 5$, while the case of $\Delta = 4$ is omitted due to space limitation. Also assume that the every non-major and non-leaf vertex in $T$ is a $(\Delta-1)$-vertex for simplicity of description. Then, we prove this lemma by showing that we can find a $(p+(\Delta-1)q)$-$(p, q)$-total labeling of $T$ according to Algorithm $(p, q)$-OPTLABELΔ5. The algorithm starts with choosing a major vertex $v_r$ as the root and assign labels to $V \cup E$ in the breadth-first-search order in a similar way to Algorithm $(p, q)$-TREE. Let $M$ denote the set of major vertices.

Observe that the labelings in each step do not violate the feasibility (note that $3q - p \geq p$ and $2q \geq p$ by $p \leq 3q/2$). Hence, for proving the correctness of the algorithm, we show that as a result of line 1 (resp., the while-loop in lines 2–19), each $c_i(v_r) \in C(v_r)$ (resp., $c_i(v) \in C(v)$) satisfies the conditions of Case-$j$ or Case-$j'$ of lines 2–19 for some $j \in \{1, 2, \ldots, 6\}$. Notice that by the condition (2), all children of each major vertex are non-major. Hence, $c_i(v_r)$, $i \in [1, \Delta-1]$ satisfies the conditions of Case-2 and $c_0(v_r)$ satisfies those of Case-5. Similarly, we can observe that the children of $v$ in Case-1 of lines 2–19 satisfy the conditions of Case-2 or Case-5. As for children of $v$ in other cases, we can prove this as follows:

(Case-2) $c_i(v)$ satisfies the conditions of Case-1, Case-3, or Case-4.
(Case-3) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, or Case-2′.
(Case-4) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, Case-5, or Case-6.
(Case-5) $c_i(v)$ satisfies the conditions of Case-1, Case-1′, Case-2, or Case-6.
(Case-6) $c_i(v)$ satisfies the conditions of Case-1 or Case-2.

Also, it is not difficult to see that Algorithm $(p, q)$-OPTLABELΔ5 can be implemented to run in linear time. □

**Algorithm** $(p, q)$-OPTLABELΔ5

**Input:** A tree $T = (V, E)$ satisfying the condition (2) and $\Delta \geq 5$ such that the degree of all non-major and non-leaf vertices is $\Delta - 1$, and two integers $p$ and $q$ with $p \leq 3q/2$.

**Output:** A $(p, q)$-total labeling $f : V \cup E \to \{0, 1, \ldots, p + (\Delta-1)q\}$ of $T$.

1: Assign label 0 to the root $v_r$; let $f(v_r) := 0$. For each $i \in [1, \Delta-1]$, let $f(v_r, c_i(v_r)) := p + iq$ and $f(c_i(v_r)) := q$, where $C(v_r) = \{c_i(v_r) \mid i = 0, 1, \ldots, \Delta-1\}$. Let $f(v_r, c_0(v_r)) := p$ and $f(c_0(v_r)) := 3q$. {Note that the root $v_r$ is major.}

2: **while** there exists a non-leaf $v \in V - \{v_r\}$ such that $f(v)$ has been determined but no label is assigned to any child of $v$ **do**

3: {Let $M$ denote the set of major vertices. Denote $C(v)$ by $\{c_i(v) \mid i = 0, 1, \ldots, |C(v)|-1\}$ such that $d(c_0(v)) \geq d(c_1(v)) \geq \cdots \geq d(c_{|C(v)|-1}(v))$ and let $j$ be an index such that $M \cap C(v) = \{c_i(v) \mid i \in [0, j]\}$ if $M \cap C(v) \neq \emptyset$, and $j = -1$ otherwise.}

4: **if** (Case-1) $v$ is major, $f(p(v), v) \in \{p + iq \mid i \in [0, \Delta-1]\}$ and $f(v) = 0$ **then**

5: Assign labels in $\{p + iq \mid i \in [0, \Delta-1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta-2]\}$ so that $f(v, c_0(v)) < f(v, c_1(v)) < \cdots < f(v, c_{\Delta-2}(v))$, and let $f(c_i(v)) := q$ for each $i \in [0, \Delta-2]$. Only if $f(v, c_0(v)) = p$, then relabel $c_0(v)$ as $f(c_0(v)) := 3q$.

6: **else if** (Case-2) $v$ is non-major, $f(p(v), v) \in \{p + iq \mid i \in [1, \Delta-1]\}$ and $f(v) = q$ **then**

7: Assign labels in $\{p + iq \mid i \in [1, \Delta-1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [0, \Delta-3]\}$ so that $f(v, c_0(v)) < f(v, c_1(v)) < \cdots < f(v, c_{\Delta-3}(v))$, let $f(c_i(v)) := 0$ for each $i \in [0, j]$ and $f(c_i(v)) := 2q$ for each $i \in [j+1, \Delta-3]$. Only if $f(v, c_0(v)) = p + q$ and $M \cap C(v) = \emptyset$, then relabel $c_0(v)$ as $f(c_0(v)) := 4q$.

8: **else if** (Case-3) $v$ is non-major, $f(p(v), v) \in \{p + iq \mid i \in [2, \Delta-1]\}$ and $f(v) = 2q$ **then**

9: Assign labels in $\{p + iq \mid i \in [2, \Delta-1]\} - \{f(p(v), v)\}$ injectively to edges $\{(v, c_i(v)) \mid i \in [1, \Delta-3]\}$, and let $f(c_i(v)) := 0$ for each $i \in [1, j]$, $f(c_i(v)) := q$ for each $i \in [\max\{1, j+1\}, \Delta-3]$, and $f(v, c_0(v)) := 0$. If $M \cap C(v) \neq \emptyset$, then let $f(c_0(v)) := p + (\Delta-1)q$ and otherwise let $f(c_0(v)) := p + (\Delta-2)q$.

10: **else if** (Case-4) $v$ is non-major, $f(p(v), v) = p + q$ and $f(v) = 4q$ **then**

11: Let $f(v, c_0(v)) := 0$, $f(v, c_i(v)) := p + (i+2)q$ for each $i \in [2, \Delta-3]$, $f(c_i(v)) := 0$ for each $i \in [2, j]$ and $f(c_i(v)) := q$ for each $i \in [\max\{2, j+1\}, \Delta-3]$. If $|M \cap C(v)| \geq 2$, then let $f(c_0(v)) := f(c_1(v)) := p + (\Delta-1)q$, and $f(v, c_1(v)) := q$. If $|M \cap C(v)| = 1$, then let $f(c_0(v)) := p + (\Delta-1)q$, $f(c_1(v)) := 3q$, and $f(v, c_1(v)) := p$. If $M \cap C(v) = \emptyset$, then let $f(c_0(v)) := 2q$, $f(c_1(v)) := 3q$, and $f(v, c_1(v)) := p$.

12: **else if** (Case-5) $v$ is non-major, $f(p(v), v) = p$ and $f(v) = 3q$ **then**

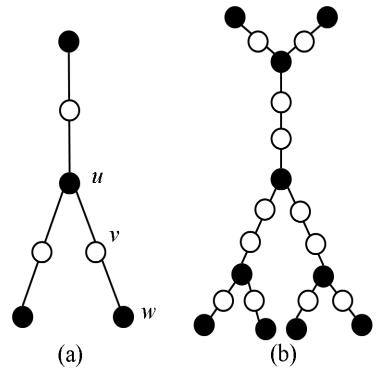


**図 1** Configurations that any tree $T$ with $\lambda_{p,q}(T) = p + 2q$ does not contain in the case of $\Delta = 3$, where each major vertex is drawn by a black circle.

Let $f(v, c_0(v)) := 0$, $f(v, c_i(v)) := p + (i+2)q$ for each $i \in [1, \Delta - 3]$, $f(c_i(v)) := 0$ for each $i \in [1, j]$, and $f(c_i(v)) := q$ for each $i \in [\max\{1, j+1\}, \Delta - 3]$. If $M \cap C(v) \neq \emptyset$, then let $f(c_0(v)) := p + (\Delta - 1)q$ and otherwise let $f(c_0(v)) := 2q$.

**else if** (Case-6) $v$ is non-major, $f(p(v), v) = 0$ and $f(v) = 2q$ **then**

Let $f(v, c_i(v)) := p + (i+2)q$ for each $i \in [0, \Delta - 3]$, $f(c_i(v)) := 0$ for each $i \in [0, j]$, and $f(c_i(v)) := q$ for each $i \in [j+1, \Delta - 3]$.

**else if** (Case-$j'$) $\overline{f}(p(v), v))$ and $\overline{f}(v)$ satisfy the conditions of Case-$j$ for $j \in [1, 6]$ **then**

After determining labels for $f(c_i(v))$ and $f(v, c_i(v))$ according to the above (Case-$j$) based on $\overline{f}(p(v), v))$ and $\overline{f}(v)$, let $f(c_i(v)) := p + (\Delta - 1)q - f(c_i(v))$ and $f(v, c_i(v)) := p + (\Delta - 1)q - f(v, c_i(v))$ for each $i$.

**end if**

**end while**

Output $f$ as a $(p, q)$-total labeling of $T$.

We remark that since the procedure of Case-4 needs the assumption of $\Delta \geq 5$, Algorithm $(p, q)$-OPTLABEL$\Delta$5 cannot be applied to the cases of $\Delta < 5$.

### 4.2 Case: $p > 3q/2$

First we consider the case of $p = 2$ and $q = 1$. In this case, the condition (2) is sufficient for $\lambda^T_{2,1}(T) = \Delta + 1$, as described in the following lemma.

**Lemma 11** Let $T$ be a tree. If $\Delta \geq 4$ and the condition (2) is satisfied, then $\lambda^T_{2,1}(T) = \Delta + 1$ and a $(\Delta + 1)$-$(2, 1)$-total labeling of $T$ can be found in linear time. □

This lemma can be proved in a similar way to the proof of Lemma 8.

On the other hand, the condition (2) is not necessary for $\lambda^T_{2,1}(T) = \Delta + 1$, in contrast to the case of $p \leq 3q/2$. For example, a tree $T$ which consists of two adjacent major vertices and $2(\Delta - 1)$ leaves satisfies $\lambda^T_{2,1}(T) = \Delta + 1$, and there are many other such instances.

We also remark that the case of $p = 2$ and $q = 1$ is linearly solvable by Hasunuma et al.'s algorithm for the $L(2, 1)$-labeling problem[13], while the algorithm proposed in the proof of this lemma is much simpler (though it can be applied only to some restricted cases).

Consider the case of $p > 3q/2$ and $p \neq 2q$. In this case, the condition (2) is not even sufficient for $\lambda^T_{p,q}(T) = p + (\Delta - 1)q$, i.e., if $p > 3q/2$ and $p \neq 2q$, then for an arbitrary $\Delta$, there exist instances $T$ with $\lambda^T_{p,q}(T) > p + (\Delta - 1)q$ even if the condition (2) holds. For example, a tree which contains the configuration (a′) is one of such instances, where the configuration (a′) denotes one obtained from (a) in Fig. 1 by replacing each vertex with degree 3 (resp., degree 2) with a vertex with degree $\Delta > 0$ (resp., degree $\Delta - 1$) (note that $v$ has $\Delta - 1$ major vertices as its neighbors).

## 5. Concluding remarks

In this paper, we have discussed the $(p, q)$-total labeling problem for general $p$ and $q$. By extending known results about the case of $q = 1$, we have derived upper and lower bounds on $\lambda_{p,q}(G)$ for some classes of graphs $G$. In particular, we provided tight bounds on $\lambda_{p,q}(T)$ for trees $T$ for all possible $p$ and $q$. Also, in the case of $p \le 3q/2$, we showed that the $(p,q)$-total labeling problem can be solved in linear time, and characterized trees $T$ achieving $\lambda^T_{p,q}(T)$ if $\Delta \ge 4$, in contrast to the counterparts of the $L(p,q)$-labeling problem. On the other hand, in the case of $3q/2 < p \le \Delta q - 1$ and $p \ne 2q$, it is left open whether the $(p,q)$-total labeling problem for trees is polynomially solvable or not.

It is also challenging to derive a tight upper bound on $\lambda^T_{p,q}(G)$ for a general graph $G$, where even the case of $q = 1$ is open. We here give the following conjecture, which is a generalization of Havet and Yu's conjecture (1).

**Conjecture 12** $\lambda^T_{p,q}(G) \le 2p + \Delta q - 1$.

By Lemma 2, Corollary 4, and the fact that $\lambda^T_{1,1}(G) \le \Delta + 1$ for any series-parallel graph $G$[26], this conjecture is true if $p > (\Delta - 1)q$ holds or $G$ is the complete graph, a bipartite graph, or a series-parallel graph.

Another interesting issue might be to investigate the case $p < q$. We actually obtain tight bounds on $\lambda^T_{p,q}(T)$ for trees $T$ about the case, though we omit the details.

## 参 考 文 献

1) F.Bazzaro, M. Montassier, and A.Raspaud. $(d,1)$-total labelling of planar graphs with large girth and high maximum degree. *Discr. Math.* 307, 2141–2151 (2007).
2) M. Behzad. Graphs and their chromatic numbers. Ph.D. Thesis, Michigan State University (1965).
3) T. Calamoneri. The $L(h,k)$-labelling problem: A survey and annotated bibliography. *The Computer Journal* 49, 585–608 (2006). (The $L(h,k)$-Labelling Problem: A Survey and Annotated Bibliography, `http://www.dsi.uniroma1.it/˜calamo/PDF-FILES/survey.pdf`, ver. Oct. 19, 2009.)
4) G.J.Chang, W.-T. Ke, D.Kuo, D.D.-F.Liu and R.K.Yeh. On $L(d,1)$-labeling of graphs. *Discr. Math.* 220, 57–66 (2000).
5) G.J. Chang and D. Kuo. The $L(2,1)$-labeling problem on graphs. *SIAM J. Discr. Math.* 9, 309–316 (1996).
6) D.Chen and W.Wang. (2,1)-Total labelling of outerplanar graphs. *Discr. Appl. Math.* 155, 2585–2593 (2007).
7) L. Dekar, B. Effantin and H. Kheddouci. $[r,s,t]$-coloring of trees and bipartite graphs. *Discr. Math.* 310, 260–269 (2010).
8) J. Fiala, P. A. Golovach and J. Kratochvíl. Computational Complexity of the Distance Constrained Labeling Problem for Trees, *Proc. 35th ICALP*, Part I, 294–305 (2008).
9) S. Fiorini. On the chromatic index of outerplanar graphs. *J. Combin. Theory Ser. B* 18, 35–38 (1975).
10) J. P. Georges and D. W. Mauro. Generalized vertex labeling with a condition at distance two. *Congr. Numer.* 109, 141–159 (1995).
11) J. P. Georges and D. W. Mauro. Labeling trees with a condition at distance two. *Discr. Math.* 269, 127–148 (2003).
12) J.R.Griggs and R.K.Yeh. Labelling graphs with a condition at distance 2. *SIAM J. Discr. Math.* 5, 586–595 (1992).
13) T. Hasunuma, T. Ishii, H. Ono and Y. Uno. A linear time algorithm for $L(2,1)$-labeling of trees. *Proc. 17th ESA*, 35–46 (2009).
14) T. Hasunuma, T. Ishii, H. Ono and Y. Uno. A tight upper bound on the (2,1)-total labeling number of outerplanar graphs. CoRR abs/0911.4590 (2009).
15) F. Havet and M. -L. Yu. $(d,1)$-Total labelling of graphs. Technical Report 4650, INRIA (2002).
16) F.Havet and M. -L. Yu. $(p,1)$-Total labelling of graphs. *Discr. Math.* 308, 496–513 (2008).
17) F.Havet and S. Thomassé. Complexity of $(p,1)$-total labelling. *Discr. Appl. Math.* 157, 2859–2870 (2009).
18) A. Kemnitz and M. Marangio. $[r,s,t]$-Colorings of graphs. *Discr. Math.* 307, 199–207 (2007).
19) K.-W. Lih, D.D.-F.Liu and W.Wang. On $(d,1)$-total numbers of graphs. *Discr. Math.* 309, 3767–3773 (2009).
20) M. Montassier and A.Raspaud. $(d,1)$-total labeling of graphs with a given maximum average degree. *J. Graph Theory* 51, 93–109 (2006).
21) M. S. Villà. $[r,s,t]$-colourings of paths, cycles and stars. Doctoral Thesis, TU Bergakademie, Freiberg (2005).
22) V. G. Vizing. Some unsolved problems in graph theory. *Russian Mathematical Surveys* 23, 125–141, (1968).
23) W.Wang and D.Chen. (2,1)-Total number of trees maximum degree three. *Inf. Process. Lett.* 109, 805–810 (2009).
24) M. A. Whittlesey, J. P. Georges, and D. W. Mauro. On the $\lambda$-number of $Q_n$ and related graphs. *SIAM J. Discr. Math.* 8, 499–506 (1995).
25) R.K.Yeh. A survey on labeling graphs with a condition at distance two. *Discr. Math.* 306, 1217–1231 (2006).
26) X. Zhou, Y. Matsuo, and T. Nishizeki. List total colorings of series-parallel graphs. *J. Discr. Algorithms* 3, 47–60, (2005).