Undecidability of a Simple Origami Problem

Ryuhei Uehara

Origami has recently attracted much attention as “computational origami”. In a sense, origami can be seen as a platform of computation. Both of tractable and intractable results have been obtained on the platform. For a computation model like Turing machine, undecidable problems are a kind of paradoxical evidence of the computational power of the model. These natural models are strong enough, and in a sense, this is why their computational power has their limit. For example, consider the following problem, that is well known as the halting problem:

Input: A program code $P$ and an input $x$ to $P$.
Output: Determine whether $P$ will halt with the input $x$ in a finite number of steps.

The halting problem is a simple undecidable problem. That is, there is no program $Q$ solving the halting problem. Since G"odel’s incompleteness theorems, such a limit of computation is a paradoxical evidence of the power of a computation system.

Then, how about origami? Is the idea “computational origami” strong enough so that it derives such a paradoxical limit? In this paper, we give an affirmative answer. In a reasonable model of computational origami, we give a natural and simple undecidable problem, that is named foldability problem.

Input: An origami with four points $p, q, r, s$ on it.
Output: Determine whether we can fold two lines $\ell_1$ and $\ell_2$ such that (1) they are folded by a finite number of operations starting from $p, q, r$, and (2) they cross at $s$.

Roughly speaking, the foldability problem asks if we can fold a given point $s$ from just other given three points $p, q, r$ in a finite steps. This is a quite natural problem as origami; but it is, surprisingly, undecidable. We can prove a simpler version of the foldability problem in 1D. That is, the following simpler foldability problem in 1D is still undecidable.

Input: A line segment and four points $p, q, r, s$ on it.
Output: Determine whether we can fold the point $s$ from the other points $p, q, r$ in a finite number of operations.
The foldability problem in 2D contains the foldability problem in 1D as a special case. Thus we will show the undecidability for the 1D version in this paper.

2. Computation Model and Undecidability

A 1D origami $P$ is a finite line segment of 0 thickness. Without loss of generality, we assume that $P$ has length 1 and put on the interval $[0, 1]$ at first. One real number in $[0, 1]$ is used to represent a point. That is, a point $p$ on a 1D origami is specified by a coordinate. We denote the coordinate of $p$ on $P$ by $P(p)$. We also abuse $P$ to denote each folded state of the origami, and $P(p)$ to denote the coordinate of the point $p$ on the folded origami. We note that each coordinate is a real number; that is crucial.

On a 2D origami, we use seven basic operations that consist of Hujita’s six axioms and Hatori’s additional axiom (see Chapter 19) for further details). That is, one step in an origami is applying one of seven basic operations, and obtain a new line segment in a finite time. Since $A$ is an algorithm that can be represented by a programming language on a Turing machine, we can define a function $t_A(p, q, r, s)$ by the number of steps required to output “Yes” or “No” for the input $p, q, r, s$. By assumption, $t_A(p, q, r, s)$ is finite for any input.

We now fix the start points $p, q, r$ and $r$ by $p = 0$, $q = 1$ and, say, $r = 1/\sqrt{2}$. Let $T_i$ be the set of points $s$ such that $|T_i| = \{s \mid t_A(p, q, r, s) = i\}$. We here prove that $|T_i|$ is countable. Here $T_i$ contains two kinds of points; let $Y_i$ be the set of points $s$ such that $A$ outputs “Yes” for the $p, q, r$ and $s$, and let $N_i$ be the set of points $s$ such that $A$ outputs “No” for the $p, q, r$ and $s$. By the definition of the operation, $Y_i$ is countable. That is, $A$ outputs “Yes” because it puts another point $s’$ onto $s$ for some $s’ < i$ with $s’ \in T_i$. However, $N_i$ might contain infinitely many points with some reason. We prove that such a case cannot occur. If $N_i$ contain infinitely many points, there is an open interval $(a, b)$ with $0 < a < b < 1$ such that all points in $(a, b)$ are in $N_i$. Then $a’ = 0$ and $b’ = 1$ are the folded points on the paper with $0 = a’ < a < b < b’ = 1$. We put $a’$ on $b’$ and make a folded point $c(= 1/2)$ with $|a’c| = |b’c|$. If $c$ is in $(a, b)$, we have a contradiction since $c$ is “Yes” instance. Thus we have either $a’ < c < a < b < b’$ or $a’ < a < b < c < b’$. If $c < a$, we replace $a’$ by $c’$; otherwise, replace $b’$ by $c$. Re-
peating this process finitely many times, we can put the center point $c$ between $a_0$ and $b_0$ in $(a, b)$. This is a contradiction. Thus $N_i$ can contain finitely many points. Thus $|T_i|$ is countable. Hence $\bigcup_{0 \leq j \leq T_j}$ is countable for any fixed integer $j$. This implies that the size of a set of decidable points by $A$ in a finite time is countable. We let $s_1 < s_2 < s_3 < \ldots$ are the points decidable by $A$.

Now, by a diagonalization, we can construct a real point $s$ which is not decidable. More precisely, we let (remind that an origami has a unit length)

$$s_1 = 0, s_1, 1, s_1, 2, s_1, 3, \ldots$$

$$s_2 = 0, s_2, 1, s_2, 2, s_2, 3, \ldots$$

$$\ldots$$

$$s_i = 0, s_i, 1, s_i, 2, s_i, 3, \ldots$$

$$\ldots$$

Then we define $s = 0, s_1, s_2, s_3, \ldots$, where $s_i = s_i, i + 1 \pmod{10}$. Then $s$ is a point on the 1D origami, but it does not appear in any $T_i$.

Hence $t_A(p, q, r, s)$ is not finite for the $s$. Consequently, the algorithm $A$ does not halt for the input $p, q, r$, and this $s$. This is a contradiction that $A$ solves the foldability problem in a finite time. Thus, the foldability problem is undecidable even in 1D origami model.

**参 考 文 献**

