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Theory of Minimizing Linear Separation Automata

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In this paper, we theoretically analyze a certain extension of a finite automaton, called a linear separation automaton (LSA). An LSA accepts a sequence of real vectors, and has a weight function and a threshold sequence at every state, which determine the transition from some state to another at each step. Transitions of LSAs are just corresponding to the behavior of perceptrons. We develop the theory of minimizing LSAs by using Myhill-Nerode theorem for LSAs. Its proof is performed as in the proof of the theorem for finite automata. Therefore we find that the extension to an LSA from the original finite automaton is theoretically natural.

1. Introduction

The computational model, the finite automaton, is an inevitably important concept in computer science. Recent advances in information technology and its related fields reveal the importance of computational models which can deal with time series of real valued data. Many researchers utilize computational tools based on these models to solve various problems including weather forecasting [7], motion recognition [4),5)], and time-sequential image analysis [13].

There are several works proposing an extension of an automaton which can deal with real values in some sense. Models in these works include a hybrid automaton [1),9)] and a timed automaton [3]. The hybrid automaton is a mathematical model for describing systems in which computational processes interact with physical processes. More formally speaking, the hybrid automaton is a finite state machine augmented with differential equations at each state. It is used for modelling various control systems and for verifying various theoretical properties of them. The timed automaton is a labeled transition system for modelling real-time systems. It has time-passage action in addition to ordinary inputs, outputs, and internal actions. It was developed as providing a formal framework for simulation proof techniques of such real-time systems.

In this paper, we will theoretically analyze a certain extension of a finite automaton, which has a weight function and a threshold sequence at every state, and accepts a sequence of real vectors. We call this automaton a **linear separation automaton (LSA)**.

Let us consider how an LSA works. Transitions of LSAs are just corresponding to the behavior of perceptrons [6),12)]. **Figure 1** shows an example M_1 of an LSA. An LSA has a weight function $w(q_i)$ and a threshold sequence $h(q_i)$ at each state q_i. If a vector $x \in \mathbf{R}^2$ is input to the current state q_i, then the next state is determined by comparing the inner product $x \otimes w(q_i)$ with each element of $h(q_i)$. If a threshold sequence $h(q_i)$ has n elements, then there can be $n+1$ transitions from a state q_i. In an LSA M_1, the transition $\delta(q_1, x)$ from q_1 with x is the following:

$$\delta(q_1, x) = \begin{cases} q_1 & \text{if} \quad x \otimes w(q_1) \le 0 \\ q_2 & \text{if} \quad 0 < x \otimes w(q_1) \le 5 \\ q_3 & \text{if} \quad 5 < x \otimes w(q_1). \end{cases}$$

We will develop the theory of minimizing LSAs by using **Myhill-Nerode the-**

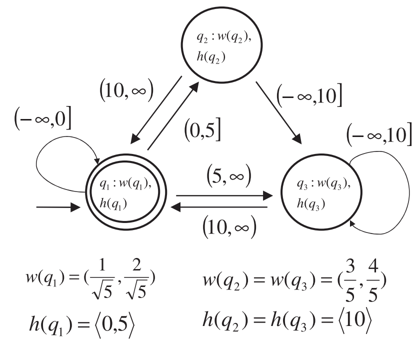

Fig. 1 LSA M_1.

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orem for LSAs. Its proof is performed as in the proof of the theorem for the original finite automaton [8),10)]. Therefore we find that the extension to an LSA from the original finite automaton is theoretically natural.

LSA-like computational models have been already utilized in some application problems. For instance, Matsunaga and Oshita [4),5)] proposed to use a state transition system for recognizing a specified motion. Their system accepts at each state feature vector values acquired from a camera or a motion capture, and determines its transition from the current state by using Support Vector Machines.

In order to develop a theory of learning such computational models, we need computational analysis on the proposed models themselves. For instance, the uniqueness of the minimum state finite automaton for a given one is crucially important in the theory of learning finite automata, because almost all of the learning algorithms try to identify the minimum state automaton of a target language [2),11)]. Therefore we will develop the theory of minimizing LSAs in this paper.

In Section 2, we will give necessary definitions and notation needed in the sequel of this paper. Section 3 introduces a linear separation automaton (LSA). We show that Myhill-Nerode theorem for LSAs is established as in the original finite automata in Section 4. The uniqueness of the minimum state LSA is shown in Section 5. In Section 6, we will characterize the minimum state LSA for a given one by using Myhill-Nerode theorem for LSAs. Section 7 includes concluding remarks and future works.

2. Preliminaries

We introduce basic definitions and notation needed later in this paper.

By $\mathbf{R}$, we denote the set of real numbers. For a positive integer d, by $\mathbf{R}^d$ we denote d-dimensional vector space over $\mathbf{R}$. For $x, y \in \mathbf{R}^d$, $x \otimes y$ denotes the inner product of x and y. We define $(\mathbf{R}^d)^*$ as the set of all finite sequences of vectors in $\mathbf{R}^d$. For a sequence $\alpha = \langle x_1, \ldots, x_n \rangle \in (\mathbf{R}^d)^*$, we denote the length of α by $|\alpha|$, that is, $|\alpha| = n$. An element in $(\mathbf{R}^d)^*$ of length 0 is called an empty sequence, and is denoted by λ. For sequences $\alpha, \beta \in (\mathbf{R}^d)^*$, we denote the concatenation of α and β by $\alpha\beta$. For $\alpha = \langle x_1, \ldots, x_n \rangle \in (\mathbf{R}^1)^*$, the sequence α is said to be increasing if the inequality $x_i < x_{i+1}$ holds for every i.

A partition $\pi = \{S_1, \ldots, S_k\}$ of $\mathbf{R}^d$ (i.e., $S_1, \ldots, S_k$ are mutually disjoint nonempty subsets of $\mathbf{R}^d$ such that $\cup_{i=1,\ldots,k} S_i = \mathbf{R}^d$) is said to be **linearly separable** iff there exists $w \in \mathbf{R}^d$ and an increasing $h = \langle h_1, \ldots, h_{k-1} \rangle \in (\mathbf{R}^1)^*$ such that, for any $x \in \mathbf{R}^d$,

$$h_{i-1} < x \otimes w \leq h_i \quad \Leftrightarrow \quad x \in S_i \quad (i = 1, \ldots, k)$$

holds, where $h_0 = -\infty$ and $h_k = \infty$.

Consider equivalence relations $\equiv, \equiv_1$, and $\equiv_2$ over $(\mathbf{R}^d)^*$. The number of the equivalence classes of $\equiv$ is called the **index** of $\equiv$. An equivalence relation $\equiv_1$ is **finer** than an equivalence relation $\equiv_2$ (or $\equiv_2$ is coarser than $\equiv_1$) iff $x \equiv_1 y$ implies $x \equiv_2 y$ for any x and y. An equivalence relation $\equiv$ is **right invariant** iff $\alpha \equiv \beta$ implies $\alpha\gamma \equiv \beta\gamma$ for any α, β and γ.

Consider partitions π_1 and π_2 of $\mathbf{R}^d$. A partition π_1 is **finer** than a partition π_2 (or π_2 is coarser than π_1) iff for any block $B \in \pi_1$, there exists a block $B' \in \pi_2$ such that $B \subseteq B'$.

3. Linear Separation Automata

This section introduces an extension of a finite automaton, called a **linear separation automaton** (**LSA**). This automaton has a weight function and a threshold sequence at every state, and accepts a sequence of real vectors. The transition from the current state to another is determined by the weight function and the threshold sequence associated with the current state.

An LSA M is defined as an 8-tuple

$$M = (d, Q, q_0, F, w, h, s, \delta),$$

where

d is a positive integer specifying the dimension of input vectors to M,

Q is a finite set of states,

q_0 is an initial state ($q_0 \in Q$),

F is a finite set of final states ($F \subseteq Q$),

w is a weight function from Q to $\mathbf{R}^d$ such that $w(q)$ is a unit vector for any $q \in Q$,

h is a threshold function from Q to $(\mathbf{R}^1)^*$ such that $h(q)$ is increasing for every $q \in Q$, and

s is a sub-transition function from Q to Q^*.

If $|s(q)| \geq 1$, then the equality $|h(q)| = |s(q)| - 1$ holds for every $q \in Q$.

δ is a transition function from $Q \times \mathbf{R}^d$ to Q; and is defined in the following way by using w, h, and s. Consider any state $q \in Q$. First, in the case of $|s(q)| = 0$, the transition function δ is undefined. Secondly, suppose that $|s(q)| \geq 1$. In order to improve the readability, we define $i_q = |h(q)|$ for any $q \in Q$. Let $s(q) = \langle p_1, \ldots, p_{i_q+1} \rangle$ and $h(q) = \langle h_1, \ldots, h_{i_q} \rangle$. The value $\delta(q, x)$ for a given $x \in \mathbf{R}^d$ is defined as follows:

$$\delta(q,x) = \begin{cases} p_1 & \text{if} \quad x \otimes w(q) \leq h_1 \\ p_2 & \text{if} \quad h_1 < x \otimes w(q) \leq h_2 \\ \vdots & \qquad \vdots \\ p_{i_q} & \text{if} \quad h_{i_q-1} < x \otimes w(q) \leq h_{i_q} \\ p_{i_q+1} & \text{if} \quad h_{i_q} < x \otimes w(q). \end{cases}$$

In the state transition diagrams of LSAs as in Fig. 1, we illustrate the condition of the transition from a state p to a state q by using an interval $I \subseteq \mathbf{R}$. Suppose that $\delta(p, x) = q$ holds if $h_i < x \otimes w(p) \leq h_j$. In the diagram, the transition from p to q is associated with an interval $(h_i, h_j]$.

For $\alpha = \langle x_1, \ldots, x_l \rangle \in (\mathbf{R}^d)^*$, we write $\delta(p, \alpha) = q$ if there exists a sequence $p_1(= p), p_2, \ldots, p_{l+1}(= q)$ of states such that $\delta(p_i, x_i) = p_{i+1}$ holds for $i = 1, \ldots, l$. We define the set of sequences accepted by an LSA M, denoted by $L(M)$, as

$$L(M) = \{\alpha \in (\mathbf{R}^d)^* \mid \delta(q_0, \alpha) \in F\}.$$

A subset L of $(\mathbf{R}^d)^*$ is said to be **regular** if there exists an LSA M such that $L = L(M)$. We define the size of M as $\mathrm{size}(M) = |Q|$.

A state $q \in Q$ is said to be **reachable** if there exists $\alpha \in (\mathbf{R}^d)^*$ such that $\delta(q_0, \alpha) = q$. A state is said to be **unreachable** if it is not reachable.

Let $M = (d, Q, q_0, F, w, h, s, \delta)$ be an LSA. We define an equivalence relation $\equiv_M$ over $(\mathbf{R}^d)^*$ as follows:

$$\alpha \equiv_M \beta \ \stackrel{\text{def}}{\Leftrightarrow} \ \delta(q_0, \alpha) = \delta(q_0, \beta).$$

Example . Consider an LSA $M_1 = (d = 2, Q = \{q_1, q_2, q_3\}, q_1, F = \{q_1\}, w, h, s, \delta)$ in Fig. 1. Let $\alpha = \langle x_1, x_2, x_3 \rangle$ be a sequence of vectors in $\mathbf{R}^2$ with $x_1 = (1,1), x_2 = (2,2)$, and $x_3 = (10,10)$. The inner product $x_1 \otimes w(q_1) = \frac{3}{\sqrt{5}}$ is in the interval $(0,5]$, which implies that $\delta(q_1, x_1) = q_2$. We see in the same way that $\delta(q_2, x_2) = q_3$ and $\delta(q_3, x_3) = q_1 \in F$. Hence the sequence α is accepted by M_1. □

4. Myhill-Nerode Theorem for LSAs

In the sequel of this paper, we will develop the theory of minimizing LSAs by using Myhill-Nerode theorem for LSAs. Its proof is performed as in the proof of the theorem for the original finite automaton.

In this section, we will show that Myhill-Nerode theorem for LSAs is established as in the original finite automata.

Myhill-Nerode theorem is originally proved by Myhill [8] and Nerode [10]. This theorem characterizes the class of languages accepted by a finite automaton. We modify this theorem in order to develop the theory of minimizing linear separation automata.

Let $\equiv$ be a right invariant equivalence relation over $(\mathbf{R}^d)^*$ and consider an equivalence class $[\alpha]_\equiv$ containing $\alpha \in (\mathbf{R}^d)^*$. An equivalence relation $R([\alpha]_\equiv)$ over $\mathbf{R}^d$ induced by $[\alpha]_\equiv$ is defined as follows:

$$x \ R([\alpha]_\equiv) \ y \ \stackrel{\text{def}}{\Leftrightarrow} \ \alpha x \equiv \alpha y.$$

For any α and β with $\alpha \equiv \beta$, the equality $R([\alpha]_\equiv) = R([\beta]_\equiv)$ holds, because $\equiv$ is right invariant.

We say that a right invariant equivalence relation $\equiv$ over $(\mathbf{R}^d)^*$ is **right linearly separable** iff for any equivalence class $[\alpha]_\equiv$, there exists a finite linearly separable partition of $\mathbf{R}^d$ that is finer than $\mathbf{R}^d/R([\alpha]_\equiv)$. The concept of the right linearly separability is newly added to the original Myhill-Nerode relation and theorem. It is the very essence of the characterization of the class of languages accepted by an LSA. It also plays an important role in the proof of Myhill-Nerode theorem for LSAs.

Lemma 1. Consider two right invariant equivalence relations $\equiv_1$ and $\equiv_2$ over $(\mathbf{R}^d)^*$ such that $\equiv_1$ is finer than $\equiv_2$. If $\equiv_1$ is right linearly separable, then $\equiv_2$ is right linearly separable.

Proof. Let $\alpha \in (\mathbf{R}^d)^*$. We have

$$x\ R([\alpha]_{\equiv_1})\ y \overset{\text{def}}{\Leftrightarrow} \alpha x \equiv_1 \alpha y,$$
$$x\ R([\alpha]_{\equiv_2})\ y \overset{\text{def}}{\Leftrightarrow} \alpha x \equiv_2 \alpha y.$$

Since $\equiv_1$ is finer than $\equiv_2$, we have $\alpha x \equiv_1 \alpha y$ implies $\alpha x \equiv_2 \alpha y$. Hence we obtain

$$x\ R([\alpha]_{\equiv_1})\ y \ \Rightarrow\ x\ R([\alpha]_{\equiv_2})\ y.$$

Thus, $R([\alpha]_{\equiv_1})$ is finer than $R([\alpha]_{\equiv_2})$. Since $\equiv_1$ is right linearly separable, there exists a finite linearly separable partition $\mathcal{P}$ of $\mathbf{R}^d$ that is finer than $\mathbf{R}^d/\ R([\alpha]_{\equiv_1})$. Then, $\mathcal{P}$ is finer than $\mathbf{R}^d/\ R([\alpha]_{\equiv_2})$, because $R([\alpha]_{\equiv_1})$ is finer than $R([\alpha]_{\equiv_2})$. We have finally proven the claim. □

Definition 1 (**Modified Myhill-Nerode Relation for LSAs).** Let $S \subseteq (\mathbf{R}^d)^*$ be a set of sequences. The equivalence relation $\equiv$ over $(\mathbf{R}^d)^*$ satisfying the following conditions is called a **modified Myhill-Nerode relation** with respect to S.

(1) The equivalence relation $\equiv$ is right invariant.
(2) The equivalence relation $\equiv$ is of finite index.
(3) The equivalence relation $\equiv$ is right linearly separable.
(4) The set S is a union of some equivalence classes of $\equiv$. □

For any subset S of $(\mathbf{R}^d)^*$, we define an equivalence relation $\approx_S$ over $(\mathbf{R}^d)^*$ as follows:

$$\alpha \approx_S \beta \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\gamma \in (\mathbf{R}^d)^* \ \ (\alpha\gamma \in S \ \textbf{iff}\ \beta\gamma \in S).$$

Theorem 1 (**Myhill-Nerode Theorem for LSAs).** Let $S \subseteq (\mathbf{R}^d)^*$ be a set of sequences. The following three statements are equivalent.

(1) The set S is regular.
(2) There exists a modified Myhill-Nerode relation with respect to S.
(3) The equivalent relation $\approx_S$ is of finite index and right linearly separable.

Proof. **(1)⇒(2)**:

Let $M = (d, Q, q_0, F, w, h, s, \delta)$ be an LSA accepting S. The relation $\equiv_M$ is right invariant because

$$\begin{aligned}\alpha \equiv_M \beta &\Rightarrow \delta(q_0,\alpha) = \delta(q_0,\beta)\\ &\Rightarrow \forall\gamma \in (\mathbf{R}^d)^*\ \ \delta(q_0,\alpha\gamma) = \delta(q_0,\beta\gamma)\\ &\Rightarrow \forall\gamma \in (\mathbf{R}^d)^*\ \ \alpha\gamma \equiv_M \beta\gamma.\end{aligned}$$

The relation $\equiv_M$ is of finite index because $|(\mathbf{R}^d)^*/\equiv_M|$ is bounded by $|Q|$.

Let $[\alpha]_{\equiv_M}$ be any equivalence class of $\equiv_M$. Consider the relation $R([\alpha]_{\equiv_M})$ induced by $[\alpha]_{\equiv_M}$. Let $p = \delta(q_0, \alpha)$ and let $h(p) = \langle h_1, \ldots, h_{i_p}\rangle$. We define a partition $\pi = \{S_1, \ldots, S_{i_p+1}\}$ of $\mathbf{R}^d$ as follows: for $k = 1, \ldots, i_p + 1$,

$$S_k \overset{\text{def}}{=} \{\, x \in \mathbf{R} \mid h_{k-1} < x \otimes w(p) \le h_k \,\},$$

where $h_0 = -\infty$ and $h_{i_p+1} = \infty$. It is clear that the partition π of $\mathbf{R}^d$ is linearly separable. Furthermore, it is straightforward to see that $x, y \in S_k$ implies $\delta(p,x) = \delta(p,y)$ implies $\alpha x \equiv_M \alpha y$ implies $x\ R([\alpha]_{\equiv_M})\ y$. Thus, π is finer than $\mathbf{R}^d/\ R([\alpha]_{\equiv_M})$, which implies that $\equiv_M$ is right linearly separable.

Finally, we have

$$\begin{aligned}S \ = \ L(M) &= \{\alpha \in (\mathbf{R}^d)^* \mid \delta(q_0,\alpha) \in F\}\\ &= \cup_{f\in F}\{\alpha \in (\mathbf{R}^d)^* \mid \delta(q_0,\alpha) = f\}\\ &= \cup_{f\in F}[\alpha_f]_{\equiv_M},\end{aligned}$$

where α_f is any representative element α such that $\delta(q_0, \alpha) = f$. Thus, S is a union of some equivalence classes of $\equiv_M$.

Therefore, $\equiv_M$ is a Myhill-Nerode relation with respect to S.

(2)⇒(3):

Let $\equiv$ be a Myhill-Nerode relation with respect to S. The relation $\equiv$ is finer than $\approx_S$ because

$$\begin{aligned}\alpha \equiv \beta &\Rightarrow \forall\gamma \in (\mathbf{R}^d)^*,\ \alpha\gamma \equiv \beta\gamma\\ &\Rightarrow \forall\gamma \in (\mathbf{R}^d)^*,\ \alpha\gamma \in S \ \textbf{iff}\ \beta\gamma \in S\\ &\Rightarrow \alpha \approx_S \beta.\end{aligned}$$

Thus, the relation $\approx_S$ is of finite index.

It is clear from the definition of $\approx_S$ that $\approx_S$ is right invariant. Therefore we deduce from Lemma 1 that $\approx_S$ is right linearly separable.

(3)⇒(1):

Let α be any element in $(\mathbf{R}^d)^*$. Since $\approx_S$ is right linearly separable, there exists a finite linearly separable partition $\pi = \{S_1, \ldots, S_k\}$ that is finer than

$\mathbf{R}^d/\ R([\alpha]_{\approx_S})$. Thus, there exist $w \in \mathbf{R}^d$ and $h = \langle h_1, \ldots, h_{k-1}\rangle \in (\mathbf{R}^1)^*$ such that, for any $x \in \mathbf{R}^d$,

$$h_{i-1} < x \otimes w \leq h_i \quad \Leftrightarrow \quad x \in S_i \ \ (i = 1, \ldots, k)$$

holds, where $h_0 = -\infty$ and $h_k = \infty$. Such w and h are denoted by w_α and h_α, respectively. Note that $h_{\alpha\ i-1} < x \otimes w_\alpha \leq h_{\alpha\ i}$ iff $x \in S_i$. Then, we define an LSA $M' = (d, Q', q'_0, F', w', h', s', \delta')$, where

$$Q' = (\mathbf{R}^d)^*/\approx_S, \qquad q'_0 = [\lambda]_{\approx_S}, \qquad F' = \{[\alpha]_{\approx_S} \mid \alpha \in S\},$$
$$\delta'([\alpha]_{\approx_S}, x) = [\alpha x]_{\approx_S}, \qquad w'([\alpha]_{\approx_S}) = w_\alpha, \qquad h'([\alpha]_{\approx_S}) = h_\alpha.$$

We will show that δ' is well-defined. Suppose that $[\alpha]_{\approx_S} = [\beta]_{\approx_S}$, that is, $\alpha \approx_S \beta$. Since $\approx_S$ is right invariant, $\alpha x \approx_S \beta x$ holds. Hence, $[\alpha x]_{\approx_S} = [\beta x]_{\approx_S}$, which implies that $\delta'([\alpha]_{\approx_S}, x) = \delta'([\beta]_{\approx_S}, x)$.

Since $\approx_S$ is of finite index, the set Q' is finite. The selection of α in the definition of w' and h' could be arbitrary. Note that for any $\alpha, \beta \in (\mathbf{R}^d)^*$, the equality $\delta'([\alpha]_{\approx_S}, \beta) = [\alpha\beta]_{\approx_S}$ holds. Finally, we have

$$\begin{aligned} \alpha \in L(M') &\Leftrightarrow \delta'(q'_0, \alpha) \in F' \\ &\Leftrightarrow \delta'([\lambda]_{\approx_S}, \alpha) \in F' \\ &\Leftrightarrow [\alpha]_{\approx_S} \in F' \\ &\Leftrightarrow \alpha \in S. \end{aligned}$$

Therefore, $L(M') = S$, which implies that S is regular. □

5. Uniqueness of Minimum State LSA

In this section, we demonstrate the uniqueness of the minimum state LSA for a given one.

Let S be any regular subset of $(\mathbf{R}^d)^*$. In the sequel, by

$$M_{\min} = (d, Q_{\min}, q_{0\min}, F_{\min}, w_{\min}, h_{\min}, s_{\min}, \delta_{\min})$$

we denote the LSA M' constructed in the proof **(3)** $\Rightarrow$ **(1)** of Theorem 1. We will prove that the minimum state LSA accepting S is determined uniquely in the sense that $M_{\min}$ is **isomorphic** to every minimum state LSA. The definition of isomorphism is described below.

Let $M = (d, Q, q_0, F, w, h, s, \delta)$ and $M' = (d, Q', q'_0, F', w', h', s', \delta')$ be LSAs. We say that M is isomorphic to M' iff there exists a bijection f from Q to Q' satisfying the following conditions:

(1) $f(q_0) = q'_0$.
(2) $f(\delta(q, x)) = \delta'(f(q), x)$ holds for any $q \in Q$ and $x \in \mathbf{R}^d$.
(3) $f(F) = F'$.

Theorem 2 (Uniqueness of Minimum State LSA). Let S be a regular subset of $(\mathbf{R}^d)^*$. The LSA $M_{\min}$ is isomorphic to every minimum state LSA accepting S.

Proof. Let M be any LSA accepting S. As shown in the proof **(2)** $\Rightarrow$ **(3)** of Theorem 1, the equivalence relation $\equiv_M$ is finer than $\approx_S$. Thus we have

$$\mathrm{size}(M_{\min}) \ = \ (\text{index of } \approx_S) \ \leq \ (\text{index of } \equiv_M) \leq \ \mathrm{size}(M).$$

Therefore, $M_{\min}$ is a minimum state LSA.

Let $N = (d, Q, q_0, F, w, h, s, \delta)$ be any minimum state LSA accepting S. Let define the mapping f from $Q_{\min}$ to Q as $f([\alpha]_{\approx_S}) = \delta(q_0, \alpha)$.

We deduce from the discussion in the proof **(2)** $\Rightarrow$ **(3)** of Theorem 1 that $\equiv_N$ is finer than $\approx_S$. Therefore, we have $[\alpha]_{\equiv_N} \subseteq [\alpha]_{\approx_S}$ for any $\alpha \in (\mathbf{R}^d)^*$. Suppose that there exist $\alpha, \beta \in (\mathbf{R}^d)^*$ such that $\alpha \approx_S \beta$ and $\alpha \not\equiv_N \beta$. Then, it immediately holds that $[\alpha]_{\approx_S}$ contains two equivalence classes $[\alpha]_{\equiv_N}$ and $[\beta]_{\equiv_N}$. This implies that the index of $\approx_S$ is less than that of $\equiv_N$, which contradicts the minimality of N. Therefore, we have $[\alpha]_{\equiv_N} = [\alpha]_{\approx_S}$ for any $\alpha \in (\mathbf{R}^d)^*$. Thus, the definition $f([\alpha]_{\approx_S}) = \delta(q_0, \alpha)$ is well-defined.

We will first demonstrate that f is a bijection.

First, we will show that f is injective. Suppose that $f([\alpha]_{\approx_S}) = f([\beta]_{\approx_S})$, that is, $\delta(q_0, \alpha) = \delta(q_0, \beta)$. We deduce from the definition of $\equiv_N$ that $[\alpha]_{\equiv_N} = [\beta]_{\equiv_N}$. Note that $\equiv_N$ is finer than $\approx_S$. We have $[\alpha]_{\approx_S} = [\beta]_{\approx_S}$.

Secondly, we will show that f is surjective. To prove that an LSA N does not have any unreachable state, let us assume the opposite and see what happens. Removing unreachable states leads to an LSA accepting S with fewer states than $M_{\min}$, which contradicts the minimality of $M_{\min}$. Therefore, for every state $p \in Q$, there exists $\alpha \in (\mathbf{R}^d)^*$ such that $p = \delta(q_0, \alpha) = f([\alpha]_{\approx_S})$.

We will next demonstrate that the mapping f satisfies the three conditions above.

The initial state $[\lambda]_{\approx_S}$ is mapped to $f([\lambda]_{\approx_S}) = q_0$.

It holds that

$$
\begin{aligned}
f(\delta_{\min}([\alpha]_{\approx_S}, x)) &= f([\alpha x]_{\approx_S}) \\
&= \delta(q_0, \alpha x) \\
&= \delta(\delta(q_0, \alpha), x) \\
&= \delta(f([\alpha]_{\approx_S}), x).
\end{aligned}
$$

It also holds that

$$
\begin{aligned}
[\alpha]_{\approx_S} \in F_{\min} &\Leftrightarrow \alpha \in S = L(M_{\min}) \\
&\Leftrightarrow \alpha \in L(N) \\
&\Leftrightarrow \delta(q_0, \alpha) \in F \\
&\Leftrightarrow f([\alpha]_{\approx_S}) \in F.
\end{aligned}
$$

Finally we conclude that f is an isomorphic mapping from $M_{\min}$ to N. □

6. Characterization of Minimum State LSA

In this section, we characterize the minimum state LSA for a given one.

Let $M = (d, Q, q_0, F, w, h, s, \delta)$ be an LSA accepting the set of sequences S with no unreachable states. For any $p, q \in Q$, there exists $\alpha, \beta \in (\mathbf{R}^d)^*$ such that $\delta(q_0, \alpha) = p$ and $\delta(q_0, \beta) = q$. We define the equivalence relation $\sim$ over Q as follows:

$$p \sim q \overset{\text{def}}{\Leftrightarrow} \alpha \approx_S \beta.$$

The choice of α and β can not be determined uniquely. However, for $\alpha', \alpha'' \in (\mathbf{R}^d)^*$ such that $\delta(q_0, \alpha') = \delta(q_0, \alpha'')$, we have $\delta(q_0, \alpha'\gamma) = \delta(q_0, \alpha''\gamma)$ for any $\gamma \in (\mathbf{R}^d)^*$. Hence, it holds that $\alpha' \approx_S \alpha''$. Therefore, $\sim$ is well-defined.

We say that p and q are **indistinguishable** iff $p \sim q$. The states p and q are said to be **distinguishable** iff $p \not\sim q$.

Example . Consider an LSA in Fig. 1. The equality $w(q_2) = w(q_3)$ holds, which implies that $x \otimes w(q_2) = x \otimes w(q_3)$ for any $x \in \mathbf{R}^d$. If $x \otimes w(q_2) = x \otimes w(q_3) \leq 10$ holds, then $\delta(q_2, x) = \delta(q_3, x) = q_3$; otherwise $\delta(q_2, x) = \delta(q_3, x) = q_1$ holds. Thus we have $\delta(q_2, x) = \delta(q_3, x)$ for any $x \in \mathbf{R}^d$, which implies that $q_2 \sim q_3$, that is, q_2 and q_3 are indistinguishable.

Let $x_1 = (-1, -1)$. We obtain $x_1 \otimes w(q_1) = -\frac{3}{\sqrt{5}}$, which implies that $\delta(q_1, x_1) = q_1$. We also obtain $x_1 \otimes w(q_2) = x_1 \otimes w(q_3) = -\frac{7}{5}$, which implies that $\delta(q_2, x_1) = \delta(q_3, x_1) = q_3$. Note that q_1 is a final state and q_3 is not. We have $q_1 \not\sim q_2$ and $q_1 \not\sim q_3$, that is, q_1 and q_2 (or q_1 and q_3) are distinguishable. □

Lemma 2.

$$p \sim q \quad \Leftrightarrow \quad \forall \gamma \in (\mathbf{R}^d)^*,\ \delta(p, \gamma) \in F \textbf{ iff } \delta(q, \gamma) \in F.$$

Proof. We have

$$
\begin{aligned}
p \sim q &\Leftrightarrow \exists \alpha, \beta \in (\mathbf{R}^d)^*,\ \delta(q_0, \alpha) = p,\ \delta(q_0, \beta) = q,\ \alpha \approx_S \beta \\
&\Leftrightarrow \exists \alpha, \beta \in (\mathbf{R}^d)^*,\ \delta(q_0, \alpha) = p,\ \delta(q_0, \beta) = q, \\
&\quad\ \forall \gamma \in (\mathbf{R}^d)^*,\ \alpha\gamma \in L(M) \textbf{ iff } \beta\gamma \in L(M) \\
&\Leftrightarrow \exists \alpha, \beta \in (\mathbf{R}^d)^*,\ \delta(q_0, \alpha) = p,\ \delta(q_0, \beta) = q, \\
&\quad\ \forall \gamma \in (\mathbf{R}^d)^*,\ \delta(q_0, \alpha\gamma) \in F \textbf{ iff } \delta(q_0, \beta\gamma) \in F \\
&\Leftrightarrow \forall \gamma \in (\mathbf{R}^d)^*,\ \delta(p, \gamma) \in F \textbf{ iff } \delta(q, \gamma) \in F.
\end{aligned}
$$

□

Furthermore, Lemma 2 immediately implies Lemma 3.

Lemma 3.

$$p \sim q \quad \Leftrightarrow \quad \forall \alpha \in (\mathbf{R}^d)^*,\ \delta(p, \alpha) \sim \delta(q, \alpha).$$

Proof. We have

$$
\begin{aligned}
p \sim q &\Leftrightarrow \forall \alpha, \beta \in (\mathbf{R}^d)^*,\ \delta(p, \alpha\beta) \in F \textbf{ iff } \delta(q, \alpha\beta) \in F \\
&\Leftrightarrow \forall \alpha, \beta \in (\mathbf{R}^d)^*,\ \delta(\delta(p, \alpha), \beta) \in F \textbf{ iff } \delta(\delta(q, \alpha), \beta) \in F \\
&\Leftrightarrow \forall \alpha \in (\mathbf{R}^d)^*,\ \delta(p, \alpha) \sim \delta(q, \alpha).
\end{aligned}
$$

□

For any $p \in Q$, by $r(p)$ we denote a **representative element** of $[p]_\sim$.

Lemma 4.

$$\delta(r(p), \alpha) \sim r(\delta(p, \alpha)).$$

Proof. For any $\alpha \in (\mathbf{R}^d)^*$, we have

$$
\begin{aligned}
\delta(r(p), \alpha) &\sim \delta(p, \alpha) \quad \text{(By Lemma 3)} \\
&\sim r(\delta(p, \alpha)).
\end{aligned}
$$

□

We will prove that the minimum state LSA is obtained by identifying indistinguishable states.

We define an LSA

$M/\sim = (d, Q', q'_0, F', w', h', s', \delta')$,

where

$Q' = Q/\sim, \quad q'_0 = [q_0]_\sim, \quad F' = \{[q]_\sim \mid q \in F\},$

$\delta'([q]_\sim, x) = [\delta(r(q), x)]_\sim, \quad w'([q]_\sim) = w(r(q)), \quad h'([q]_\sim) = h(r(q)).$

Lemma 5. For $\alpha \in (\mathbf{R}^d)^*$,

$\delta'([p]_\sim, \alpha) = [\delta(p, \alpha)]_\sim.$

Proof. We will prove this Lemma by induction on $|\alpha|$.

In the case of $|\alpha| = 0$, i.e., $\alpha = \lambda$, we have

$$\begin{aligned}\delta'([p]_\sim, \lambda) &= [p]_\sim \\ &= [\delta(p, \lambda)]_\sim.\end{aligned}$$

Assume that the claim holds for $|\alpha| \leq k$ and consider the case of $|\alpha| = k + 1$. Let $\alpha = \beta x \ \ (\beta \in (\mathbf{R}^d)^*, x \in \mathbf{R}^d)$. Then, we have

$$\begin{aligned}\delta'([p]_\sim, \alpha) &= \delta'(\delta'([p]_\sim, \beta), x) \\ &= \delta'([\delta(p, \beta)]_\sim, x) \quad \text{(By induction hypothesis)} \\ &= [\delta(r(\delta(p, \beta)), x)]_\sim \\ &= [r(\delta(\delta(p, \beta), x))]_\sim \quad \text{(By Lemma 4)} \\ &= [r(\delta(p, \beta x))]_\sim \\ &= [\delta(p, \beta x)]_\sim \\ &= [\delta(p, \alpha)]_\sim.\end{aligned}$$ □

Lemma 6.

$p \in F$ **iff** $[p]_\sim \in F'$.

Proof. From the definition of F', it is clear that $p \in F$ implies $[p]_\sim \in F'$. Suppose that $[p]_\sim \in F'$. Then, there exists $q \in F$ such that $p \sim q$. We deduce from $\delta(q, \lambda) \in F$ and Lemma 2 that $p = \delta(p, \lambda) \in F$ holds. □

Lemma 7.

$L(M/\sim) = L(M)$.

Proof. For any $\alpha \in (\mathbf{R}^d)^*$, we have

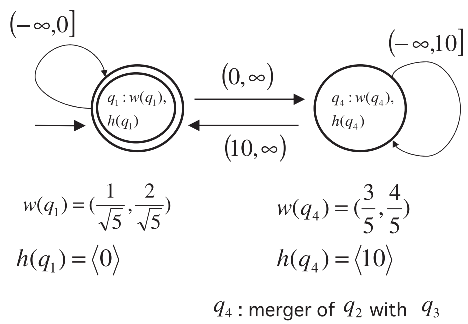

Fig. 2 LSA $M_1/\sim$.

$$\begin{aligned}\alpha \in L(M/\sim) &\Leftrightarrow \delta'(q'_0, \alpha) \in F' \\ &\Leftrightarrow \delta'([q_0]_\sim, \alpha) \in F' \\ &\Leftrightarrow [\delta(q_0, \alpha)]_\sim \in F' \quad \text{(By Lemma 5)} \\ &\Leftrightarrow \delta(q_0, \alpha) \in F \quad \text{(By Lemma 6)} \\ &\Leftrightarrow \alpha \in L(M).\end{aligned}$$ □

Theorem 3 (Characterization of Minimum State LSA). Let M be an LSA. The LSA $M/\sim$ is a minimum state LSA for M such that $L(M/\sim) = L(M)$.

Proof. Lemma 7 implies that $L(M/\sim) = L(M)$ holds.

It is clear that $\sim$ is an equivalence relation. From the definition of $\sim$, the index $|Q/\sim|$ of $\sim$ is equal to $|(\mathbf{R}^d)^*/\approx_S|$. Therefore we conclude that $\text{size}(M/\sim) = |Q/\sim| = |(\mathbf{R}^d)^*/\approx_S| = \text{size}(M_{\min})$. □

Example . Consider an LSA M_1 in Fig. 1. From the example above, the states q_2 and q_3 are indistinguishable; and the states q_1 and q_2 (or q_1 and q_3) are distinguishable. Let q_4 be a state obtained by merging q_2 with q_3. Thus, we obtain the minimum state LSA for M_1, $M_1/\sim$, illustrated in **Fig. 2**. □

7. Conclusions

In this paper, we theoretically analyzed a certain extension of a finite automaton, called a linear separation automaton (LSA). We developed the theory of minimizing LSAs by using Myhill-Nerode theorem for LSAs. Myhill-Nerode theorem for LSAs is established as in the original finite automata. The minimum state LSA for a given one is unique, and is characterized by using Myhill-Nerode theorem for LSAs.

In order to develop a theory of learning computational models like LSAs, we need computational analysis on the models themselves. The theory of minimizing LSAs will play an important role in the theory of learning LSAs as in the original finite automata [2),11)].

Some of our future works are the following.

In this paper, we do not give algorithms for minimizing LSAs. Therefore in the next paper, we will present some algorithms for minimizing LSAs, which will be the naive algorithm directly induced by Myhill-Nerode theorem for LSAs, and a more efficient algorithm.

The development of the theory of learning LSAs is one of the future research topics. Its theory will help us solve some application problems including weather forecasting, motion recognition, and time-sequential image analysis.

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