Vol.2009-AL-127 No.5
2009/11/27

順列制約をみたす模調要求をもつ正モジュラシステムについて

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有限集合 V，正モジュラ関数 $f : 2^V \to \mathbb{R}$ および模調関数 $r : 2^V \to \mathbb{R}$ からなるシステム (V, f, r) において，すべての $X \subseteq V - R$ に対し $f(X) \geq r(X)$ が成り立つような要素数最小の集合 $R \subseteq V$ を求める問題を考える．この問題は横断問題と呼ばれ，Sakashita ら[6] により無向グラフまたは無向ハイパーグラフにおける辺連結度要求をもつ供給点配置問題および外部ネットワーク問題を一般化した枠組みとして導入された．

本論文では，任意の模調関数 r が $r(X) = \max\{p_r(v, W) \mid v \in X \subseteq V - W\}$ をみたす関数 $p_r : V \times 2^V \to \mathbb{R}$ により特徴づけられることを示し，さらに供給点配置問題に対する Tamura らの結果[8] を一般化し，r が π-単調であるとき横断問題が簡潔な貪欲法により解けることを示す．ここで，すべての $W \subseteq V$ と $\pi(u) \geq \pi(v)$ であるすべての 2 要素 $u, v \in V$ に対し $p_r(u, W) \geq p_r(v, W)$ が成り立つ V の順列 π が存在するとき，r は π-単調であるという．

また，r が π-単調であるときの横断問題における極小不足集合族 $\mathcal{W}$ に関する構造的性質も示す．すなわち，すべての点 u とその親 v に対し $\pi(u) \leq \pi(v)$ が成り立つような $\mathcal{W}$ に対する基本木が存在することを示す．この性質は，供給点配置問題に対する貪欲法の正当性の別の証明を与える．

さらに，フラクショナル横断問題が，横断問題に対するアルゴリズムと同様の手法により解けることを示す．

Posi-modular Systems with Modulotone Requirements under Permutation Constraints

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Given a system (V, f, r) on a finite set V consisting of a posi-modular function $f : 2^V \to \mathbb{R}$ and a modulotone function $r : 2^V \to \mathbb{R}$, we consider the problem of finding a minimum set $R \subseteq V$ such that $f(X) \geq r(X)$ for all $X \subseteq V - R$. The problem, called the transversal problem, was introduced by Sakashita et al.[6] as a natural generalization of the source location problem and external network problem with edge-connectivity requirements in undirected graphs and hypergraphs.

By generalizing[8] for the source location problem, we show that the transversal problem can be solved by a simple greedy algorithm if r is π-monotone, where a modulotone function r is π-monotone if there exists a permutation π of V such that the function $p_r : V \times 2^V \to \mathbb{R}$ associated with r satisfies $p_r(u, W) \geq p_r(v, W)$ for all $W \subseteq V$ and $u, v \in V$ with $\pi(u) \geq \pi(v)$. Here we show that any modulotone function r can be characterized by p_r as $r(X) = \max\{p_r(v, W) \mid v \in X \subseteq V - W\}$.

We also show the structural properties on the minimal deficient sets $\mathcal{W}$ for the transversal problem for π-monotone function r, i.e., there exists a basic tree T for $\mathcal{W}$ such that $\pi(u) \leq \pi(v)$ for all arcs (u, v) in T, which, as a corollary, gives an alternative proof for the correctness of the greedy algorithm for the source location problem.

Furthermore, we show that a fractional version of the transversal problem can be solved by the algorithm similar to the one for the transversal problem.

1. Introduction

Given a system (V, f, r) on a finite set V consisting of a posi-modular function $f : 2^V \to \mathbb{R}$ and a modulotone function $r : 2^V \to \mathbb{R}$ with $f(\emptyset) \geq r(\emptyset)$, we consider the following problem:

$$\begin{array}{lll} \text{Minimize} & |R| & \\ \text{subject to} & f(X) \geq r(X) \text{ for all } X \subseteq V - R & \\ & R \subseteq V. & \end{array} \quad (1)$$

Here $f(\emptyset) \geq r(\emptyset)$ is necessary for the problem to have a feasible solution. This problem was first introduced by Sakashita et al.[6] as a generalized framework of the source location problem and external network problem with edge-connectivity requirements in undirected graphs and hypergraphs[3),5),8]. They showed that the family of minimal *deficient* sets of (V, f, r) forms a tree hypergraph, and that conversely any tree hypergraph can be represented by minimal deficient sets of (V, f, r) for some posi-modular function f and some modulotone function r, where a set $X \subseteq V$ with $f(X) < r(X)$

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is called deficient. Note that Problem (1) asks to find a minimum set hitting all deficient sets. By combining these results with properties shown in[2),3)], it follows that Problem (1) can be solved in $O(|V|^3\rho(|V|))$ time, where $\rho(|V|)$ is the time required to check the feasibility (i.e., a given $R \subseteq V$ satisfies $f(X) \geq r(X)$ for all $X \subseteq V - R$), while it is still open whether the feasibility can be checked in polynomial time. They also gave a polynomial time algorithm for Problem (1) by utilizing a basic tree for the tree hypergraph, under the assumption that f is submodular and r is given by either $r(X) = \max\{d_1(v) \mid v \in X\}$ for a function $d_1 : V \to \mathbb{R}_+$ or $r(X) = \max\{d_2(u,v) \mid u \in X, v \in V - X\}$ for a function $d_2 : V \times V \to \mathbb{R}_+$. We here remark that these assumptions are necessary only for executing the algorithm in polynomial time. Both of the source location problem and external network problem satisfy these assumptions, and hence are polynomially solvable. On the other hand, it was shown by Tamura et al.[8)] that the source location problem can be solved in polynomial time by a much simpler greedy algorithm without using any basic tree for the tree hypergraph.

Then natural questions arise: (i) is there some relationship between Sakashita et al.'s algorithm and Tamura et al.'s greedy one? (ii) if so, how can we characterize cases where such a greedy algorithm works? In this paper, we show that there exists a basic tree for the family of all minimal deficient sets for which Sakashita et al.'s algorithm can perform in the same way as Tamura et al.'s algorithm does. In other words, Sakashita et al.'s algorithm includes Tamura et al.'s one as its special case. Furthermore, we show that this relationship can be extended to Problem (1) in which a modulotone function r has a property called π*-monotonicity*.

The π-monotonicity of a modulotone function is defined as follows. An arbitrary modulotone function r can be characterized by using a function $p_r : V \times 2^V \to \mathbb{R}$, which is a slight generalization of similar properties shown in[4)]. A modulotone function is called π-monotone if there exists a permutation π of V such that for all $u, v \in V$ and $W \subseteq V - \{u, v\}$, $\pi(u) \geq \pi(v)$ if and only if $p_r(u, W) \geq p_r(v, W)$. A modulotone function r in the above source location problem satisfies $r(X) = \max\{d_1(v) \mid v \in X\}$, $X \subseteq V$ for some function $d_1 : V \to \mathbb{R}_+$, and hence is π-monotone. Also, Problem (1) with a π-monotone modulotone function includes problems whose requirements are based on a function q on V; we will discuss these problems later in Subsection 3.2. We then show that if r is π-monotone, then there exists a tree hypergraph whose basic tree satisfies $\pi(u) \leq \pi(v)$ for each pair of u and its parent v. This interesting property enables that Sakashita et al.'s algorithm[6)] can be executed in a simple greedy manner without computing any basic tree for the tree hypergraph.

Furthermore, we consider a fractional version of Problem (1):

$$\begin{aligned} &\text{Minimize} && x(V) \\ &\text{subject to} && f(X) + x(X) \geq r(X) \text{ for all } X \subseteq V \\ &&& x : V \to \mathbb{R}, \end{aligned} \qquad (2)$$

where $x(X) = \sum_{v \in X} x(v)$ for all $X \subseteq V$. This problem can be regarded as a generalization of a capacitated type of the source location problem with edge-connectivity requirements in undirected graphs. Then we show that Sakashita et al.'s algorithm can be extended to this problem.

The rest of this paper is organized as follows. In Section 2, after giving basic definitions, we review properties and applications of Problem (1) shown in[6)]. In Section 3, we define a π-monotonicity of a modulotone function. Furthermore, we show a structural property of minimal deficient sets of Problem (1) with a π-monotone modulotone function, which enables a greedy algorithm. Section 4 discusses Problem (2) as a fractional version of Problem (1). Finally, we give some concluding remarks in Section 5.

2. Preliminaries

Let V be a finite set. For two sets $X, Y \subseteq V$, we say that X and Y *intersect* each other if $X \cap Y \neq \emptyset$, $X - Y \neq \emptyset$, and $Y - X \neq \emptyset$. For a family $\mathcal{E} \subseteq 2^V$, the hypergraph $(V, \mathcal{E})$ may be written as $\mathcal{E}$ simply. Let $V(\mathcal{E})$ denote the vertex set of a hypergraph $\mathcal{E}$. For a hypergraph $\mathcal{E}$, a subset $R \subseteq V$ is called a *transversal* (or *hitting set*) of $\mathcal{E}$ if $R \cap E \neq \emptyset$ for all $E \in \mathcal{E}$. A hypergraph $\mathcal{E}$ is called a *tree hypergraph* (or *hypertree*) if there exists a tree T with a vertex set V such that each hyperedge in $\mathcal{E}$ induces a subtree of T. We call such a tree T a *basic tree* for $\mathcal{E}$, and we may regard T as a rooted tree in describing algorithms. For a subset U of vertices in a tree T, $T[U]$ denotes the subgraph of T induced by U. For a vertex v in a rooted tree T, $T(v)$ denotes the subtree of T rooted at v.

2.1 Posi-modular Systems

In this subsection, let us review several properties about Problem (1) shown by Sakashita et al.[6]. A set function $f : 2^V \to \mathbb{R}$ is called *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \tag{3}$$

for arbitrary two subsets X, Y of V, and *posi-modular* if

$$f(X) + f(Y) \geq f(X - Y) + f(Y - X) \tag{4}$$

for arbitrary two subsets X, Y of V. A set function $r : 2^V \to \mathbb{R}$ is called *modulotone* if for any nonempty subset X of V, there exists an element $v \in X$ such that all subsets Y of X with $Y \ni v$ satisfies $r(Y) \geq r(X)$.

Observe that Problem (1) is equivalent to that of asking to find a minimum transversal R of $\{X \subseteq V \mid f(X) < r(X)\}$. A set $X \subseteq V$ is called *deficient* if $f(X) < r(X)$. A deficient set X is called *minimal* if any proper subset Y of X is not deficient. We denote the family of all minimal deficient sets by $\mathcal{W}(f, r)$. It is known that the posi-modular systems have the following interesting property, where a *Sperner* family denotes a family of sets in V in which arbitrary two distinct sets E, E' satisfy neither $E \subseteq E'$ nor $E' \subseteq E$.

Theorem 2.1 [6] A Sperner family $\mathcal{E} \subseteq 2^V$ is a tree hypergraph if and only if $\mathcal{E} = \mathcal{W}(f, r)$ holds for a posi-modular function $f : 2^V \to \mathbb{R}$ and a modulotone function $r : 2^V \to \mathbb{R}$. □

By this theorem and properties observed in[2,3], it follows that Problem (1) can be solved in $O(|V|^3 \rho(|V|))$ time, where $\rho(|V|)$ is the time required to check the feasibility, while it is still open whether the feasibility can be checked in polynomial time. On the other hand, we can solve Problem (1) more efficiently (more precisely, quadratically faster) by the following algorithm MinTransversal, under the assumption that f is submodular and r is given as

$$r(X) = \begin{cases} \max\{d_1(v) \mid v \in X\} & \text{if } X \neq \emptyset \\ 0 & \text{if } X = \emptyset, \end{cases} \tag{5}$$

for a function $d_1 : V \to \mathbb{R}_+$ or

$$r(X) = \begin{cases} \max\{d_2(u, v) \mid u \in X, v \in V - X\} & \text{if } X \neq \emptyset, V \\ 0 & \text{if } X = \emptyset \text{ or } V \end{cases} \tag{6}$$

for a function $d_2 : V \times V \to \mathbb{R}_+$.

Algorithm 1 Algorithm MinTransversal[6]

Require: A posi-modular function $f : 2^V \to \mathbb{R}$, a modulotone function $r : 2^V \to \mathbb{R}$ with $f(\emptyset) \geq r(\emptyset)$.
Ensure: A minimum transversal R of $\mathcal{W}(f, r)$.

Compute a basic tree T for $\mathcal{W}(f, r)$.
Initialize $R := \emptyset$ and $U := V$.
while $U \neq \emptyset$ **do**
  Choose a leaf v of $T[U]$ and $U := U - \{v\}$.
  if $R \cup U$ is not a transversal **then**
    $R := R \cup \{v\}$.
  end if
end while
Output R as a solution.

It is not difficult to observe that both functions defined as (5) and (6) are modulotone. Also, we remark that these assumptions are necessary only for executing this algorithm in polynomial time.

2.2 Applications of Problem (1)

We here introduce the source location problem and the external network problem in undirected graphs discussed as applications of Problem (1) in[6].

Let $G = (V, E, c)$ be an undirected graph with a set V of vertices, a set E of edges, and a capacity function $c : E \to \mathbb{R}_+$. Suppose that each vertex $v \in V$ has a demand $d_1(v) \in \mathbb{R}_+$. The source location problem is defined as follows:

$$\begin{array}{ll} \text{Minimize} & |S| \\ \text{subject to} & \lambda_G(S, v) \geq d_1(v) \text{ for all } v \in V \\ & S \subseteq V, \end{array} \tag{7}$$

where $\lambda_G(S, v)$ denotes the maximum flow value (or edge-connectivity) between S and

Vol.2009-AL-127 No.5
2009/11/27

v in G, and we define $\lambda_G(S,v) = +\infty$ if $v \in S$. This problem has been studied as a location problem with requirements measured by a network flow amount or network connectivity[1),5),7),8)].

In a multimedia network, a set S of some specified network nodes, such as the so-called mirror servers, may have functions of offering the same services for users. A user at a node v can use the service by communicating with at least one node $s \in S$ through a path between s and v. The edge-connectivity between S and v measures the robustness of the service against network link failures. Thus, location problems with such a fault-tolerancy can be formulated as the source location problem.

By the max-flow min-cut theorem, it is not difficult to see that the constraint of Problem (7) is equivalent to $u(X) \geq r(X)$ for all subsets X of $V - S$, where $u(X) = \sum\{c(u,v) \mid u \in X, v \in V - X, (u,v) \in E\}$ (i.e., u is a cut function in G) and r is given as (5). Since u is posi-modular, it follows that Problem (7) is a special case of Problem (1).

Given an undirected graph $G = (V, E, c)$ and a demand function $d_2 : V \times V \to \mathbb{R}_+$, the external network problem is given by:

$$\begin{array}{ll} \text{Minimize} & |S| \\ \text{subject to} & \lambda_{G/S}(u,v) \geq d_2(u,v) \text{ for all } u, v \in V \\ & S \subseteq V, \end{array} \qquad (8)$$

where G/S denotes the graph obtained from G by contracting S into a single vertex s, and if $u \in S$, we define $\lambda_{G/S}(u,v) = \lambda_{G/S}(s,v)$. This problem has been studied as a problem of finding access points to some highly reliable external network while taking into account a network flow amount or connectivity[3)].

In a communication network N, each pair of nodes may have some requirements measured by a network flow amount or connectivity. Suppose that we can use a highly reliable external network N' in which neither node nor link failures occurs. Then we can improve the reliability of N by adding access points to N'. The problem of asking to find a minimum set S of access points to N' in order to satisfy the connectivity requirements can be formulated as Problem (8).

Again by the max-flow min-cut theorem, we can see that the constraint of Problem (8) is equivalent to $u(X) \geq r(X)$ for all subsets X of $V - S$, where r is given as (6). Thus, Problem (8) is also a special case of Problem (1).

Furthermore, since a cut function u is submodular, both problems can be solved in polynomial time by Algorithm MinTransversal. In particular, for the source location problem, a much simpler greedy algorithm without using any basic tree for the tree hypergraph was proposed[8)]. This algorithm is described as Algorithm MinSourceSet.

Algorithm 2 Algorithm MinSourceSet[8)]

Require: An undirected graph $G = (V, E, c)$ and a demand function $d_1 : V \to \mathbb{R}_+$.
Ensure: A minimum set S satisfying $\lambda_G(S,v) \geq d_1(v)$ for all $v \in V$.

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Order vertices of V such that d_1(v_1) ≤ ··· ≤ d_1(v_n).
Initialize S := ∅ and U := V.
for j = 1 to n do
    U := U − {v_j}.
    if S ∪ U is infeasible then
        S := S ∪ {v_j}.
    end if
end for
Output S as a solution.
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3. Modulotone Function with π-Monotonicity

From the previous section, we can observe that as for Problem (7), if there exists a basic tree T for the family $\mathcal{W}(f,r)$ of minimal deficient sets such that $d_1(u) \leq d_1(v)$ holds for each pair of a vertex u and its parent v in T, then Algorithm MinTransversal can be executed in the same way as Algorithm MinSourceSet does; that is, in such cases we need not prepare any basic tree for the tree hypergraph. In this section, we will prove the existence of such a basic tree in a more general setting.

For this, we first characterize a modulotone function by using a function $p : V \times 2^V \to \mathbb{R}$ in Subsection 3.1. In Subsection 3.2, we define Problem (1) with a function r called

π-monotone which is a generalization of Problem (7), discuss its applications, and prove the existence of basic trees for $\mathcal{W}(f,r)$ defined as above.

3.1 Characterization of a Modulotone Function

We here show that an arbitrary modulotone function can be characterized by using a function $p : V \times 2^V \to \mathbb{R}$. This is a slight generalization of similar properties observed in[4]. For a nonempty subset X of V and a function $p : V \times 2^V \to \mathbb{R}$, let

$$p^*(X) = \max\{p(v,U) \mid U \subseteq V, v \in X \subseteq V - U\}. \quad (9)$$

Lemma 3.1 (i) Let $p : V \times 2^V \to \mathbb{R}$ be a function. Then, the set function $p^* : 2^V \to \mathbb{R}$ given as (9) is modulotone.
(ii) Let $p^* : 2^V \to \mathbb{R}$ be a modulotone function. Then, there exists a function $p : V \times 2^V \to \mathbb{R}$ that satisfies (9). □

3.2 π-Monotonicity

For a modulotone function r, we denote by p_r a function $p : V \times 2^V \to \mathbb{R}$ such that r is given as (9). A modulotone function r is called *π-monotone* if there exist a function p_r and a permutation $\pi : V \to [|V|]$ of V such that for all $u, v \in V$ and $U \subseteq V - \{u, v\}$, $\pi(u) \geq \pi(v)$ if and only if $p_r(u,U) \geq p_r(v,U)$. In this section, we focus on Problem (1) in the case where r is π-monotone.

We first observe that the function r defined as (5) is π-monotone. Let $p_r(v,U) = d_1(v)$ for all $v \in V$ and $U \subseteq V$, and π be a permutation of V such that $\pi(u) \geq \pi(v)$ if and only if $d_1(u) \geq d_1(v)$ for each pair of two vertices u and v. Thus, r is clearly π-monotone. It follows that Problem (7) is a special case of Problem (1) with a π-monotone r.

For the function r defined as (6), if $d_2(u,v)$ is defined as a function of $(q(u), q(v))$ such as $q(u)+q(v)$ or $q(u)q(v)$ for a given function $q : V \to \mathbb{R}$, then we can observe that r is π-monotone. For example, it is natural to consider a situation where a user who pays more cost (or money) can communicate with a higher reliablity; $d_2(u,v)$ may be considered as a value proportional to $q(u)+q(v)$ where $q(u)$ is a payment of a user u. In another situation where each node u corresponds to a city whose population is $q(u)$, the reliablility requirement between two cities u and v may be assumed to be proportional to $q(u)q(v)$. In these settings, Problem (8) becomes a special case of Problem (1) with a π-monotone r.

On the other hand, we remark that even if r is given as (6), then r is not necessarily π-monotone. Consider p_r in the case where $V = \{v_1, v_2, v_3, v_4\}$, $d_2(v_1, v_2) = 1$, $d_2(v_3, v_4) = 2$, and $d_2(v_i, v_j) = 0$ otherwise. For $X_1 = \{v_1, v_3\}$, $p_r(v_1, U) \leq 1$ holds for all nonempty subsets U of $V - X_1$, since otherwise $r(v_1) > 2$, a contradiction. It follows by $r(X_1) = 2$ that $p_r(v_3, U') = 2$ for some $U' \subseteq V - X_1$. For $X_2 = \{v_1, v_3, v_4\}$, $p_r(v, V - X_2\ (= \{v_2\})) = 0$ for all $v \in \{v_3, v_4\}$ by $r(\{v_1, v_2\}) = 0$. It follows by $r(X_2) = 1$ that $p_r(v_1, V - X_2) = 1$. Thus, by $p_r(v_3, U') > p_r(v_1, U')$ and $p_r(v_3, V - X_2) < p_r(v_1, V - X_2)$, we can see that this r is not π-monotone.

In the rest of this subsection, we will show the following interesting structural property about $\mathcal{W}(f,r)$.

Theorem 3.2 For a posi-modular function $f : 2^V \to \mathbb{R}$ and a π-monotone modulotone function $r : 2^V \to \mathbb{R}$, there exists a basic tree T for $\mathcal{E} = \mathcal{W}(f,r)$ (which is a tree hypergraph) such that for any pair of two vertices u and v in T,

$$\text{if } u \text{ is a child of } v, \text{ then } \pi(u) \leq \pi(v). \quad (10)$$

This property enables us to execute Algorithm MinTransversal greedily based on π without any basic tree for $\mathcal{W}(f,r)$. Indeed, if we pick up all elements in V in nondecreasing order of their π-values, then it follows that we pick up a leaf of $T[U]$ for the current U in each iteration of the while loop of Algorithm MinTransversal. Also notice that this greedy procedure based on π is a generalization of Algorithm MinSourceSet.

Corollary 3.3 If a modulotone function r is π-monotone, then Problem (1) can be solved in a greedy manner based on π as described in Algorithm 3. □

Before proving this theorem, we show several preparatory lemmas. For a set $X \subseteq V$, let $\pi(X) = \max\{\pi(v) \mid v \in X\}$.

Lemma 3.4 If W_1 and W_2 in $\mathcal{W}(f,r)$ satisfy $W_1 \cap W_2 \neq \emptyset$, then W_1 and W_2 intersect each other and we have $\pi(W_1 \cap W_2) > \pi(W_1 - W_2)$ or $\pi(W_1 \cap W_2) > \pi(W_2 - W_1)$. □

Lemma 3.5 Let $\mathcal{W} = \{W_1, W_2, \ldots, W_p\}$ be a family of sets in $\mathcal{W}(f,r)$ with $W_1 \cap W_2 \cap \ldots \cap W_p \neq \emptyset$. Then there exists a set $W_q \in \mathcal{W}$ such that all elements $w \in W_q$ with $\pi(w) = \pi(W_q)$ are contained in $W_1 \cap W_2 \cap \ldots \cap W_p$. □

Proof of Theorem 3.2. Let $\mathcal{E}$ be a tree hypergraph with $\mathcal{E} = \mathcal{W}(f,r)$ and T_1 be its basic tree. Let v_r be a vertex with the maximum π-value (i.e., $\pi(v_r) = \max\{\pi(v) \mid v \in V\}$)

Algorithm 3 Algorithm MinTransversal2

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Order elements of V such that π_1(v_1) ≤ ··· ≤ π_1(v_n).
Initialize S := ∅ and U := V.
for j = 1 to n do
    U := U − {v_j}.
    if S ∪ U is infeasible then
        S := S ∪ {v_j}.
    end if
end for
Output S as a solution.
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and regard T_1 as a tree rooted at v_r. Assume that T_1 does not satisfy (10). Let u, v, and w be three vertices in T_1 such that $u = p_{T_1}(v)$, $v = p_{T_1}(w)$, $\pi(u) \geq \pi(v) < \pi(w)$, and $depth(w; T_1)$ is the minimum, where $p_T(x)$ denotes the parent of x in T, and $depth(x; T)$ denotes the length of the simple path connecting r and x in T rooted at r. For a tree T rooted at r, define $F_T = \{(x, p_T(x)) \mid \pi(x) > \pi(p_T(x))\}$, and a potential function

$$\Phi(T) = depth(x^*_T; T) + \sum_{x:\ (x,p_T(x)) \in F_T} n(\pi(x) - \pi(p_T(x))),$$

where $n = |V|$ and x^*_T is a vertex x with $(x, p_T(x)) \in F_T$ such that $depth(x; T)$ is the minimum. Notice that if (10) is satisfied, $\Phi(T) = 0$, otherwise $\Phi(T) > 0$. Below, we will prove this theorem by showing the existence of a basic tree T' for $\mathcal{W}(f, r)$ such that $\Phi(T') < \Phi(T_1)$. Let $C(v)$ denote the set of all children of v other than w in T_1, and $\mathcal{W}^*$ denote the family of sets in $\mathcal{W}(f, r)$ containing v. Partition $\mathcal{W}^*$ into $\mathcal{X}_1 = \{X \in \mathcal{W}(f, r) \mid v, w \in X, u \notin X, C(v) \cap X \neq \emptyset\}$, $\mathcal{X}_2 = \{X \in \mathcal{W}(f, r) \mid v, w \in X, u \notin X, C(v) \cap X = \emptyset\}$, $\mathcal{Y}_1 = \{X \in \mathcal{W}(f, r) \mid u, v \in X, w \notin X, C(v) \cap X \neq \emptyset\}$, $\mathcal{Y}_2 = \{X \in \mathcal{W}(f, r) \mid u, v \in X, w \notin X, C(v) \cap X = \emptyset\}$, $\mathcal{Z}_1 = \{X \in \mathcal{W}(f, r) \mid u, v, w \in X, C(v) \cap X \neq \emptyset\}$, and $\mathcal{Z}_2 = \{X \in \mathcal{W}(f, r) \mid u, v, w \in X, C(v) \cap X = \emptyset\}$. Notice that every two sets in $\mathcal{W}^*$ intersect each other since every set is a minimal deficient set. There are the following three possible cases: (Case-1) $\mathcal{X}_1 \cup \mathcal{X}_2 = \emptyset$, (Case-2) $\mathcal{Y}_1 \cup \mathcal{Y}_2 = \emptyset$, and (Case-3) otherwise.

(Case-1) Let T_2 denote the tree from T_1 by deleting the edge (v, w) and adding a new edge connecting u and w (i.e., $p_{T_2}(w) := u$). T_2 is also a basic tree for $\mathcal{W}(f, r)$ because otherwise there exists a set $X \in \mathcal{W}(f, r)$ with $v, w \in X$ and $u \notin X$, contradicting $\mathcal{X}_1 \cup \mathcal{X}_2 = \emptyset$. Also, we can observe that $\Phi(T_2) < \Phi(T_1)$. Indeed, if $\pi(u) < \pi(w)$, then we have $\Phi(T_2) - \Phi(T_1) = depth(w; T_2) - depth(w; T_1) + n(-\pi(u) + \pi(v)) < 0$ because $x^*_{T_1} = x^*_{T_2} = w$, $depth(w; T_2) = depth(w; T_1) - 1$, and $\pi(u) \geq \pi(v)$. If $\pi(u) \geq \pi(w)$, then we have $\Phi(T_2) - \Phi(T_1) = depth(x^*_{T_2}; T_2) - depth(w; T_1) + n(-\pi(u) + \pi(v)) < 0$ because $depth(x^*_{T_2}; T_2) \leq n - 1$ and $\pi(u) > \pi(v)$ (by $\pi(u) \geq \pi(w) > \pi(v)$).

(Case-2) Let T_2 denote the tree from T_1 by deleting the edge (u, v), adding a new edge connecting u and w (i.e., $p_{T_2}(w) := u$), and making the parent of v the vertex w (i.e., $p_{T_2}(v) := w$). T_2 is also a basic tree for $\mathcal{W}(f, r)$ because otherwise there exists a set $X \in \mathcal{W}(f, r)$ with $u, v \in X$ and $w \notin X$, contradicting $\mathcal{Y}_1 \cup \mathcal{Y}_2 = \emptyset$.

Also, we can observe that $\Phi(T_2) < \Phi(T_1)$. Indeed, if $\pi(u) < \pi(w)$, then we have $\Phi(T_2) - \Phi(T_1) = depth(w; T_2) - depth(w; T_1) + n(-\pi(u) + \pi(v)) < 0$ because $x^*_{T_1} = x^*_{T_2} = w$, $depth(w; T_2) = depth(w; T_1) - 1$, $\pi(u) \geq \pi(v)$, and $(v, w) \notin F_{T_2}$. If $\pi(u) \geq \pi(w)$, then we have $\Phi(T_2) - \Phi(T_1) = depth(x^*_{T_2}; T_2) - depth(w; T_1) + n(-\pi(u) + \pi(v)) < 0$ because $depth(x^*_{T_2}; T_2) \leq n - 1$, $\pi(u) > \pi(v)$ (by $\pi(u) \geq \pi(w) > \pi(v)$), and $(v, w) \notin F_{T_2}$.

(Case-3) If $\mathcal{X}_2 \neq \emptyset$, then $X \in \mathcal{X}_2$ and $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ satisfy $X \cap Y = \{v\}$, $\pi(X - Y) \geq \pi(w) > \pi(v)$ $(= \pi(X \cap Y))$, and $\pi(Y - X) \geq \pi(u) \geq \pi(v)$ $(= \pi(X \cap Y))$, contradicting Lemma 3.4. If $\mathcal{Y}_2 \neq \emptyset$, then $X \in \mathcal{X}_1 \cup \mathcal{X}_2$ and $Y \in \mathcal{Y}_2$ satisfy $X \cap Y = \{v\}$, $\pi(X - Y) \geq \pi(w) > \pi(v)$ $(= \pi(X \cap Y))$, and $\pi(Y - X) \geq \pi(u) \geq \pi(v)$ $(= \pi(X \cap Y))$, contradicting Lemma 3.4. Hence, assume that $\mathcal{X}_2 = \emptyset = \mathcal{Y}_2$. Then the following property holds, where its proof is omitted due to space limitation.

Claim 3.6 $\mathcal{Z}_2 = \emptyset$. □

Let Y_1 be a set in $\mathcal{Y}_1$ and $y_1 \in Y_1$ be a vertex with $\pi(y_1) = \pi(Y_1)$. We first claim that $\pi(y_1) > \pi(v)$. This follows since otherwise $\pi(y_1) = \pi(u) = \pi(v)$ (by $\pi(y_1) \geq \pi(u) \geq \pi(v)$) and it follows that for $X_1 \in \mathcal{X}_1$, $\pi(Y_1) = \pi(Y_1 - X_1) = \pi(X_1 \cap Y_1)$ (by $u \in Y_1 - X_1$ and $v \in X_1 \cap Y_1$) and $\pi(X_1 - Y_1) \geq \pi(w) > \pi(v) = \pi(X_1 \cap Y_1)$, implying that X_1 and Y_1 would contradict Lemma 3.4. Then if every $X \in \mathcal{X}_1 \cup \mathcal{Z}_1$ contains y_1, then construct the tree T_2 from T_1 by deleting the edge (v, w), adding a new edge connecting y_1 and w, and making the parent of w the vertex y_1 (i.e., $p_{T_2}(w) := y_1$). Observe that in this case, T_2 is also a basic tree for $\mathcal{E}$ and $\Phi(T_2) < \Phi(T_1)$, because

$\Phi(T_2) - \Phi(T_1) = depth(x^*_{T_2}; T_2) - depth(w; T_1) + n(-\pi(y_1) + \pi(v)) < 0$ by $\pi(y_1) > \pi(v)$.

Consider the remaining cases. Then, there exists a set $X_1 \in \mathcal{X}_1 \cup \mathcal{Z}_1$ with $y_1 \notin X_1$. Here, we assume that X_1 contains a vertex x_1 with $\pi(x_1) = \pi(X_1)$ such that all sets $X \in \mathcal{X}_1 \cup \mathcal{Z}_1$ with $y_1 \notin X$ contains x_1 (this is possible by Lemma 3.5). Notice that $\pi(Y_1 - X_1) = \pi(Y_1)$ by $y_1 \in Y_1 - X_1$. Hence, we have $\pi(X_1 \cap Y_1) > \pi(X_1 - Y_1)$, since otherwise (i.e., if $\pi(X_1 \cap Y_1) = \pi(X_1)$) X_1 and Y_1 would contradict Lemma 3.4. Therefore, we have $x_1 \in X_1 \cap Y_1$.

If every $X \in \mathcal{X}_1 \cup \mathcal{Z}_1$ contains x_1, then construct the tree T_2 from T_1 by deleting the edge (v, w), adding a new edge connecting x_1 and w, and making the parent of w the vertex x_1 (i.e., letting $p_{T_2}(w) := x_1$). Observe that in this case, T_2 is also a basic tree for $\mathcal{W}(f, r)$ and $\Phi(T_2) < \Phi(T_1)$, because $\pi(x_1) = \pi(X_1) \geq \pi(w) > \pi(v)$ indicates that $\Phi(T_2) - \Phi(T_1) = depth(x^*_{T_2}; T_2) - depth(w; T_1) + n(-\pi(x_1) + \pi(v)) < 0$. Consider the case where some $X_2 \in \mathcal{X}_1 \cup \mathcal{Z}_1$ with $x_1 \notin X_2$. By the choice of X_1, we have $y_1 \in X_2$. Then, $\pi(X_1 \cap X_2) > \pi(X_2 - X_1)$ holds since otherwise (i.e., if $\pi(X_2 - X_1) = \pi(X_2)$) by $x_1 \in X_1 - X_2$, we have $\pi(X_1 - X_2) = \pi(X_1)$ and hence X_1 and X_2 would contradict Lemma 3.4. Hence, by $x_1 \in X_1 \cap Y_1$, $y_1 \in X_2$ and this, we have $\pi(X_1) \geq \pi(X_1 \cap X_2) = \pi(X_2) \geq \pi(y_1) = \pi(Y_1) \geq \pi(x_1) = \pi(X_1)$. It follows that $\pi(X_1) = \pi(X_2) = \pi(X_1 \cap X_2)$ holds, and X_1 and X_2 would contradict Lemma 3.4. □

In Algorithm MinSourceSet, if there exist two vertices u and v with $d_1(u) = d_1(v)$, then we can choose u before v or vice versa, depending on the sorting in line 1. The following corollary shows that there exists a basic tree for $\mathcal{W}(f, r)$ corresponding to each of these cases.

Corollary 3.7 Let $\mathcal{E} = \mathcal{W}(f, r)$ be a tree hypergraph whose basic tree T satisfies (10), and (v, w) be an edge in T with $v = p_T(w)$ and $\pi(v) = \pi(w)$. Then there exists a basic tree T' for $\mathcal{W}(f, r)$ satisfying (10) such that $w = p_{T'}(v)$ or v and w have a common parent in T'. □

Finally, we give a much simpler proof of the property that the above greedy algorithm based on π works. Notice that we scan all elements in nondecreasing order of their π-values. When we pick up an element v and $R \cup U - \{v\}$ is not a transversal of $\mathcal{W}(f, r)$ for the current transversal $R \cup U$, there exists a minimal deficient set $W_v \in \mathcal{W}(f, r)$ such that $W_v \cap U = \{v\}$. Then notice that all elements in W_v other than v have been scanned and deleted, and this implies that v has the maximum π-value among all elements in W_v; $\pi(v) = \pi(W_v)$. It follows that for each $v \in R$, there exists such a set W_v; let $\mathcal{W} = \{W_v \mid v \in R\}$. Then we can prove that every two sets in $\mathcal{W}$ are disjoint, which implies that R is a minimum transversal. Indeed, if two sets W_u and W_v satisfy $W_u \cap W_v \neq \emptyset$, then by $u \in W_u - W_v$, $\pi(u) = \pi(W_u)$, $v \in W_v - W_u$, and $\pi(v) = \pi(W_v)$, we have $\pi(W_u - W_v) = \pi(W_u)$ and $\pi(W_v - W_u) = \pi(W_v)$, contradicting Lemma 3.4.

4. Algorithm for Problem (2)

In this section, we consider Problem (2). This can be regarded as a generalization of a variant of the source location problem; given an undirected graph $G = (V, E, c)$ and a demand function $d_1 : V \to \mathbb{R}_+$, find an $x : V \to \mathbb{R}_+$ such that $x(X) + u(X) \geq \max\{d_1(v) \mid v \in X\}$ holds for all nonempty subsets X of V and $x(V)$ is the minimum. In applications of multimedia networks, we can locate a mirror server v with an arbitrarily finite capacity $x(v)$ and we want to minimize the total capacity $x(V)$ of servers to be located. Notice that in a setting discussed in Subsection 2.2, each server to be located has an infinite capacity.

For $f : 2^V \to \mathbb{R}$, $r : 2^V \to \mathbb{R}$, and $\mathcal{W}(f, r)$, $x : V \to \mathbb{R}$ is called a *cover* of $\mathcal{W}(f, r)$ if $x(X) + f(X) \geq r(X)$ for all $X \subseteq V$. Then we can find a minimum cover of $\mathcal{W}(f, r)$ by Algorithm MinCover, similar to Algorithm MinTransversal.

Lemma 4.1 Algorithm MinCover finds a minimum cover x of $\mathcal{W}(f, r)$.

Proof. Let x^* be a function obtained by Algorithm MinCover. Since x^* is clearly a cover of $\mathcal{W}(f, r)$, we only prove the optimality of x^*. Consider the iteration of the while loop in which v is chosen. Note that immediately before this iteration starts, $x + x'$ is a cover. As a result of line 4, if $x + x'$ is not a cover, then it follows by line 6 that some set $W_v \in \mathcal{W}(f, r)$ with $v \in W_v$ satisfies $x(W_v) + f(W_v) = r(W_v)$ and $W_v \subseteq V(T(v))$. The latter property follows since if W contains the parent u of v, then $x(W) \geq x(u) = \max\{r(W) \mid W \in \mathcal{W}(f, r)\} \geq r(W)$, a contradiction.

Then we claim that there exists a family $\mathcal{W} \subseteq \mathcal{W}(f, r)$ of pairwise disjoint minimal deficient sets such that each set $W \in \mathcal{W}$ satisfies $x^*(W) + f(W) = r(W)$ and each v with $x^*(v) > 0$ is included in some set in $\mathcal{W}$. Such a family $\mathcal{W}$ can be obtained by the

Algorithm 4 Algorithm MinCover

Require: A posi-modular function $f : 2^V \to \mathbb{R}$, a modulotone function $r : 2^V \to \mathbb{R}$ with $f(\emptyset) \geq r(\emptyset)$.
Ensure: A minimum cover $x : V \to \mathbb{R}$ of $\mathcal{W}(f, r)$.
Compute a basic tree T for $\mathcal{W}(f, r)$.
Initialize $x(v) := 0$ and $x'(v) := \max\{r(W) \mid W \in \mathcal{W}(f, r)\}$ for all $v \in V$ and $U := V$.
while $U \neq \emptyset$ **do**
  Choose a leaf v of $T[U]$, $x'(v) := 0$, and $U := U - \{v\}$.
  if $x + x'$ is not a cover **then**
    $x(v) := \max\{r(W) - x(W) - f(W) \mid W \in \mathcal{W}(f, r)\}$.
  end if
end while
Output x as a solution.

following procedure (i) and (ii):

(i) Initialize $U := \{v \in V \mid x^*(v) > 0\}$ and $\mathcal{W} := \emptyset$.

(ii) While scanning all vertices v in T from the root in breadth-first order, only if $v \in U$, then $\mathcal{W} := \mathcal{W} \cup \{W_v\}$, $U := U - \{u \in W_v \mid x^*(u) > 0\}$.

It is not difficult to see that every two sets in the resulting $\mathcal{W}$ are pairwise disjoint because every W_v satisfies $W_v \subseteq V(T(v))$. Since any cover x satisfies $x(V) \geq \sum_{W \in \mathcal{W}}(r(W) - f(W)) = x^*(V)$, it follows that x^* is optimal. □

Finally, we remark that if f is submodular and r is given as (5) or (6), Algorithm MinCover can be implemented to run in the same complexity as Algorithm MinTransversal. Furthermore, similarly to the discussion in the previous section, if r is π-monotone, then we can execute it greedily based on π without any basic tree for $\mathcal{W}(f, r)$.

5. Concluding Remarks

In this paper, we consider the problem of finding a minimum transversal of a given system (V, f, r) on a finite set V consisting of a posi-modular function f and a modulotone function r with $f(\emptyset) \geq r(\emptyset)$. We define the π-monotonicity of a modulotone function, and derive an interesting structural property for a basic tree for $\mathcal{W}(f, r)$ under the assumption that r is π-monotone, which enables Algorithm MinTransversal to perform in the same way as a simple greedy algorithm for the source location problem (i.e., Algorithm MinSourceSet) does. This also shows that Algorithm MinTransversal is a generalization of Algorithm MinSourceSet. Also, we define a fractional version of the problem, and show that the discussion about algorithms and properties of the original problem can be extended to this problem. For both problems, the greedy algorithms can be implemented to run by checking the feasibility O($|V|$) times. On the other hand, as pointed out in[6], it is still open whether the feasibility can be checked in polynomial time, unless f is submodular and r is given as (5) or (6).

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