Vol.2009-AL-126 No.3
2009/9/15

劣モジュラ費用集合被覆問題

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本研究では，頂点被覆問題，枝被覆問題，集合被覆問題，それぞれの目的関数を線形な関数から非負な劣モジュラ関数で置き換えて一般化した問題を扱う．これらの問題に対し，劣モジュラ関数の離散凸性を用いた近似アルゴリズムを設計する．まず，劣モジュラ頂点被覆問題に対しては，連続緩和問題の半整数性を証明し，それ利用したラウンディングによる 2 近似アルゴリズムを与えた．さらに半整数的な最適解が劣モジュラ関数最小化によって求まることを示した．劣モジュラ費用集合被覆問題に対しては，最大重複度が定数でおさえられる場合，ラウンディングアルゴリズムと主双対アルゴリズムの両方で定数近似率が達成されることを示した．劣モジュラ枝被覆問題に対しては，近似可能性の本質的にタイトな下界を与えた．

Submodular Cost Set Cover Problem

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This paper addresses the problems of minimizing nonnegative submodular functions under covering constraints, which generalize the vertex cover, edge cover, and set cover problems. We give approximation algorithms for these problems exploiting the discrete convexity of submodular functions. We first present a rounding 2-approximation algorithm for the submodular vertex cover problem based on the half-integrality of the continuous relaxation problem, and show that the rounding algorithm can be performed by one application of submodular function minimization on a ring family. We also show that a rounding algorithm and a primal-dual algorithm for the submodular cost set cover problem are both constant factor approximation algorithms if the maximum frequency is fixed. In addition, we give an essentially tight lower bound on the approximability of the submodular edge cover problem.

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1. Introduction

Let N be a finite nonempty set of cardinality n. A real-valued set function ρ on N is *submodular* if it satisfies $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for all $X, Y \subseteq N$. In the areas of combinatorial optimization, game theory, and machine learning and various other fields, submodular set functions are recognized as fundamental tools and interesting subjects of research. Besides, submodular functions and convex functions are closely related: a set function is submodular if and only if its Lovász extension is convex[23].

The first polynomial algorithm for submodular function minimization, described by Grötschel, Lovász, and Schrijver[12),13], relies on the ellipsoid method. Combinatorial strongly polynomial algorithms for minimizing submodular functions were developed later by Iwata, Fleischer, and Fujishige[18] and by Schrijver[30]. These combinatorial algorithms have been improved in time complexity[17),19),29].

In contrast, the maximization problem of submodular functions is NP-hard, as it contains the maximum cut problem. Approximation algorithms for the maximization have been extensively studied even under some constraints including knapsack and matroid constraints[6),22),34].

Constrained submodular function minimization problems have also been investigated in various contexts. It is easy to see that we can find a nonempty proper subset X that minimizes $\rho(X)$ in polynomial time. When the feasible region $\mathcal{F} \subseteq 2^N$ is a ring family, that is, $\mathcal{F}$ is a collection of subsets of N closed with respect to union and intersection, the minimization problem can be solved in polynomial time (see[30]). Goemans and Ramakrishnan[11] dealt with the case when the feasible region is a parity family. Recently, Svitkina and Fleischer[32] considered the submodular function minimization problem with cardinality lower bound and gave an $o(\sqrt{n/\ln n})$ lower bound for the approximability.

This paper addresses the problems of minimizing nonnegative submodular functions under covering constraints. These problems described below generalize the classical covering problems: the set cover, vertex cover, edge cover problems.

Submodular Cost Set Cover:

Let U be a finite set of cardinality k and $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a collection of its subsets indexed by $N = \{1, \ldots, n\}$. For a subset $X \subseteq N$ we denote $S_X = \bigcup\{S_i \mid i \in X\}$. We say that a subset $X \subseteq N$ is a set cover if $S_X = U$. Given a nonnegative cost function $c : N \to \mathbf{R}_+$, the set cover problem asks for finding a set cover $X \subseteq N$ that minimizes the cost $c(X) = \sum_{i \in X} c(i)$. This problem is known to be solved approximately in polynomial time within a factor of $\mathrm{O}(\ln k)$ or the maximum frequency $\eta = \max_{u \in U} |\{i \mid u \in S_i\}|$. Given a nonnegative submodular function $\rho : 2^N \to \mathbf{R}_+$, the *submodular set cover problem* asks for finding a set cover $X \subseteq N$ that minimizes the cost $\rho(X)$.

Submodular Vertex Cover:

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A vertex subset $X \subseteq V$ is called a *vertex cover* in G if every edge in E is incident to a vertex in X. Given a nonnegative cost function $c : V \to \mathbf{R}_+$, the vertex cover problem asks for finding a vertex cover $X \subseteq V$ that minimizes the cost $c(X) = \sum_{v \in X} c(v)$. This problem is known to be NP-hard, and efficient 2-approximation algorithms are known[1]. Given a nonnegative submodular function $\rho : 2^V \to \mathbf{R}_+$, the *submodular vertex cover problem* asks for finding a vertex cover $X \subseteq V$ that minimizes the cost $\rho(X)$. This is a special case of a submodular cost set cover problem with $U = E$ and $N = V$.

Submodular Edge Cover:

Let $H = (W, F)$ be a graph with vertex set W and edge set F. An edge subset $X \subseteq F$ is called an *edge cover* in H if every vertex in W is incident to an edge in X. Given a nonnegative cost function $c : F \to \mathbf{R}_+$, the edge cover problem asks for finding an edge cover $X \subseteq F$ that minimizes the cost $c(X) = \sum_{e \in X} c(e)$. This problem is known to be polynomial time solvable by graph matching (see, e.g.,[31] [§19.3]). Given a nonnegative submodular function $\rho : 2^F \to \mathbf{R}_+$, the *submodular edge cover problem* asks for finding an edge cover $X \subseteq F$ that minimizes the cost $\rho(X)$. This is a special case of the submodular cost set cover problem with $U = W$ and $N = F$.

A different type of generalization of the set cover problem was introduced by Hayrapetyan, Swamy, and Tardos[14], in which a submodular penalty cost was imposed. Chudak and Nagano[3] developed an approximation algorithm for this problem using the Lovász extension and the non-smooth convex minimization algorithms of Nesterov[27),28]. The present paper aims at providing another effective use of the Lovász extension in design of approximation algorithms.

In this paper, we first introduce a natural convex programming relaxation of the submodular vertex cover problem using the Lovász extension and prove that the relaxation problem has a half-integral optimal solution. This extends the result of Nemhauser and Trotter[26] for the vertex cover problem. Accordingly, a rounding algorithm for the vertex cover problem achieves an approximation factor of 2, and we further show that it can be performed by one application of submodular function minimization.

In addition, we describe approximation algorithms for the submodular cost set cover problem. Extending the algorithm of Hochbaum[15], we devise a rounding algorithm based on a convex programming relaxation. We also present a primal-dual algorithm that extends the algorithm of Bar-Yehuda and Even[1]. Both of these algorithms successfully achieve an approximation guarantee of η. The analysis of our rounding algorithm implies that the nonnegative submodular function ρ can be replaced by another submodular function ρ° that is monotone, i.e., $\rho^\circ(X) \leq \rho^\circ(Y)$ for $X \subseteq Y \subseteq N$. Each evaluation of ρ°, however, requires submodular function minimization. For the sake of efficiency, our approximation algorithms directly deal with general nonnegative submodular function without relying on this reduction.

Assuming the unique games conjecture, these approximation factors are optimal even for the vertex cover problem and the set cover problem (Khot and Regev[20]).

One can obtain a k-approximation solution for the submodular cost set cover problem in a very simple way. Interestingly, we will see that this bound k is an essentially tight bound on the approximability for the submodular cost set cover problem. This will be shown by exhibiting the difficulty of the submodular edge cover problem. Our analysis depends on a technique similar to those of Goemans *et al.*[10] and Svitkina and Fleischer[32], and utilizes a celebrated result of Erdős and Rényi[5] on random graphs. We also show that the submodular edge cover problem is NP-hard, whereas the edge cover problem can be solved efficiently by weighted matching algorithms.

A recent paper by Koufogiannakis and Young[21] provides algorithms for a general framework that include the set cover problem with monotone submodular cost function. Their algorithm achieves the same approximation guarantee of η. In contrast, we deal with nonnegative submodular cost function that is not necessarily monotone.

Simultaneously with us, Goel, Karande, Tripathi, and Wang[9] deal with the problems of minimizing monotone submodular functions under various combinatorial constraints and their extensions. In particular, they give a 2-approximation algorithm for the submodular vertex cover based on the ellipsoid method, and proved that the optimal approximation factor is indeed 2.

The present paper is organized as follows. Section 2 provides preliminaries on submodular functions and associated polyhedra. In Section 3, we give an efficient rounding 2-approximation algorithm for the submodular vertex cover problem. In Section 4, we describe approximation algorithms for the submodular cost set cover problem. Finally, in Section 5, we present hardness results on the submodular edge cover problem.

2. Submodular Functions and Convexity

In this section, we provide preliminaries on submodular functions and associated polyhedra.

We denote $N = \{1, \ldots, n\}$. Let $f : 2^N \to \mathbf{R}$ be a set function with $f(\emptyset) = 0$. The function f is called *nonnegative* if $f(X) \geq 0$ for each $X \subseteq N$. The function f is called *monotone* if $f(X) \leq f(Y)$ for each pair of subsets $X, Y \subseteq N$ with $X \subseteq Y \subseteq N$. Obviously, a monotone function f with $f(\emptyset) = 0$ is nonnegative. Throughout this paper, we assume that $\rho : 2^N \to \mathbf{R}$ is a nonnegative submodular function with $\rho(\emptyset) = 0$, which is not necessarily monotone. We also assume that the function ρ is given by a value-giving oracle. Note that the nonnegative submodularity of ρ implies the *subadditivity*, that is, we have $\rho(X) + \rho(Y) \geq \rho(X \cup Y)$ for all $X, Y \subseteq N$. For a vector $z \in \mathbf{R}^N$ and a subset $X \subseteq N$, we denote $z(X) = \sum_{i \in X} z(i)$.

Associated with the submodular function ρ, we consider a polyhedron

$$\mathrm{P}(\rho) = \{z \mid z \in \mathbf{R}^N,\ z(Y) \leq \rho(Y),\ \forall Y \subseteq N\},$$

which is called a *submodular polyhedron*. A vector in $\mathrm{P}(\rho)$ is called a *subbase*. For any subbase z, we say $X \subseteq N$ is *z-tight* if $z(X) = \rho(X)$. The submodularity of ρ implies that for any subbase z the collection of all z-tight subsets is closed under union and intersection.

Linear optimization over the submodular polyhedron can be solved efficiently by the greedy algorithm of Edmonds[4]. Given a nonnegative vector $p \in \mathbf{R}^N_+$, consider a linear ordering $L = (i_1, \cdots, i_n)$ such that $p(i_1) \geq p(i_2) \geq \cdots \geq p(i_n)$. For any $i_j \in N$, we denote $L(i_j) = \{i_1, \cdots, i_j\}$. The greedy algorithm with respect to L generates a vector $z_L \in \mathbf{R}^N$ determined by

$$z_L(i) := \rho(L(i)) - \rho(L(i) \setminus \{i\}). \tag{1}$$

Then z_L is an extreme point of $\mathrm{P}(\rho)$ maximizing the inner product $\langle p, z\rangle = \sum_{i\in N} p(i)z(i)$ among $z \in \mathrm{P}(\rho)$.

Let $p_1 > p_2 > \cdots > p_m$ be the distinct values of p. For each $j = 1, \ldots, m$, we denote $N_j = \{i \mid p(i) \geq p_j\}$. We now define $\widehat{\rho}(p)$ by

$$\widehat{\rho}(p) = \sum_{j=1}^m (p_j - p_{j+1})\rho(N_j),$$

where $p_{m+1} = 0$. The function $\widehat{\rho} : \mathbf{R}^N_+ \to \mathbf{R}$ is known as the Lovász extension.

Note that the above definition of $\widehat{\rho}$ is free from the submodularity of ρ. For a set function $f : 2^N \to \mathbf{R}$ in general, we define $\widehat{f} : \mathbf{R}^N_+ \to \mathbf{R}$ in the same way. Then $\widehat{f}(\chi_X) = f(X)$ holds for any $X \subseteq N$, where $\chi_X \in \mathbf{R}^N$ is the characteristic vector defined by $\chi_X(i) = 1$ for $i \in X$ and $\chi_X(i) = 0$ for $i \in N \setminus X$. Hence we may regard $\widehat{f}$ as a natural extension of f. Moreover, by definition, $\widehat{f}$ is positively homogeneous, that is, $\widehat{f}(\alpha p) = \alpha \widehat{f}(p)$ holds for any $\alpha > 0$ and $p \in \mathbf{R}^N_+$.

For submodular functions, the validity of the greedy algorithm[4] shows that

$$\widehat{\rho}(p) = \max\{\langle p, z\rangle \mid z \in \mathrm{P}(\rho)\}, \tag{2}$$

which implies the convexity of $\widehat{\rho}$.

The restriction of $\widehat{f}$ to the hypercube $[0,1]^N$ can be interpreted as follows. A linear ordering L corresponds to the simplex whose extreme points are given by the characteristic vectors of $L(i)$ for $i \in N$ and the empty set. Since there are $n!$ linear orderings of N, the hypercube $[0,1]^N$ can be partitioned into $n!$ congruent simplices obtained by this way. Determine the function values of $\widehat{f}$ in each simplex by the linear interpolation of the values at the extreme points. The resulting function $\widehat{f}$ is a continuous function on the hypercube.

The following theorem provides a connection between submodularity and convexity.

Theorem 1 (Lovász[23]). *A set function f is submodular if and only if $\widehat{f}$ is convex.*

3. The Submodular Vertex Cover Problem

In this section, we introduce a natural continuous relaxation of the submodular vertex cover problem using the Lovász extension of $\rho : 2^V \to \mathbf{R}$. We prove that the relaxation has a half-integral optimal solution and the rounding algorithm achieves an approximation guarantee of 2 for the submodular vertex cover problem. Furthermore, we show that a half-integral optimal solution can be obtained by one execution of submodular function minimization over a ring family.

3.1 Half-integrality

We start with the vertex cover problem, which can be formulated as an integer programming problem. The linear programming relaxation is given as follows.

$$\begin{array}{lll}
\text{(LPR)} & \text{Minimize} & \sum_{v\in V} c(v)x(v) \\
& \text{subject to} & x(u) + x(v) \geq 1 \quad ((u,v) \in E) \\
& & x(v) \geq 0 \quad (v \in V).
\end{array}$$

Nemhauser and Trotter[26] showed that (LPR) has a half-integral optimal solution. This can be derived from the following lemma in matrix theory.

Lemma 2 ([16] [Lemma 6.1]). *Let A be a nonsingular $\{0, \pm 1\}$-matrix each row and each column of which has at most two nonzero entries. Then every entry of the inverse matrix A^{-1} is a half integer.*

The half-integrality result on (LPR) naturally leads to an LP-rounding 2-approximation algorithm for the vertex cover problem. Bar-Yehuda and Even[26] developed a primal-dual 2-approximation algorithm that runs in $\mathrm{O}(|E|)$ time.

We now introduce a continuous relaxation (CPR) of the submodular vertex cover:

$$\begin{array}{lll}
\text{(CPR)} & \text{Minimize} & \widehat{\rho}(x) \\
& \text{subject to} & x(u) + x(v) \geq 1 \quad ((u,v) \in E) \\
& & x(v) \geq 0 \quad (v \in V).
\end{array}$$

This problem can be solved in polynomial time by the ellipsoid method.

Lemma 3. *The relaxation problem* (CPR) *has a half-integral optimal solution.*

Proof. Let x° be an optimal solution of (CPR). Consider a linear ordering $L = (v_1, \ldots, v_n)$ such that $x^\circ(v_1) \geq x^\circ(v_2) \geq \cdots \geq x^\circ(v_n)$. Then x° is an optimal solution to the following linear programming problem.

$$\begin{array}{lll}
\text{(SLP)} & \text{Minimize} & \widehat{\rho}(x) \\
& \text{subject to} & x(u) + x(v) \geq 1 \quad ((u,v) \in E) \\
& & x(v_j) - x(v_{j+1}) \geq 0 \quad (j = 1, \ldots, n-1) \\
& & x(v_n) \geq 0.
\end{array}$$

Note that the objective function is linear in the feasible region. The coefficient matrix of (SLP) is a $\{0, \pm 1\}$-matrix, each row of which has at most two nonzero entries. By Lemma 2, any nonsingular submatrix has a half-integral inverse matrix. Hence (SLP) has a half-integral optimal solution x^*, which is also optimal to (CPR). □

3.2 A rounding algorithm

Let x^* be a half-integral optimal solution to (CPR). Then $X^* := \{v \mid x^*(v) \geq \frac{1}{2}\}$ is a vertex cover. The following theorem shows that X^* is a 2-approximation solution for the submodular vertex cover problem.

Theorem 4. *The vertex cover X^* satisfies $\rho(X^*) \leq 2\rho(X)$ for any vertex cover X.*

Proof. The half-integral optimal solution x^* can be expressed by $x^* = \frac{1}{2}\chi_{X'} + \frac{1}{2}\chi_{X^*}$, where $X' := \{v \mid x^*(v) = 1\}$. Then $\widehat{\rho}(x^*) = \frac{1}{2}\rho(X') + \frac{1}{2}\rho(X^*) \geq \frac{1}{2}\rho(X^*)$ holds. Since $\widehat{\rho}(x^*)$ is the optimal value of the relaxation problem (CPR), we have $\widehat{\rho}(x^*) \leq \widehat{\rho}(\chi_X) = \rho(X)$ for any vertex cover X in G. Therefore, we obtain $\rho(X^*) \leq 2\widehat{\rho}(x^*) \leq 2\rho(X)$. □

We now discuss a combinatorial algorithm for finding a half-integral optimal solution to the relaxation problem (CRP). Let V^+ and V^- be the copies of V. We denote by $v^+ \in V^+$ and $v^- \in V^-$ the copies of $v \in V$. We also denote the copies of $X \subseteq V$ by $X^+ \subseteq V^+$ and $X^- \subseteq V^-$. Construct a bipartite graph $G^\pm = (V^+, V^-; E^\pm)$ with vertex sets V^+ and V^-. The edge set $E^\pm$ is given by $E^\pm = \{(u^+, v^-), (v^+, u^-) \mid (u, v) \in E\}$. For a vertex cover (X^+, Y^-), we define its rank by $\rho(X) + \rho(Y)$. Observe that if (X^+, Y^-) is a vertex cover, then $(X^+ \cap Y^+, X^- \cup Y^-)$ and $(X^+ \cup Y^+, X^- \cap Y^-)$ are also vertex covers.

Lemma 5. *Let (X^+, Y^-) be a vertex cover in $G^\pm$ with minimum rank. Then $x = \frac{1}{2}(\chi_X + \chi_Y)$ is a half-integral optimal solution of* (CPR).

Proof. For any half-integral feasible solution x of (CPR), we assign a pair of vertex subsets $X = \{v \mid x(v) = 1\}$ and $Y = \{v \mid x(v) \geq \frac{1}{2}\}$. Then (X^+, Y^-) is a vertex cover in $G^\pm$, and $\widehat{\rho}(x) = \frac{1}{2}[\rho(X) + \rho(Y)]$ holds. Conversely, for any vertex cover (X^+, Y^-) in $G^\pm$, $x = \frac{1}{2}(\chi_X + \chi_Y)$ is a feasible solution of (CPR), and $\widehat{\rho}(x) = \frac{1}{2}[\rho(X \cap Y) + \rho(X \cup Y)] \leq \frac{1}{2}[\rho(X) + \rho(Y)]$ holds. Therefore, a half-integral optimal solution x^* of (CPR) can be obtained by $x^* = \frac{1}{2}(\chi_X + \chi_Y)$ from a minimum rank vertex cover (X^+, Y^-) in $G^\pm$. □

For a vertex subset $Z \subseteq V$, let $\Gamma(Z)$ denote the set of vertices adjacent to Z in G, namely $\Gamma(Z) = \{v \mid \exists u \in Z, (u, v) \in E\}$. For any X, Y, $Z \subseteq V$ with $Z = V \setminus X$, the pair (X^+, Y^-) is a vertex cover in $G^\pm$ if and only if $\Gamma(Z) \subseteq Y$. We now consider a family $\mathcal{D}$ of subsets $D = Z^+ \cup Y^-$ of $V^+ \cup V^-$ such that $\Gamma(Z) \subseteq Y$. Then $\mathcal{D}$ forms a ring family, i.e., $\mathcal{D}$ is closed with respect to union and intersection. Note that (X^+, Y^-) is a vertex cover in $G^\pm$ if and only if $X^+ = V^+ \setminus D$ and $Y^- = D \cap V^-$ for some $D \in \mathcal{D}$. For each $D = Z^+ \cup Y^-$ in $\mathcal{D}$, we assign $f(D) := \rho(V \setminus Z) + \rho(Y)$. Then f is a submodular function on $\mathcal{D}$. Thus finding a minimum rank vertex cover reduces to minimizing the submodular function f on the ring family $\mathcal{D}$. Therefore, by Lemma 5, a half-integral optimal solution of (CPR) can be obtained by one execution of submodular function minimization over a ring family.

4. The Submodular Cost Set Cover Problem

In this section, we present approximation algorithms for the submodular cost set cover problem. For each $u \in U$, we denote $N_u = \{i \mid u \in S_i\}$. The maximum frequency η is given by $\eta = \max\{|N_u| \mid u \in U\}$. Note that the special case with $\eta = 2$ is essentially the submodular vertex cover problem, for which we have presented a 2-approximation algorithm in §3.

For the standard set cover problem (which means $\rho = c$), it is known that the greedy algorithm achieves an approximation guarantee of $\mathrm{O}(\ln k)$ (see, e.g.,[33]). As for the submodular cost set cover problem, the performance of the greedy set cover algorithm is no better than a simple k-approximation algorithm of §4.1. In contrast, the LP-rounding algorithm of Hochbaum[15] can be extended to achieve the same performance guarantee of the maximum frequency η for the submodular cost set cover problem. The resulting algorithm, presented in §4.2, requires solving a convex optimization problem by the ellipsoid method. To avoid this, we also devise a factor η primal-dual approximation algorithm in §4.3 by extending the algorithm of Bar-Yehuda and Even[1].

4.1 A simple algorithm

We start with a simple approximation algorithm. For $u \in U$, let $X_u \subseteq N$ denote a minimizer of $\rho(X)$ among all the subsets $X \subseteq N$ that covers u. Then $X^\bullet = \bigcup_{u \in U} X_u$ is a set cover.

Proposition 6. *The set cover $X^\bullet$ satisfies $\rho(X^\bullet) \leq k\rho(X)$ for any set cover X.*

Proof. Let X be an arbitrary set cover. By the definition of X_u, we have $\rho(X_u) \leq \rho(X)$ for each $u \in U$. The subadditivity of ρ implies that $\rho(X^\bullet) \leq \sum_{u \in U} \rho(X_u) \leq k\rho(X)$. □

For each $u \in U$, X_u can be computed by applying submodular function minimization $|N_u|$ times. Thus, Proposition 6 suggests a strongly polynomial k-approximation algorithm for the submodular cost set cover problem.

4.2 A rounding algorithm

Consider a convex programming relaxation of the submodular cost set cover:

$$\text{(SCP)}\quad \begin{array}{ll} \text{Minimize} & \widehat{\rho}(x) \\ \text{subject to} & \sum_{i \in N_u} x(i) \geq 1 \quad (u \in U) \\ & x(i) \geq 0 \qquad (i \in N). \end{array}$$

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This problem can be solved in polynomial time with the aid of the ellipsoid method. Let $\rho^\circ : 2^N \to \mathbf{R}$ be defined by

$$\rho^\circ(X) = \min\{\rho(Z) \mid X \subseteq Z \subseteq N\} \qquad (X \subseteq N).$$

Clearly, ρ° is monotone. It is known that ρ° is submodular (see, e.g.,[8] [Section 3.1(b)]). By definition, $\rho(X) \geq \rho^\circ(X)$ holds for all $X \subseteq N$. Therefore, for all $x \in \mathbf{R}^N_+$ we have $\widehat{\rho}(x) \geq \widehat{\rho}^\circ(x)$. For each $X \subseteq N$, let X° denote the unique minimal subset Z such that $X \subseteq Z \subseteq N$ and $\rho(Z) = \rho^\circ(X)$. Then $\rho^\circ(X) = \rho(X^\circ)$ holds for any $X \subseteq N$.

Let $x^* \in \mathbf{R}^N$ be an optimal solution to (SCP). Then $T = \{i \mid x^*(i) \geq 1/\eta\}$ is a set cover, and so is T°. Note that T° can be obtained by executing submodular function minimization. The following theorem shows that T° is an η-approximate solution for the submodular cost set cover problem.

Theorem 7. *The set cover T° satisfies $\rho(T^\circ) \leq \eta\rho(X)$ for any set cover X.*

Proof. Since $\widehat{\rho}(x^*)$ is the optimal value of the relaxation problem (SCP), we have $\widehat{\rho}^\circ(x^*) \leq \widehat{\rho}(x^*) \leq \rho(\chi_X) = \rho(X)$ for any set cover X. The function $\widehat{\rho}^\circ$ is monotone and positively homogeneous. Then it follows from $\eta x^* \geq \chi_T$ that $\eta\widehat{\rho}^\circ(x^*) = \widehat{\rho}^\circ(\eta x^*) \geq \widehat{\rho}^\circ(\chi_T) = \rho^\circ(T) = \rho(T^\circ)$. Thus, we obtain $\rho(T^\circ) \leq \eta\rho(X)$. □

Note that replacing ρ by ρ° does not change the optimal values of submodular cost set cover problems. Therefore, it suffices to consider a monotone submodular function as an objective function. One thing we should be careful of is that we must execute some submodular function minimization algorithm to obtain the value of $\rho^\circ(X)$ for each $X \subseteq N$. In this paper, we treat the nonnegative submodular function ρ directly and do not use the monotonized function ρ° in algorithms.

4.3 A primal-dual algorithm

We now present a primal-dual algorithm using the relaxation problem (SCP). Given a vector $x \in \mathbf{R}^V_+$, we have $\widehat{\rho}(x) = \max\{\langle x, z\rangle \mid z \in \mathrm{P}(\rho)\}$. Thus, the value $\widehat{\rho}(x)$ is equal to the optimal value of the following dual problem with variables $\xi(X)$ for all $X \subseteq N$.

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{X\subseteq N} \rho(X)\cdot\xi(X) \\ &\text{subject to } \textstyle\sum_{X: i\in X\subseteq N} \xi(X) = x(i) \quad (i \in N) \\ &\qquad \xi(X) \geq 0 \quad (X \subseteq N). \end{aligned}$$

Therefore, the problem (SCP) can be written as a linear program:

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{X\subseteq N} \rho(X)\cdot\xi(X) \\ &\text{subject to } \textstyle\sum_{i\in N_u} x(i) \geq 1 \quad (u \in U) \\ &\qquad \textstyle\sum_{X: i\in X\subseteq N} \xi(X) = x(i) \quad (i \in N) \\ &\qquad \xi(X) \geq 0 \quad (X \subseteq N). \end{aligned}$$

Here, we neglect the redundant nonnegativity constraint of $x(i)$ for $i \in N$. Therefore, the dual problem to (SCP) is given as follows.

$$\begin{aligned} \text{(DCP)}\ &\text{Maximize } \textstyle\sum_{u\in U} y(u) \\ &\text{subject to } z \in \mathrm{P}(\rho), \\ &\qquad \textstyle\sum_{u\in S_i} y(u) = z(i) \quad (i \in N), \\ &\qquad y(u) \geq 0 \quad (u \in U). \end{aligned}$$

The primal-dual algorithm keeps a feasible solution (y, z) of (DCP) and a subset $T \subseteq N$ that is z-tight. The algorithm starts with $y := 0$, $z := 0$ and $T := \emptyset$. Since ρ is a nonnegative submodular function with $\rho(\emptyset) = 0$, this gives a feasible solution of (DCP) and we have $z(T) = \rho(T)$. While T is not a set cover, there must be an element $u \in U$ which is not covered by T. The algorithm augments $y(u)$ and $z(i)$ for $i \in N_u$ as much as possible without violating the constraints in (DCP). Then the algorithm updates T to be the unique maximal set with $z(T) = \rho(T)$. The algorithm iterates this procedure until T becomes a set cover. The algorithm is now described more precisely as follows.

Primal-dual algorithm for the submodular cost set cover

Step 0: Put $y := 0$, $z := 0$, and $T := \emptyset$.

Step 1: Repeat the following (1-1) to (1-4) until T covers all elements of U.

- **(1-1)** Select an element $u \in U \setminus S_T$ and put $Y := N_u$.
- **(1-2)** Compute $\alpha := \max\{\lambda \mid z + \lambda\chi_Y \in \mathrm{P}(\rho)\}$.
- **(1-3)** Put $y(u) := y(u) + \alpha$ and $z := z + \alpha\chi_Y$.
- **(1-4)** Update T to be the unique maximal set with $z(T) = \rho(T)$.

Step 2: Return T.

It is easy to see that the primal-dual algorithm indeed keeps a feasible solution of (DCP) and a z-tight set $T \subseteq N$. We now analyze the running time of the primal-dual algorithm. Since

$$z + \lambda\chi_Y \in \mathrm{P}(\rho) \quad \text{if and only if} \quad \lambda|X \cap Y| \leq \rho(X) - z(X) \ (X \subseteq N),$$

the computation of α in Step (1-2) is tantamount to minimizing $[\rho(X) - z(X)]/|X \cap Y|$ subject to $X \cap Y \neq \emptyset$. This minimization problem can be solved by the Newton method within the same running time as submodular function minimization[7),25)]. The obtained minimizer X satisfies $\rho(X) = z(X) + \alpha|X \cap Y|$ and $|X \cap Y| \geq 1$. Note that $\rho(T) = z(T)$ and $|T \cap Y| = 0$. Then, after the subsequent update of z in Step (1-3), it holds that $z(X) = \rho(X)$ and $z(T) = \rho(T)$, which implies $z(X \cup T) = \rho(X \cup T)$ by the submodularity of ρ. Therefore, T gets larger as a result of Step (1-4). Thus, the algorithm terminates after at most n iterations.

We now analyze the approximation ratio of the primal-dual algorithm. The following theorem shows that the primal-dual algorithm is an η-approximation algorithm.

Theorem 8. *At the termination of the primal-dual algorithm, $\rho(T) \leq \eta\rho(X)$ holds for any set cover $X \subseteq N$, where η is the maximum frequency.*

Proof. By the definition of η, for any set cover X, we have

$$\sum_{u \in U} y(u) \leq \sum_{i \in X} \sum_{u \in S_i} y(u) \leq \eta \sum_{u \in U} y(u).$$

Since $T \subseteq N$ is a set cover with $z(T) = \rho(T)$, it follows from the feasibility of (y, z) in (DCP) that

$$\rho(T) = z(T) = \sum_{i \in T} \sum_{u \in S_i} y(u) \leq \eta \sum_{u \in U} y(u).$$

On the other side, for any set cover $X \subseteq N$, we have

$$\rho(X) \geq z(X) = \sum_{i \in X} \sum_{u \in S_i} y(u) \geq \sum_{u \in U} y(u).$$

Thus we obtain $\rho(T) \leq \eta\rho(X)$ for any set cover X. □

5. The Submodular Edge Cover Problem

This section is devoted to hardness results on the submodular edge cover problems. The edge cover problem is solvable in polynomial time by weighted matching algorithms. In contrast, we now show that the submodular edge cover problem is NP-hard.

Theorem 9. *The submodular edge cover problem is NP-hard.*

In what follows, we examine the inapproximability of the submodular edge cover problem. Our analysis is based on a framework similar to those of Goemans *et al.*[10] and Svitkina and Fleischer[32], and uses a sophisticated result on random graphs.

The simple algorithm of §4.1 achieves an approximation guarantee of k for general submodular cost set cover problems. We will see that this factor is essentially optimal even for the submodular edge cover problem with monotone submodular cost functions. The following theorem is the main result of this section.

Theorem 10. *Let $\varepsilon > 0$ be any positive real number. In the value-giving oracle model, there is no $\mathrm{O}(|W|^{1-\varepsilon})$-approximation algorithm with polynomial number of oracle calls for the submodular edge cover problem on a graph $H = (W, F)$. More precisely, the submodular edge cover problem cannot be approximated within a factor of $o(|W|/\ln^2|W|)$.*

This result immediately implies that the submodular cost set cover problem cannot be approximated within a factor of $o(k/\ln^2 k)$. The proof of Theorem 10 will be given below. The following lemma of[32] is used for obtaining the inapproximability result.

Lemma 11 ([32] [Lemma 2.1]). *Let f_1 and f_2 be two functions defined on 2^N, where f_2 is parametrized by a string of random bits R but f_1 is not. Suppose that for any subset $X \subseteq N$, chosen without knowing R, the probability over R that $f_1(X) \neq f_2(X)$ is $n^{-\omega(1)}$. Then, any algorithm that calls a value-giving oracle a polynomial number of times can find a subset $X^* \subseteq N$ such that $f_1(X^*) \neq f_2(X^*)$ with probability at most $n^{-\omega(1)}$.*

To prove Theorem 10, we will give a graph $H = (W, F)$ and two monotone submodular functions ρ_1 and ρ_2 defined on 2^F such that

- The function ρ_2 is parametrized by a random subset $R \subseteq F$ but ρ_1 is not.
- Without knowledge of R, it is difficult to find $X \subseteq F$ such that $\rho_1(X) \neq \rho_2(X)$.
- It holds that $\mathrm{OPT}_1 = \Omega(|W|)$ and $\mathrm{OPT}_2 = \mathrm{O}(\ln^2|W|)$ with probability at least $3/4$, where OPT_i is the optimal value of the submodular edge cover problem for H and ρ_i for each $i = 1, 2$.

Then, the existence of a factor $o(|W|/\ln^2|W|)$ algorithm would lead to a contradiction.

A random subgraph

Let k be an even number, and let $H = (W, F)$ be a complete graph with $|W| = k$. The edge set F is of cardinality $n = \frac{1}{2}k(k-1)$. If $X \subseteq F$ is a perfect matching in H, then $X \subseteq F$ satisfies the edge cover constraint with respect to H.

Let $R \subseteq F$ be a random subset for which each $e \in F$ is chosen independently with an identical probability $\pi \in [0, 1]$, where π is the parameter that will be defined below. We show some properties of a random subgraph $H_\pi = (W, R)$. Denote $\mu = \mathbf{E}[|R|] = \frac{1}{2}k(k-1)\pi$.

The parameter π will be defined so that R contains a perfect matching with high probability. Erdős and Rényi proved the following result on random graphs (cf.[2] [Theorem VII.14]).

Theorem 12 (Erdős and Rényi[5]). *If $\pi \geq (\ln k + 3\ln\ln k)/2k$, then the probability (over the choice of R) that $H_\pi = (W, R)$ does not have a perfect matching is $o(1)$.*

In order to evaluate the cardinality of the random subset R, we need to know the tail distribution of the sum of Bernoulli trials. The following well-known bound is referred to as a Chernoff bound.

Lemma 13 (Chernoff bounds (see, e. g.,[24])). *Let $\beta_1, \ldots, \beta_m$ be independent random variables such that $\Pr(\beta_i = 1) = \pi$ and $\Pr(\beta_i = 0) = 1 - \pi$. Let $\beta = \sum_{i=1}^m \beta_i$ and $\mu_\beta = \mathbf{E}[\beta] = \pi m$. For $\alpha \geq 8\mu_\beta$, we have $\Pr(\beta \geq \alpha) \leq \exp(-\alpha)$.*

Now, we set $\pi = \ln k/k$. Theorem 12 implies that there exists an integer k_0 such that

$$\Pr(H_\pi = (W, R) \text{ has a perfect matching}) > \tfrac{3}{4},$$

for all $k \geq k_0$. Since $\mu = \mathbf{E}[|R|] = \frac{1}{2}(k-1)\ln k$, we see from Lemma 13 that

$$\begin{aligned}\Pr(|R| \geq 8\mu) &\leq \exp(-8\mu) \\ &= \exp(-4(k-1)\ln k) = k^{-4(k-1)}.\end{aligned} \tag{3}$$

Comparison of two submodular functions

Consider the following two set functions defined on 2^F:

$$\rho_1(X) = \min\{\mu, |X|\} \quad (X \subseteq F),$$
$$\rho_2(X) = \min\{\mu, |X \setminus R| + \min\{36 \ln^2 k, |X \cap R|\}\} \quad (X \subseteq F).$$

The function ρ_2, but not ρ_1, is parametrized by the random subset R. Regardless of the choice of R, $\rho_2(X) \leq \rho_1(X)$ for all $X \subseteq F$, and both ρ_1 and ρ_2 are monotone submodular functions. Let OPT_i denote the optimal value of the monotone submodular edge cover problem for $H = (W, F)$ and ρ_i for $i = 1, 2$. We now evaluate the gap between OPT_1 and OPT_2, which plays an important role to prove Theorem 10.

Lemma 14. *If $H_\pi = (W, R)$ has a perfect matching, it holds that $\mathrm{OPT}_1 = \Omega(k)$ and $\mathrm{OPT}_2 = \mathrm{O}(\ln^2 k)$.*

Proof. For any edge cover $X \subseteq F$, we have $|X| \geq k/2$. Since $\mu = \frac{1}{2}(k-1)\ln k$, we have $\mathrm{OPT}_1 \geq \min\{\mu, k/2\} = \Omega(k)$. Consider the case where $H_\pi = (W, R)$ has a perfect matching X. Since $X \subseteq R$, we have $\mathrm{OPT}_2 \leq \rho_2(X) = \min\{\mu, 36 \ln^2 k, k/2\} = \mathrm{O}(\ln^2 k)$. □

Note that the probability (over the choice of R) that H_π has a perfect matching is at least $\frac{3}{4}$ for a sufficiently large k.

The other crucial element towards the proof of Theorem 10 is that, for any fixed $X \subseteq F$, the probability (over the choice of R) that $\rho_1(X) \neq \rho_2(X)$ is quite small.

Lemma 15. *Fix any subset $X \subseteq F$. Let R be a random subset of F for which each $e \in F$ is chosen independently with probability $\pi = \ln k/k$. Then, the probability (over the choice of R) that $\rho_1(X) \neq \rho_2(X)$ is at most $k^{-\omega(1)}$.*

Proof. To show the assertion, we consider the case that $|X| \geq 9\mu$ and the case that $|X| \leq 9\mu$, separately. We assume that k is sufficiently large.

(i) Suppose that $|X| \geq 9\mu$. Then, $|R| \leq 8\mu$ implies $\mu \leq |X \setminus R|$. Furthermore, $\mu \leq |X \setminus R|$ implies $\rho_1(X) = \rho_2(X) = \mu$. Thus, in view of (3), we obtain

$$\begin{aligned}\Pr(\rho_1(X) \neq \rho_2(X)) &\leq \Pr(|R| > 8\mu) \\ &\leq k^{-4(k-1)} = k^{-\omega(1)}.\end{aligned}$$

(ii) Suppose that $|X| \leq 9\mu$. By the definitions of ρ_1 and ρ_2, $\rho_1(X) \neq \rho_2(X)$ implies $|X \cap R| > 36 \ln^2 k$. Thus,

$$\Pr(\rho_1(X) \neq \rho_2(X)) \leq \Pr(|X \cap R| > 36 \ln^2 k). \tag{4}$$

Clearly, the right hand side of inequality (4) is maximized with respect to X when $|X| = \lfloor 9\mu \rfloor$. Let T be an arbitrary subset of F such that $|T| = \lfloor 9\mu \rfloor$ and let $\mu' = \mathbf{E}(|T \cap R|) = \pi \cdot \lfloor 9\mu \rfloor$. Since $9\mu\pi = \frac{9}{2}\frac{k-1}{k}\ln^2 k$, we have $4 \ln^2 k \leq \mu' \leq \frac{9}{2} \ln^2 k$. Lemma 13 implies

$$\begin{aligned}\Pr(|T \cap R| > 36 \ln^2 k) &= \Pr(|T \cap R| > 8 \cdot (\tfrac{9}{2} \ln^2 k)) \\ &\leq \Pr(|T \cap R| > 8\mu') \\ &\leq \exp(-8\mu') \leq \exp(-32 \ln^2 k).\end{aligned}$$

Hence, for any subset $X \subseteq F$ with $|X| \leq 9\mu$, we have

$$\begin{aligned}\Pr(|X \cap R| > 36 \ln^2 k) &\leq \Pr(|T \cap R| > 36 \ln^2 k) \\ &\leq k^{-32 \ln k} = k^{-\omega(1)}.\end{aligned} \tag{5}$$

By (4) and (5), we have $\Pr(\rho_1(X) \neq \rho_2(X)) \leq k^{-\omega(1)}$, completing the proof. □

Using Lemmas 11 and 15, we obtain the following.

Corollary 16. *For any algorithm that calls a polynomial number of value-giving oracle, the probability (over the choice of R) that it can find a subset $X \subseteq F$ such that $\rho_1(X) \neq \rho_2(X)$ is at most $k^{-\omega(1)}$.*

Proof of the inapproximability

Finally, we give a proof of Theorem 10.

PROOF OF THEOREM 10: Let $k = |W|$. Assume, to the contrary, that there is a polynomial γ-approximation algorithm $\mathcal{A}$ for the submodular edge cover problem, where $\gamma = o(k/\ln^2 k)$, which succeeds with high probability. Then, we can suppose w.l.o.g. that $\mathcal{A}$ succeeds with probability at least 3/4.

We suppose that k is sufficiently large. Apply the algorithm $\mathcal{A}$ to the submodular edge cover problem for ρ_2 and H, and let X be an edge cover given by $\mathcal{A}$. We only consider the case where $\mathcal{A}$ succeeds and H_π has a perfect matching, which occurs with probability at least $1 - (1 - \frac{3}{4}) - (1 - \frac{3}{4}) = \frac{1}{2}$. It follows from Lemma 14 that $\rho_1(X) \geq \mathrm{OPT}_1 = \Omega(k)$ and $\rho_2(X) \leq \gamma \cdot \mathrm{OPT}_2 = \mathrm{O}(\ln^2 k \cdot \gamma) = o(k)$. As a result, we obtain $\rho_1(X) \neq \rho_2(X)$ with probability at least 1/2, which contradicts Corollary 16. □

Acknowledgment

The authors thank Magnús Halldórsson and Lex Schrijver for their helpful comments.

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