A Systematic Code for Non-Independent Errors

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1. *Abstract*

A class of systematic codes is described; these codes are designed to correct every burst of length no greater than 3. It is shown that the codes considered are highly efficient. A pair of linear feedback shift registers may be used for the purpose of constructing this class of codes.

Let the code length and the number of check digits be denoted by n and m, respectively. Then for an even number m, the complete codes are given whose parity-check matrices are formed by using two sequences of the following type: the maximum-length sequence of period 3 and a suitable maximum-length sequence of period $2^{m-2}-1$. A simplified test for determing whether a particular choice of the maximum-length sequence is suitable or not is devised. Two examples of codes are illustrated.

For an odd number m, a similar method is proposed which permits the systematic construction of codes. For example, this method yields a (27, 20) code and a (121, 112) code, both of which are more efficient than the respective Reiger code[1] and are as easily realized by electronic devices.

2. *Construction of the Complete Codes with Even m*

Let us designate the codes which correct every burst of length no greater than 3 as $(n, k)_3$ code. Here k denotes the number of information digits. In this section, for example, the double error in the first and last positions will be regarded as a burst of length 2. For such $(n, k)_3$ codes, the following relation is known:

$$n \leq 2^{n-k-2}-1 = 2^{m-2}-1 . \qquad (1)$$

If the equality holds in (1), the code is called a complete $(n, k)_3$ code.

Assume that m is an even integer, and we put

$$m = 2p .$$

Since

$$2^{m-2}-1 = (2^{p-1}+1)(2^{p-1}-1) ,$$

we have

$$2^{m-2}-1 \equiv 0 \qquad (\text{mod. } 3) . \qquad (2)$$

Let $(\beta_0, \beta_1, \cdots, \beta_{2^{m-2}-2})$ be a binary maximum-length sequence of period

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$2^{m-2}-1$ which satisfies the recurrence relation,

$$\sum_{j=0}^{m-2} \varphi_j \beta_{i-j} = 0 .$$

Then the characteristic polynomial $\phi(x)$ is defined by

$$\phi(x) = \sum_{i=0}^{m-2} \varphi_{m-2-i} x^i .$$

Now, let us write

$$B_j = \begin{pmatrix} \beta_j \\ \beta_{j+1} \\ \vdots \\ \beta_{j+m-3} \end{pmatrix}, \quad j=0, 1, \cdots, 2^{m-2}-2 ;$$

$$B = (B_0, B_1, \cdots, B_{2^{m-2}-2}) . \qquad (3)$$

As is well known, there exist the integers d_{11}, d_{101}, d_{111} such that

$$B_j + B_{j+1} = B_{j+d_{11}} ,$$
$$B_j + B_{j+2} = B_{j+d_{101}} ,$$
$$B_j + B_{j+1} + B_{j+2} = B_{j+d_{111}} ,$$
$$0 \leq j \leq n-1 ; \quad 0 \leq d_{11},\ d_{101},\ d_{111} \leq n-1 .$$

A simple consideration shows that

$$d_{101} \equiv 2d_{11} \qquad (\text{mod. } 2^{m-2}-1) .$$

Now, from the maximum-length sequence of period 3, (1, 1, 0), with the characteristic polynomial $1+x+x^2$, we shall form the $2\times(2^{m-2}-1)$ matrix such that

$$C = \begin{pmatrix} 1 & 1 & 0 & \cdots\cdots \\ 1 & 0 & 1 & \end{pmatrix} .$$

And let us write

$$A = \begin{pmatrix} B \\ C \end{pmatrix} .$$

Then, the following lemma is obtained.

Lemma 1: A code whose parity-check matrix has the from A is a $(2^{m-2}-1, 2^{m-2}-m-1)_3$ code, if and only if

$$d_{11} \not\equiv 2 \qquad (\text{mod. } 3) .$$

For example, consider the maximum-length sequence (0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0,) of period 15 with the characteristic polynomial $1+x^3+x^4$. Then, we have

$$d_{11} = 12 .$$

Hence, from Lemma 1 a $(15, 9)_3$ code is obtained by using this maximum-length sequence. Similarly, a $(63, 57)_3$ code is derived from the maximum-length sequence of period 63 with the characteristic polynomial $1+x+x^6$. These codes have been found independently by Melas[(2)].

3. *Construction of the Codes with Odd m*

Now, assume that m is an odd integer. Then, since Relation (2) does not hold, the approach stated in Section 2 does not apply to this case.

Let B' denote an $(m-2)\times n$ minor matrix of B defined by (3), and let C' denote the $2\times n$ matrix such that

$$\begin{pmatrix} 0\ 1\ 0 & \cdots\cdots \\ 1\ 0\ 1 & \end{pmatrix}=C' .$$

Let us write

$$A'=\begin{pmatrix} B' \\ C' \end{pmatrix} .$$

Then, n is determined by the following lemma.

Lemma 2: The necessary and sufficient condition that a code whose parity-check matrix has the form A' shall be an $(n, n-m)_3$ code is as follows:

If d_{111} is even,

$$n\leq 2^{m-2}+1-d_{111} ,$$

and if d_{111} is odd,

$$n \leq d_{111} .$$

By d'_{111}, we shall mean d_{111} of the reverse sequence with the characteristic polynomial $x^{m-2}\phi(x^{-1})$. It is easily shown that if d_{111} is odd, d'_{111} is even.

Consequently, it is sufficient to consider only the case where d_{111} is even. Thus, from Lemma 2, it follows that $\phi(x)$ is to be chosen so that d_{111} is even and is as small as possible.

Since $d_{111}\geq m-2$ and m is odd, we have

$$d_{111} \geq m-1 .$$

Hence, if $d_{111}=m-1$, $n(=2^{m-2}+1-d_{111})$ takes the maximum value. If the polynomial

$$\phi(x)=(x^{m-1}+x^2+x+1)(x+1)^{-1}$$

is a maximal-period polynomial, it follows immediately from the definition of d_{111} that

$$d_{111} = m-1 .$$

For examples, consider the polynomials

$$(1+x+x^2+x^6)(1+x)^{-1}=1+x^2+x^3+x^4+x^5 ,$$

$$(1+x+x^2+x^3)(1+x)^{-1}=1+x^2+x^3+x^4+x^5+x^6+x^7\,.$$

These are maximal-period polynomials. Consequently, from Lemma 2 we obtain a $(27, 20)_3$ code and a $(121, 112)_3$ code, which are more efficient than the respective Reiger code.

The codes stated in Sections 2 and 3 can be implemented easily by employing linear feedback shift-registers and decoding operations are straightforward.

References

(1) S. H. Reiger; "Codes for the Correction of Clastered Errors", IRE Trans. on Information Theory, Vol. IT-6, p. 16, (March, 1960)

(2) C. M. Melas; "A New Group of Code for Correction of Dependent Errors in Data Transmission", IBM J. Research Develop., **4**, 58–65, (Jan., 1960)

(3) P. Fire; "A Class of Multiple-Error Correcting Binary Codes for Non-Independent-Errors", Stanford Electronics Labs. Tech. Rept. No. 55, (April, 1959)