Numerical Solution of Algebraic Equations

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1. *Foreword*

We frequently encounter difficulty on solving algebraic equations, which occurs as the phenomena that the accuracy of calculation and that of the solution are quite different. For example, we have to calculate in 20 digits in order to get the root of 1 percent accuracy, in case of a special example stated later. In such a problem, it is of no use to modify a method variously, and it is rather effective to transform the problem itself.

2. *Characters of Equations*

To seek for the roots of an equation is equal to search for 'x' satisfying the equation

$$f(x)=0\,.$$

We can find such x's that make $f(x)$ vahish in a definite number of calculating digits, as we like, or, in other words, a root is determined to within a certain width. In the vicinity of the root, if $f'(x)$ is large, $f(x)$ is much changed for a slight change of x, so that the root is not so much changed for a large change of $f(x)$. If $f'(x)$ is small, we must not transform $f(x)$ recklessly, as the effect of the alteration of coefficients on the solution is large.

3. *Solution of Good-Charactered Equations*

Good-charactered equations give good results by any method, so that the quality of a method is determined by the speed of convergency. Consequently, the method in which the error for roots is of the second order is better than the method in which the error is of the first order, that is, the Newton method and the Hitchcock method (sometimes called Bairstow's method) are better than 'Regula Falsi' and the Horner method. The Graeffe method is inferior to the Newton method and the Hitchcock method in briefness. We shall call the following method the 'Modified Newton method', which is the Newton method to algebraic equations modified as follows:

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$$f(x)=(x+p_n)Q(x)+R\,,$$
$$Q(x)=(x+p_n)Q'(x)+S\,,$$
$$p_{n+1}=p_n+R/S\,,$$
$$\text{convergence test:}\qquad |R/p_{n+1}S|\,.$$

This method is remarkably effective for seeking for all real roots of an equation with real coefficients, and for all roots of an equation with complex coefficients.

The modified Newton method or the Hitchcock method is the method of factorizing into linear or quadratic factors, respectively. The characteristics of these methods are the capability of obtaining the next root by using the quotient by the root obtained directly before, and that of economizing the labor of calculation as the order of the quotient descends down.

4. *Solution of Bad-Charactered Equations*

The roots of the equation

$$(x-100)^{10}-1=0$$

are

$$x_k=100+\left(\cos\frac{2\pi k}{10}+i\sin\frac{2\pi k}{10}\right)$$
$$k=1, 2, \cdots, 10\,,$$

where 1 is much less than the magnitude of the constant term in the expanded form of the polynomial $(x-100)^{10}$. The former is equal to the 20th digit of the latter. Nevertheless it is enough to alter the root by 1 percent. In such an equation we must calculate in more than 20 digits in order to gain the accuracy of 1 percent.

Though this example is an extreme case, it shows that we must sometimes calculate in high precision in order to settle the root. Therefore, we often fail to obtain the root by any method in a definite-digit calculation, where the effect of the change of the coefficients on the root is naturally severe.

In order to solve such an equation, there is no other way than to increase the number of calculating digits and to calculate the function value precisely. However, it is quite extravagant to calculate in high precision throughout, so it is necessary to do the equivalent with as less labor as possible. To do this, we need only to transfer the origin; e.g. in our example, if we put $X=x-100$, the equation becomes $X^{10}-1=0$, which is not a difficult one any more. It is necessary to shut out by high precision calculation the error caused from using the Horner method.

When we calculate in ordinary digits after the origin transfer, the accuracy of the root is not so much different from the result in case of calculating directly from the given equation, but the speed of convergence is increased.

5. *Examples*

Example 1:— We cannot get the solution of the following equation

$$(x-100)^{10}-1=0$$

corect to a sufficient number of digits unless we calculate in more than 20 digits.

Besides this extreme one, we cannot obtain the precise roots of the following equation in 8-digit calculation.

$$f(x)=x^4-4.86x^3+8.8571x^2 -7.173846x+2.1788712=0 .$$

$$\text{roots:}\quad x_1=1.20, \quad x_2=1.21, \quad x_3=1.22, \quad x_4=1.23.$$

We treat this problem with the following two methods (a) and (b).

(a) The method making use of the function value.

We need only to show that x sasisfying the equation $f(x)=0$ in 8 digits is not always the root. (See Table I.)

In Table I, we show the coefficients of $F(x)$ for various values of a, where we put $f(x+a)=F(x)$. We can conclude that the zero-region of $f(x)$ in 8 digits extends approximately between 1.195 and 1.235, where we may assume the extremum value of $f(x)$ as zero in 8-digits calculation.

Table I

a	x^4	x^3	x^2	x	x^0
0.000	1.000	−4.86	+8.8571	−7.173846	+2.1788712
1.100	1.000	−0.46	+0.0791	−0.006026	+0.0001716
1.180	1.000	−0.14	+0.0071	−0.000154	+0.0000012
1.190	1.000	−0.10	+0.0035	−0.000050	$+2.4\times10^{-7}$
1.195	1.000	−0.08	+0.00215	−0.000022	$+6.5625\times10^{-8}$
1.2038···			(extremum)		$-1.014\cdots\times10^{-8}$
1.215			(〃)		$+5.625\times10^{-9}$
1.2261···			(〃)		$-1.00\cdots\times10^{-8}$
1.235	1.000	+0.08	+0.00215	+0.000022	$+6.5625\times10^{-8}$
1.240	1.000	+0.10	+0.0035	+0.000050	$+2.4\times10^{-7}$

(b) The method making use of the Graeffe method.

We need only to see how the solution varies in one operation.

$$f(x)f(-x)=x^8-5.9054x^6+13.07617969x^4$$
$$-12.867106220676x^2+4.74747970618944.$$

We get different imaginary roots than the correct solution, if we solve the equation after rounding the coefficients in 8 digits. Consequently we cannot get the precise roots by repetition of the Graeffe method.

To tell the truth, this equation can be changed into the following equation

$$X^4-2.5\times 10^{-4}X^2+5.625\times 10^{-9}=0$$

by putting $x=X+1.215$, and then the new equation is not so difficult to solve, for the coefficients are much less than the original. Actually, we had a fairly good result, where one of the roots disagrees with the correct one only in the 8th digit by one unit, as the consequence of making use of 1 as the starting value.

Example 2:—The Modified Newton method.

By the Modified Newton method, as stated before, we can always get the new roots one by one, where we had better begin with as small a root as possible, in order to prevent the loss of significant digits in isolating the root.

7. *Conclution*

We could understand the importance of high precision calculation on the solution of algebraic equations. The settlement of difficulties of solution of algebraic equations must depend upon high precision calculation in the end. In fact, we have often to resort to high precision calculation, even when we are going to gain the root in a few digits. The fact means that we must carefully treat such a problem as has the coefficients given in a small number of digits.