Applications of Dynamic Programming

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It is well known that most of the variational problems, even in the classical sense, are difficult to solve by means of their Euler equations, and moreover, in recent years, we have so many variational problems not having analiticity, with constraints, rejecting the classical formulations, or having stochastic elements, etc., in various programming and control problems.

Dynamic programming technique may be one of the most effective formulating and computational methods for solving these novel problems because of its extensive versatility.

However, dynamic programming itself is so general in view of practical numerical computations that it is necessary to devise some additional computational techniques corresponding to the characteristic features of each problem. In particular, it is sometimes difficult to solve multi-dimensional variational problems (which have many argument functions).

The reservoirs control problem in a power system is a rather easier variational problem, but has many constraints, includes many control elements (that is, many reservoirs to be controlled) and sometimes has the stochastic elements, so that it is a typical variational problem in the modern sense.

We discuss in this summary a method of successive approximations for stochastic control process. Even though methods of successive approximations are supposed to be effective in many reservoirs deterministic case, we shall show that they are indeed very powerful methods in many reservoirs stochastic control case and that probably an application of the orthodox method (not a successive approximation) would practically be impossible.

1. *A Reservoirs Control Problem and its Functional Equations*

We consider a reservoirs control problem of which the aim is the minimization of the expected fuel cost of a thermal system during a period considered, under the conditions that the inflows into the reser-

This paper first appeared in Japanese in the Journal of Information Processing Society of Japan, Vol. 2, No. 3 (1961), pp. 138-145.

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voirs at all stages are random variables and their distributions are independent of each other.

Expressing the problem in the problem in the discrete form with N equal time intervals, we have the following stochastic variational problem:

$$\mathrm{Min}\cdot\underset{({}_1J)}{E}\cdot\underset{({}_2J)}{E}\cdot\cdots\cdot\underset{({}_NJ)}{E}[{}_1F({}_1G)+{}_2F({}_2G)+\cdots+{}_NF({}_NG)]\,, \tag{1}$$

$${}_{k+1}S_i+{}_kJ_i-{}_kQ_i \qquad \text{for} \quad {}_{k+1}S_i\leq S_{i,\max}\,, \tag{2}$$

$${}_{k+1}S_i=S_{i,\max} \qquad \text{for} \quad {}_kS_i+{}_kJ_i-{}_kQ_i>S_{i,\max}\,, \tag{2'}$$

$${}_kG+\sum_{i=1}^{n}{}_kP_i={}_kP_R\,, \tag{3}$$

$${}_kP_i={}_kP_i({}_kS_i,\ {}_{k+1}S_i({}_kJ_i),\ {}_kQ_i)\,, \tag{4}$$

$$\left.\begin{array}{l} S_{i,\min}\leq{}_kS_i\leq S_{i,\max} \\ Q_{i,\min}\leq{}_kQ_i\leq Q_{i,\max} \end{array}\right\}, \tag{5}$$

$$(k=1,\ 2,\ \cdots,\ N;\ i=1,\ 2,\ \cdots,\ n),$$

Boundary conditions: The $d_1\boldsymbol{K}_i({}_1\boldsymbol{S}_i)$ of the initial time and the desirable $d_{N+1}\boldsymbol{K}_i({}_{N+1}\boldsymbol{S}_i)$ of the end time are given.

The left-lower index for a letter indicates the stage or time number, while the right-lower index indicates the hydro-plant number, and the letter F denotes the fuel cost per unit time, G the thermal system output (power), P the hydro-plant output, P_R the load, S the water-storage (state), Q the available discharge (decision or control), J the inflow into a reservoir, and ${}_k\boldsymbol{J}=({}_kJ_1,\ {}_kJ_2,\ \cdots,\ {}_kJ_n)$ is a set of inflows into the reservoirs at the k-th stage. $\boldsymbol{E}$ indicates the expectation-operator with respect to the distributions of the quantities indicated below $\boldsymbol{E}$.

$d_1\boldsymbol{K}_i({}_1\boldsymbol{S}_i)$ and $d_{N+1}\boldsymbol{K}_i({}_{N+1}\boldsymbol{S}_i)$ are distributions of ${}_1\boldsymbol{S}_i$ and ${}_{N+1}\boldsymbol{S}_i$ respectively. Due to (3) and (4), (1) may be written in the form

$$\mathrm{Min}\cdot\underset{({}_1J)}{E}\cdot\underset{({}_2J)}{E}\cdot\cdots\cdot\underset{({}_NJ)}{E}[{}_1F({}_1\boldsymbol{S},\ {}_1\boldsymbol{Q},\ {}_1\boldsymbol{J})+\cdots+{}_NF({}_N\boldsymbol{S},\ {}_N\boldsymbol{Q},\ {}_N\boldsymbol{J})]\,, \tag{1'}$$

and moreover, from the independence of distributions of inflows it may be rewritten as

$$\mathrm{Min}\cdot\underset{({}_1J)}{E}\{{}_1F+\mathrm{Min}\cdot\underset{({}_2J)}{E}[{}_2F+\cdots+\mathrm{Min}\cdot\underset{({}_NJ)}{E}[{}_NF]\cdots]\}\,. \tag{1''}$$

For such a stochastic multistage decision process we define the "minimum expected cost functions" f as follows:

$f_{N-k+1}({}_k\boldsymbol{S})=$the minimum expected cost with an optimal policy over the period from the k-th time to the end time ($N-k+1$ stages) and with the initial conditions of states ${}_k\boldsymbol{S}$ at the k-th time.

Then we obtain the following set of functional equations:

$$f_1({}_N S)=\underset{{}_N Q \;\; ({}_N J)}{\mathrm{Min}}\left[\boldsymbol{E}\{{}_N F({}_N \boldsymbol{S},\ {}_N \boldsymbol{Q},\ {}_N \boldsymbol{J})\}\right], \tag{6}$$

$$f_i({}_{N-i+1}\boldsymbol{S})=\underset{{}_{N-i+1}Q}{\mathrm{Min}}\left[\underset{({}_{N-i+1}J)}{\boldsymbol{E}}\{{}_{N-i+1}F+f_{i+1}({}_{N-i+2}\boldsymbol{S})\}\right],\ (i=2,3,\cdots,N). \tag{7}$$

A complete calculation of dynamic programming consists of

(I) a step in which we obtain a set of the minimum expected cost functions $f_1, f_2, \cdots, f_N$ in order, by the functional equations (6) and (7), and

(II) a step in which we determine the all states (distributions in general) due to the informations obtained in (I).

They are called "calculation I" and "calculation II" respectively. Calculation II is the determination of an optimal Markov-process in the stochastic case.

2. *A Method of Successive Approximations*

Starting with some suitable feasible initially estimated policy (solution), we correct the policy with small corrections repeatedly, and approach the optimal policy successively. We can reduce approximately the many reservoirs control (multi-dimensional) problem to a sequence of one reservoir control (one-dimensional) problems of which policies are corrected one by one in a suitable order as follows.

(1) Suppose "feasible" policies and probabilty distributions of the states (water storages) at each time, where all the hydroplants are known initially.

(2) Fixing the policies of all the hydro-plants except hydro-plant No. 1 at the initially estimated policies, we consider a local optimization problem consisting of hydro-plant No. 1, the thermal system and the "Equivalent Load" obtained by subtracting the expected powers of all plants except plant No. 1 from the proper load, and we correct the policy of plant No. 1 to obtain more economy in the suitably narrow "permissible" domain. We take this new policy as an initial policy in the next.

We repeat the similar processes for plant No. 2, No. 3, $\cdots$, No. n, and again back to the policy-correction step of No. 1, and so forth. We must notice the following remarks in this method.

(A) *Equivalent Problem and Equivalent Load*

Let us now consider the correction step of policy of plant No. 1 as an example. If the fuel cost characteristic of the thermal system is expressed by a quadratic (convex) function of its output power G, $F(G)=\alpha_0+\alpha G+\beta G^2$, we canreplace the original problem by an "equivalent" problem of which total expected cost is

$$\underset{(J_1)}{\boldsymbol{E}}[{}_1F({}_1P_R'-{}_1P_1)+{}_2F({}_2P_R'-{}_2P_1)+\cdots+{}_NF({}_NP_R'-{}_NP_1)]$$
$$+\beta\sum_{k=1}^{N}\sum_{i=2}^{n}\{\textbf{var.}\,{}_kP_i\}, \tag{8}$$

$$\textbf{var.}\,{}_kP_i\equiv\underset{(J_i)}{\boldsymbol{E}}[{}_kP_i^2]-(\underset{(J_i)}{\boldsymbol{E}}[{}_kP_i])^2, \tag{9}$$

$$\underset{(J_i)}{\boldsymbol{E}}[\quad]=\int_{-\infty}^{\infty}d_1H_i({}_1J_i)\int_{-\infty}^{\infty}d_2H_i({}_2J_i)\cdots\int_{-\infty}^{\infty}d_NH_i({}_NJ_i)\cdot[\quad].$$

P_R' is the "equivalent load":

$${}_kP_R'={}_kP_R-\sum_{i=2}^{n}\underset{(J_i)}{\boldsymbol{E}}[{}_kP_i]. \tag{10}$$

The similar replacements are valid for other hydro-plants.

(B) *Feasible Policies and Phase I, Phase II*

A policy which satisfies the given constraints is generally called a feasible policy. It is necessary to prepare a feasible policy as an initial estimation for each plant in our successive approximations method, because, starting with an initial policy, we are going to correct the policy successively to obtain more economy in the "feasible" domain. The first computational stage in which we get the initial feasible policies for all plants and the second stage in which we correct these policies successively are designated as "Phase I" and "Phase II" respectively, after the nomenclature in linear programming.

Although an initial feasible policy for each plant can be obtained easily in the deterministic case, getting it in the stochastic case will be rather difficult, because in the policy of the stochastic case we must define a decision for each state (water storage) at each time, and every decision must be feasible for all possible inflow values.

We proposed an easier method of Phase I, of which computation program is almost the same as Phase II. It is, however, quasi-deterministic, that is, we consider the problem as a deterministic control process having expected inflows, but all the decisions of the process are feasible not only for the expected values of inflows but also for all possible inflow values.

3. *Flow-Chart of Computation*

See Fig. 1.

4. *An Example*

Fig. 2 shows the expected cost reduction in a typical control problem including two reservoirs using our successive approximation method.

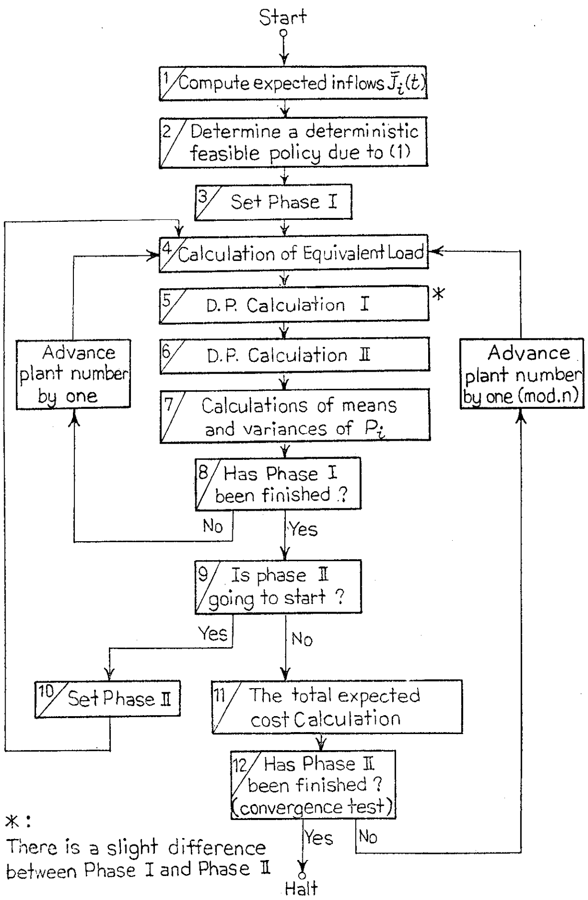
Start
1 Compute expected inflows $\bar{J}_i(t)$
2 Determine a deterministic feasible policy due to (1)
3 Set Phase I
4 Calculation of Equivalent Load
5 D.P. Calculation I
*
6 D.P. Calculation II
7 Calculations of means and variances of P_i
Advance plant number by one
Advance plant number by one (mod.n)
8 Has Phase I been finished ?
No
Yes
9 Is phase II going to start ?
Yes
No
10 Set Phase II
11 The total expected cost Calculation
12 Has Phase II been finished ? (convergence test)
Yes
No
Halt
*:
There is a slight difference between Phase I and Phase II

Fig. 1

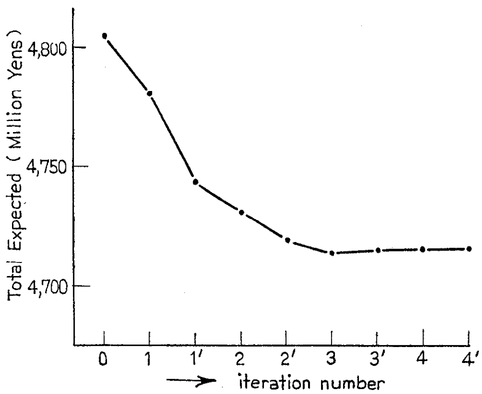
Total Expected (Million Yens)
4,800
4,750
4,700
0
1
1′
2
2′
3
3′
4
4′
iteration number

Fig. 2

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